Sample Exam 1
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods (60 points)

Consider the following optimization problems:

P1)	$\min_{x\in\mathbb{R}^2} (x_1-5)^2+(x_2-8)^4$	$x_1^2+x_2^2 \geqslant 1$
P2)	$\min_{x\in\mathbb{R}^3} \lvert x_1-4x_2+x_3-4\rvert$	$x\geqslant 0\,,\ x_1+x_2\geqslant 25$
P3)	$\min_{x\in\mathbb{R}^3} \int_0^{x_2}\left(\frac{e^{x_3^2 z}\cos(z)}{x_3^4 z^4+1}+z^{x_1}\right)dz$	$x_2^2+\log(x_1^2+1)\geqslant 10$
P4)	$\min_{x\in\mathbb{R}^2} e^{x_1^2+x_2^2}+x_1^2x_2^2$	
P5)	$\min_{x\in\mathbb{R}^2} \cosh(x_1^2+x_2^2)+14x_2^4$	$x_1^2+(x_2-10)^2\leqslant 50$
P6)	$\min_{x\in\mathbb{R}^2} \cosh(x_1^2+x_2^2)+14x_2^4$	$x_1^2+(x_2-10)^2\geqslant 50$
P7)	$\min_{x\in\mathbb{R}^2} e^{x_1}-x_2^3$	$x_1+x_2\geqslant 15$
P8)	$\min_{x\in\mathbb{R}^2} 2^{5x_1^2+2x_2^2+6x_1x_2+3x_1-x_2}$	$x_1+x_2\leqslant 10\,,\ x_1,x_2\geqslant 0$
P9)	$\min_{x\in\mathbb{R}^2} ((x_1-1)^2+(x_2-2)^2)^2-\log(2-x_1^2-x_2^2)$	$\lVert x\rVert_1\leqslant 1$
P10)	$\min_{x\in\mathbb{R}^2} (x_1+x_2)^3-\log(2-x_1^2-x_2^2)$	$\lVert x\rVert_2\leqslant 1$

a) Determine for each of the problems **P1** up to **P10** which of the following optimization methods is best suited:

M1. Simplex method for linear programming.
M2. Modified simplex method for quadratic programming.
M3. Cutting-plane algorithm.
M4. Davidon-Fletcher-Powell quasi-Newton algorithm.
M5. Levenberg-Marquardt algorithm.
M6. Perpendicular method with cubic line search.
M7. Penalty function method with Nelder-Mead algorithm.
M8. Gradient projection method with Fibonacci line search.
M9. Sequential quadratic programming.

– ***Explain*** why the selected algorithm is the *best suited* for the given problem.

- Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.
- If it is necessary to use the selected algorithm in a *multistart local minimization* procedure, you have to ***indicate*** this clearly.

b) Determine an appropriate stopping criterion for each of the problems **P1** up to **P10** and the corresponding selected optimization method. Explain your choice!

QUESTION 2: Optimization methods II (20 points)

a) Consider the following optimization problem:

$$\min_{x,y\in\mathbb{R}} f(x,y)$$

where

$$f(x,y) = |2x-2y| + |x+y-2| \ .$$

- Prove that the function f is convex.
- Prove that $(x,y) = (1,1)$ is the global optimum of this optimization problem.
- Show that the perpendicular search method fails when starting at the initial point $(x_0, y_0) = (0,0)$.

b)
- Show that the condition

$$\forall y \in [0,1] : \begin{bmatrix} \gamma - y - 1 & x \\ x & y+1 \end{bmatrix} \geqslant 0$$

with $x, y \in \mathbb{R}$ is satisfied if and only if

$$\begin{bmatrix} \gamma - 1 & x \\ x & 1 \end{bmatrix} \geqslant 0$$

$$\begin{bmatrix} \gamma - 2 & x \\ x & 2 \end{bmatrix} \geqslant 0 \ .$$

- Now consider the LMI-based optimization problem

$$\min_{\gamma,x\in\mathbb{R}} \gamma$$

$$\begin{bmatrix} \gamma - 1 & x \\ x & 1 \end{bmatrix} \geqslant 0$$

$$\begin{bmatrix} \gamma - 2 & x \\ x & 2 \end{bmatrix} \geqslant 0$$

and show that this optimization problem is equivalent to the optimization problem

$$\min_{x\in\mathbb{R}} \max_{y\in[0,1]} f(x,y)$$

where

$$f(x,y) = y + 1 + \frac{x^2}{y+1} \ .$$

<u>Hint</u>: Use the Schur complement property for positive *semi*-definite matrices: if $R(\theta)$ is invertible, then

$$\begin{bmatrix} Q(\theta) & S(\theta) \\ S^T(\theta) & R(\theta) \end{bmatrix} \geqslant 0$$

is equivalent to

$$R(\theta) \geqslant 0 \quad , \quad Q(\theta) - S(\theta)R^{-1}(\theta)S^T(\theta) \geqslant 0 \quad .$$

QUESTION 3: Controller design using optimization (20 points)

a) Consider the system described by

$$y(k) = \frac{a}{1-q^{-1}} u(k)$$

with $a \in \mathbb{R}$ and where q is the forward shift operator. The initial condition is $y(0) = y_0$. Let the input signal u be defined by $u(k) = 1$ for all k.
Is $y(k)$ convex as a function of a? Why?

Let N be a positive integer and define

$$y_{\text{av}} = \frac{1}{N} \sum_{k=1}^{N} y(k) \quad . \tag{1}$$

Is y_{av} convex as a function of a? Why?

b) Consider the system described by

$$y(k) = \frac{1}{1-bq^{-1}} u(k)$$

with $b \in \mathbb{R}$. The initial condition is $y(0) = y_0$. Let the input signal u be defined by $u(k) = 1$ for all k.
Is $y(k)$ convex as a function of b? Why?
Is y_{av} (cf. (1)) convex as a function of b? Why?

c) Consider the system described by

$$y(k) = \frac{d}{1-bq^{-1}-cq^{-2}} u(k)$$

with $b, c, d \in \mathbb{R}$. The initial condition is $y(0) = y_0$, $y(-1) = y_{-1}$. Let the input signal u be defined by $u(k) = 1$ for all k.
Is $y(k)$ convex as a function of b, c, and d? Why?
Is y_{av} (cf. (1)) convex as a function of b, c, and d? Why?