1

$$\min_x f(x)$$
$$g(x) \leq 0$$
$$h(x) = 0$$

2024-10-17

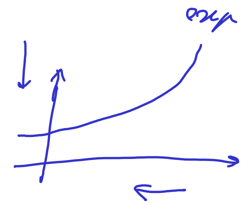
linear
quadratic
convex

} tractable → can be solved efficiently

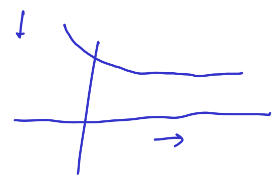
non linear non convex
integer / mixed-integer

* slide 2

$$\min_x \exp\left(x_1^2 + x_1 x_2 + 2x_2^2 + x_3^2\right)$$
$$(x_1 + x_2 + x_3)^2 \leq 9$$
$$\arctan(x_1 - x_2 x_3) \geq \frac{\pi}{4}$$



f increasing → minimize argument



f decreasing → maximize argument

$$\min_x x_1^2 + x_1 x_2 + 2x_2^2 + x_3^2$$ → convex (see 2)
$$-3 \leq x_1 + x_2 + x_3 \leq 3$$
$$x_1 - x_2 x_3 \geq 1$$ → see 3 for simplification

② $\min_x \; x_1^2 + x_1 x_2 + 2x_2^2 + x_3^2$

? convex
↓
3 methods

a) check H

$$H = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} x_1 & x_2 & x_3 \\ \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 4 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 2 \end{array}\right] \end{array} \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}$$

$\frac{1}{2} x^T H x$

$H_{\{1\}\{1\}} = 2 \quad \rightarrow \quad 2 > 0$

$H_{\{1,2\}\{1,2\}} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \rightarrow \quad 7 > 0$

$H_{\{1,2,3\}\{1,2,3\}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow 16 - 2 = 14 > 0$

→ H is positive definite → f is convex

$5x_1^2 + 8x_1x_2 + 3x_2^2 + 3x_1x_3 + 6x_3^2$

$$H = \begin{bmatrix} 10 & 8 & 3 \\ 8 & 6 & 0 \\ 3 & 0 & 12 \end{bmatrix}$$

b) positive sum of squares

$$\underbrace{x_1^2 + x_1x_2 + 2x_2^2}_{} + x_3^2$$

$(x_1 + \frac{1}{2} x_2)^2 + \frac{7}{4} x_2^2 + x_3^2$ → convex quadratic function

$\frac{1}{4} x_2^2$

$(x_1 + x_2 + 5x_3)^2$ → convex

$\sum_{w_i > 0} w_i$ convex → convex

c) compute eigenvalues of H → all positive → convex

(3) $9x_1^2 + 4x_1x_2 + x_2^2$

$= (2x_1 + x_2)^2 + 5x_1^2$ → conclude convex

$= (x_1 + 2x_2)^2 + 8x_1^2 - 3x_2^2$ → no conclusion

for positive sum of squares → sometimes you need to try several possible combinations

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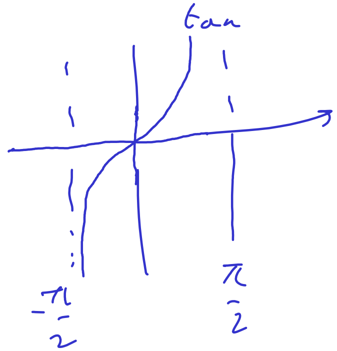
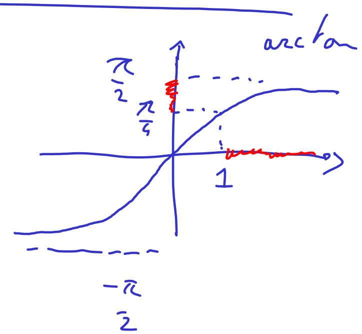
$\arctan(\underline{x_1 - x_2 x_3}) \geq \frac{\pi}{4}$



inverse →



$x_1 - \underbrace{x_2 x_3}_{} \geq 1$

↳ not linear

$\arctan(\dots) \geq \frac{\pi}{4}$

$\Updownarrow$

$(\dots) \geq 1$

standard form:

$g(x) \leq 0$

$\underbrace{x_2 x_3}_{} - x_1 + 1 \leq 0$ → not convex

↳ convex?

