# Sample Exam 2
# "Optimization in Systems and Control" (SC4091)

## QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x\in\mathbb{R}^4} |2x_1 - x_2 + x_3 - 3x_4| - 6\log(5 + x_1 + x_2 + x_3) + x_4^2$$
$$\text{s.t. } x_1^2 + x_2^4 + x_3^2 + x_4^4 \leqslant 1$$

P2. $$\max_{x\in\mathbb{R}^4} \frac{x_1 + x_2^2 - x_1x_2 - x_3^3 + 2x_4}{1 + 5\sqrt{x_1^4 + x_2^6 + x_3^8 + x_4^4}}$$

P3. $$\max_{x\in\mathbb{R}^5} x_1^3 + x_2^4 - 3\cos(x_3x_4x_5) + x_2$$
$$\text{s.t. } -2 \leqslant \log(1 + 3x_1^2 + 2x_2^2 + 8x_3^2) \leqslant 3$$
$$|x_i| \leqslant 4 \quad \text{for } i = 1, 2, \ldots, 5$$

P4. $$\min_{x\in\mathbb{R}^4} \max(|x_1|, 2|x_2|, 3|x_3|, 4|x_4|)$$
$$\text{s.t. } e^{x_1}e^{x_2}e^{x_3}e^{x_4} \leqslant e^9$$
$$(x_1 + x_2 - x_3 - 6x_4)^2 \leqslant 36$$

P5. $$\min_{\theta\in\mathbb{R}^7} \left\| \begin{pmatrix} f(x_1,\theta) \\ f(x_2,\theta) \\ \vdots \\ f(x_{2000},\theta) \end{pmatrix} - \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{2000} \end{pmatrix} \right\|_2$$

where $\theta$ are the (unknown) parameters of a nonlinear function $f$ of the form

$$f(x,\theta) = \theta_1 + \theta_2\cos(x - \theta_3)e^x + \theta_4\sin(2x - \theta_5)e^{2x} + \theta_6\cos(3x - \theta_7)e^{3x}$$

with $x$ a scalar variable, and where $\{x_k\}_{k=1}^{2000}$ is a sequence of data points, and $\{\hat{f}_k\}_{k=1}^{2000}$ is the corresponding sequence of measurements points for the function $f$.

P6. $\min_{x\in\mathbb{R}^3} 4x_1^2+4x_1x_2+6x_1x_3+5x_2^2+2x_2x_3+3x_3^2-x_1+x_2-1$

s.t. $x_1^2+2x_2^2+6x_3^2 \leqslant 4$

$|x_1+x_2+x_3| \leqslant 8$

P7. $\min_{v\in\mathbb{R}^{10}} F(v)$

s.t. $0 \leqslant v_i \leqslant 1$ for all $i$

where

$$F(v) = \int_0^{v_1}\int_{v_2}^{1}\int_0^{v_3} \frac{\cos(v_4t_1)+v_5(1+t_2)^{v_6}+\exp(2t_3)(1+\sin(v_7t_1^2))}{|2+t_2\sin(v_6t_2)+\cos(v_8t_1+v_9t_3)^2+\exp(v_{10}t_2)|}\,dt_1dt_2dt_3$$

P8. $\min_{x\in\mathbb{R}^5} 7x_1^3+x_2^2-3x_3x_4+2x_3+7x_5-9$

s.t. $\exp(1+x_1^4x_2^2)+x_2+5\sin(x_3^2)+\cos(x_1x_5-x_4)-\log\left(1+(x_4+x_1x_3x_5)^2\right)=1$

P9. $\min_{x\in\mathbb{R}^3} (7x_1^2+2x_2^2+x_3^2-4x_1x_2+2x_2x_3+1)^{\frac{1}{3}}$

s.t. $\|x\|_1 \leqslant 1$

P10. $\min_{x\in\mathbb{R}^3} 2^{|x_1|+|x_2|+|x_3|}$

s.t. $\min(x_1-x_2+8, 9-2x_1+3x_2) \geqslant 5$

$\cosh(x_1+x_2-x_3-6)^2 \leqslant 100$

Questions:

1. For each of the problems P1 up to P10:

   a) Determine the best suited optimization method for each of the given problems. Select one of the following methods:

      M1 : Simplex algorithm for linear programming
      M2 : Modified simplex algorithm for quadratic programming
      M3 : Sequential quadratic programming
      M4 : Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm
      M5 : Gauss-Newton least-squares algorithm
      M6 : Levenberg-Marquardt algorithm
      M7 : Line-search method with Powell directions and Fibonacci line minimization
      M8 : Nelder-Mead method
      M9 : Lagrange method + Levenberg-Marquardt algorithm
      M10 : Barrier function method + line search method with steepest descent direction and variable-step line minimization
      M11 : Penalty function method + line search method with Powell directions and line minimization using parabolic interpolation
      M12 : Ellipsoid algorithm
      M13 : Genetic algorithm

   - ***Explain*** why the selected algorithm is the *best suited* for the given problem.
   - Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.
   - If it is necessary to use the selected algorithm in a *multi-start local minimization* procedure, you have to ***indicate*** this clearly.

   b) Determine the most appropriate stopping criterion for the selected optimization method and ***explain*** why it is the most appropriate!

2. Determine the gradient and the Hessian of the objective function for problem P6.

# QUESTION 2: Optimization methods II (20 points)

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2^2 - x_1 x_2 - 4x_1 - 2x_2 \quad . \tag{1}$$

This problem is solved using the Nelder-Mead method.

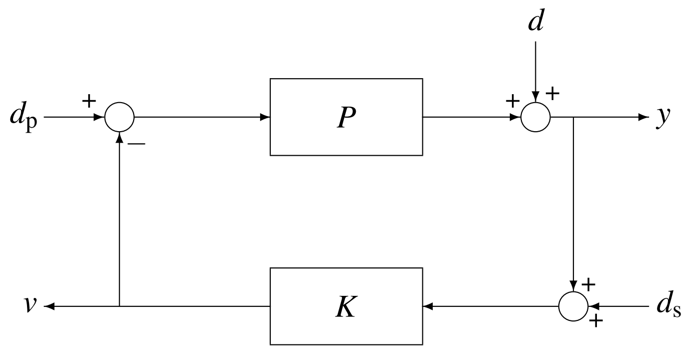
Questions:

a) Briefly explain how the Nelder-Mead method works (both the basic Nelder-Mead method and the extensions).
Illustrate your explanation by drawing one or more pictures.

b) Consider the optimization problem (1) given above.
Compute the first three points obtained using the basic Nelder-Mead method (without contraction or expansion) starting from the initial simplex defined by the points $(0,0)$, $(0,1)$, and $(1,0)$. Also give the values of the objective function for each of the points.

# QUESTION 3: Controller design (20 points)

Consider the following closed-loop system consisting of a system $P(q)$ and a controller $K(q)$ where $q$ denotes the forward shift operator:




There are two output signals $\big(y(k)$ and $v(k)\big)$ and three disturbance signals $\big(d(k)$, $d_\text{p}(k)$, and $d_\text{s}(k)\big)$.

<u>Questions:</u>

a) Determine the transfer function matrix $M$ such that

$$\begin{bmatrix} y(k) \\ v(k) \end{bmatrix} = M(q) \begin{bmatrix} d(k) \\ d_\text{p}(k) \\ d_\text{s}(k) \end{bmatrix} .$$

b) Assume that $P(q)$ is a rational stable transfer function and that $K(q)$ is parameterized as

$$K(q) = \frac{Q(q)}{1 - P(q)Q(q)}$$

with $Q(q)$ a rational stable transfer function. Show that all entries of $M(q)$ are stable.

c) Prove — using the *definition* of closed-loop convexity — that the constraint $\|M_{11}\|_\infty \leqslant 1$ is closed-loop convex in the parameter $Q$.