$$H = \begin{array}{cc} & \begin{array}{cc} x_2 & x_3 \end{array} \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{array}{c} x_2 \\ x_3 \end{array} \end{array}$$

$H_{11} \to 0$

$H \to -1$

⇒ simplified problem is not convex

(4) $\|x\|_1$ $\quad \min_{Ax \le b} \sum_i |x_i|$ $\quad \rightarrow \quad \min_{x,\alpha} \sum \alpha_i \quad \alpha_i \geq |x_i|$, $\quad Ax \le b$

$\Downarrow$ $\quad \alpha_i^* = |x_i^*|$

$\min \alpha \quad \alpha \geq |4|$, $\quad \alpha = 5 \; ? \; 4 = x^*$

$|v| = \max(v, -v) \quad \rightarrow$

$$\boxed{\begin{array}{lll} \min \sum \alpha_i & \alpha_i \geq x_i & \alpha_i \geq -x_i \\ Ax \le b & & \end{array}} \rightarrow LP$$

(*) $\alpha_i \geq 0$ ✓

$\|x\|_\infty$ $\quad \min_{Ax \le b} \max_i |x_i| \quad \rightarrow \quad \min_{x,t} t$, $\quad Ax \le b \quad t \geq |x_i|, \forall i \rightarrow t \geq \max_i |x_i|$

$\rightarrow t^* = \max_i |x_i^*|$

$$\boxed{\begin{array}{lll} \min t & t \geq \underset{\alpha_i}{x_i}, & t \geq \underset{\alpha_i}{-x_i} \\ Ax \le b & & \end{array}} \rightarrow LP$$

(*) + also works if there is a positive weight α_i in front of $|x_i|$

5

$$\min_{x} \max(x_1, 2x_2, 4x_3, 8x_4) \rightarrow \min_{x,t} t$$

$$t \geq \max(x_1, 2x_2, 4x_3, 8x_4)$$

$$t^* = \max(x_1^*, x_2^*, x_3^*, x_4^*)$$

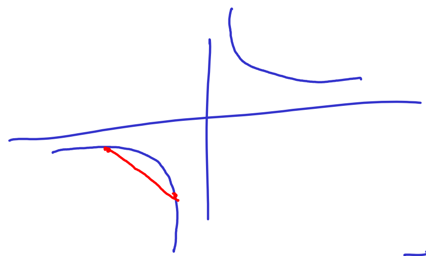
LP $\rightarrow$

$$\boxed{\begin{array}{l} \min_{x,t} t \\ t \geq x_1 \quad t \geq 2x_2 \quad t \geq 4x_3 \quad t \geq 8x_4 \end{array}}$$

$$\min_{x_1,x_2} \frac{1}{1 + x_1 + x_2}$$

$\rightarrow$ not convex

$$\frac{1}{x}$$

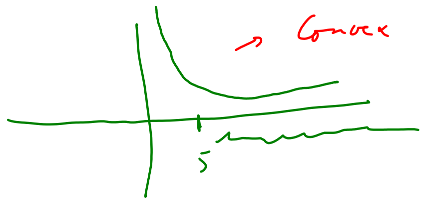


$x < 0$
$\rightarrow$ not convex

$0 \notin \operatorname{dom} f$

$\rightarrow \frac{1}{x}$ not convex

$$\left.\begin{array}{l} \min_{x_1 x_2} \frac{1}{1 + x_1 + x_2} \\ x_1 + x_2 \geq 4 \end{array}\right\} \text{Convex}$$



$\rightarrow$ Convex if $x > 0$

⑥ f analytical (exp, sin, cos, tan, ...) $\to$ compute gradient "easily"
Hessian

$x \to \boxed{?} \to f$

$$\left.\begin{array}{l}\text{few variables}\\ \text{computation is fast}\end{array}\right\} \to \text{use numerical gradient / Hessian}$$

$\sim \frac{n^2}{2}$

$$\frac{\partial f}{\partial x_i}(x_0) \approx \frac{f\left(x_0 + \delta \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\right) - f(x_0)}{\delta}$$

$\to n$ extra function evaluations

$$\left.\begin{array}{l}\text{n is very large}\\ \text{computation is very slow}\end{array}\right\} \to \text{do \underline{not} use numerical gradient / Hessian}$$

$\to$ use gradient-free method $\to$ e.g. genetic

a) Levenberg-Marquardt $\to H, Df$
b) steepest descent $\to Df$

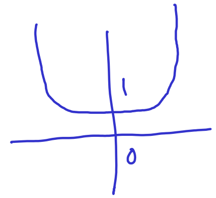
if only possible choice is a or b $\to$ b

② Home work

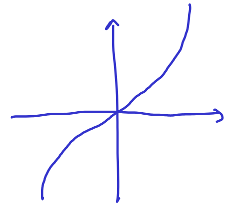
$$\min_x \; e^x\left(3\cosh x - 4\sinh x + 8\right)$$

$$\text{s.t.} \quad x \le \log 2$$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$



$\sinh = \dfrac{e^x - e^{-x}}{2}$



after simplification?

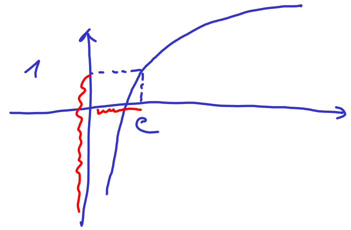
1) LP
2) QP
3) convex
4) nonlinear nonconvex

5) $$\min_x \; e^{x_1 - 2x_2 + 3x_3} - \log\left(-x_1 + 2x_2 + 5x_3\right)$$

$$(x_1 + x_2 + x_3 - 7)^5 \le 1024 = 2^{10}$$

8

$$\max_x \quad 4x_1 + 5x_2 - 6x_3$$



$$\log |2x_1 + 7x_2 + 5x_3| \leq 1 \qquad x_1, x_2, x_3 \geq 0.01$$

$$\rightarrow \quad \min_x \quad -4x_1 - 5x_2 + 6x_3 \qquad x_1, x_2, x_3 \geq 0.01$$

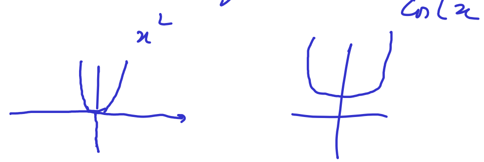
$$|2x_1 + 7x_2 + 5x_3| \leq e$$

$$-e \leq 2x_1 + 7x_2 + 5x_3 \leq e$$

lin. obj. + lin. constraint → LP → simplex, finite number of steps

(9) $$\min_x \; \max\left(\cosh(x_1+x_2+x_3),\ (5x_1-6x_2+7x_3+6)^2\right)$$

$$\|x\|_2 \le 10$$



A ~~lin~~ ~~quadratic~~
convex?

$$\left.\begin{array}{l} \cosh:\ \text{convex} + \text{affine argument} \to \text{convex} \\ (\ \ )^2:\ x^2\ \text{convex} + \text{affine argument} \to \text{convex} \end{array}\right\} \xrightarrow{\max} \text{convex!}$$

$g(x) \le 0$ $\qquad \|x\|_2 - 10 \le 0$

↳ ? convex $\qquad \|\cdot\|$ is convex in argument → convex!

⇒ convex optimization problem → cutting plane, ellipsoid, interior point

$$|f(x^*) - f(x_k)| \le \varepsilon_f$$

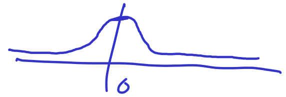
$$g(x_k) \le \varepsilon_g$$

+ (ellipsoid): $\|x^* - x_k\| \le \varepsilon_x$

(10)

$$\max_{x_1 x_2} \; e^{-x_1^2 - x_2^2}\left(x_1^2 + x_1 x_2 + 6x_1\right)$$

e^{-x^2}



$$\min_{x_1 x_2} \; \underbrace{-\,e^{-x_1^2 - x_2^2}}_{\text{nonconvex}} \cdot \underbrace{\left(x_1^2 + x_1 x_2 + 6x_1\right)}_{}$$

$H = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{matrix} 2 \\ -1 \end{matrix} \rightarrow$ not $H > 0$

nonconvex

$\rightarrow$ nonconvex, nonlinear $\longrightarrow$ Newton, Levenberg-Marquardt

$\longrightarrow \|\nabla f(x_k)\|_2 \leq \varepsilon$