Sample Exam 3
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I (60 poimts)

Consider the following optimization problems:

P1. $\min_{x \in \mathbb{R}^3} \max\left(\cosh(x_1 + x_2 + x_3), (5x_1 - 6x_2 + 7x_3 + 6)^2\right)$

subject to $\|x\|_2 \leqslant 10$

P2. $\min_{x \in \mathbb{R}^5} \exp\left(\max_i |x_i|\right)$

subject to $2 \leqslant x_1 + x_2 + x_3 \leqslant 6$

$3 \leqslant x_3 - x_4 - x_5 \leqslant 7$

P3. $\min_{x \in \mathbb{R}^2} (1 - x_1 - x_2)\sin(x_1^2 + x_2^2) - x_1 x_2 \cos(x_1 - x_2)$

subject to $x_1, x_2 \in \{-10, -9, \ldots, 9, 10\}$

P4. $\min_{x \in \mathbb{R}^3} 2x_1^2 + 2x_1 x_3 + x_3^2 + 6x_2 + 7$

subject to $-2 \leqslant x_1, x_2, x_3 \leqslant 2$

P5. $\min_{x \in \mathbb{R}^2} \int_0^{x_1} \int_1^{x_2} \left\lfloor \frac{t_1 \sin(x_1 + x_2 + t_2)}{\exp(t_1 t_2 - x_1 x_2)} \right\rfloor dt_1 dt_2$

where $\lfloor x \rfloor$ with $x \in \mathbb{R}$ is the largest integer number less than or equal to x,

P6. $\max_{x \in \mathbb{R}^3} 4x_1 + 5x_2 - 6x_3$

subject to $\log|2x_1 + 7x_2 + 5x_3| \leqslant 1$

$x_1, x_2, x_3 \geqslant 0$

P7. $\max_{x \in \mathbb{R}^2} e^{-x_1^2 - x_2^2}(x_1^2 + x_1 x_2 + 6x_1)$

P8. $\max_{x \in \mathbb{R}^3} \dfrac{x_1 x_2 x_3}{1 + x_1^6 + x_2^4 + x_3^2}$

subject to $x_1 + x_2 + x_3 = 1$

Questions:

a) Determine for each of the problems P1 up to P8 which of the following optimization method is best suited to solve the problem:

M1 : Simplex algorithm for linear programming

M2 : Ellipsoid algorithm

M3 : Davidon-Fletcher-Powell quasi-Newton algorithm

M4 : Line search method with steepest descent direction and line minimization with fixed step length

M5 : Line search method with Powell directions and quadratic line minimization

M6 : Nelder-Mead method

M7 : Lagrange method & line search method with Fletcher-Reeves directions and line minimization based on cubic interpolation

M8 : Gradient projection method with Fibonacci line minimization

M9 : Sequential Quadratic Programming

M10 : Simulated annealing

- ***Explain*** why the selected algorithm is the *best suited* for the given problem.
- Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.
- If it is necessary to use the selected algorithm in a *multi-run* or a *multi-start local minimization* procedure, you have to ***indicate*** this clearly.

b) Determine an appropriate stopping criterion for each of the problems P1 up to P8 and the corresponding selected optimization method. Explain your choice!

c) Determine the gradient and the Hessian of the objective function for problems P4 and P7.

QUESTION 2: Optimization methods II (25 points)

a) Consider the following optimization problem:

$$\max_{x \in \mathbb{R}} f(x) \quad \text{subject to} \quad g(x) \geqslant 0$$

for

$$f(x) = -\sin(x) + 3$$
$$g(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} .$$

Prove that there is an extremum in $x = 0$.

b) Consider the following linear programming problem:

$$\min_{x \in \mathbb{R}^3} 3x_1 + 4x_2 + 8x_3$$

subject to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} x - \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 0$$
$$x \geqslant 0 \ .$$

Make a partition of A in a matrix B and a matrix N (cf. the simplex method) and compute the corresponding basic cost z_0.

c) Consider the following quadratic programming problem:

$$\min_{x \in \mathbb{R}^2} 4x_1^2 + x_1 x_2 + 4x_2^2$$

subject to

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \leqslant 0$$
$$x_1, x_2 \geqslant 0 \ .$$

This is a QP problem of Type 1. Show how this QP problem can be converted into a QP problem of Type 2 (i.e., a QP problem where the only inequality constraint is that all optimization parameters are nonnegative and where all the others constraints are equality constraints).

QUESTION 3: Controller design using optimization (15 points)

a) Given the sensitivity function $S(q)$ of a closed loop system:

$$S(q) = \frac{1}{1+P(q)K(q)}$$

i. Show that for sensitivity function is affine in the parameter Q for the following parameterization for the controller:

$$K(q) = \frac{Q(q)}{1-P(q)Q(q)} \quad .$$

ii. Show that for a given weight function $W(q)$, the robust stability constraint

$$\| W(q)S(q) \|_\infty < 1$$

holds if,

$$\| S(q) \|_\infty < \frac{1}{\| W(q) \|_\infty} \quad .$$

b) Let α be a positive real number ($\alpha > 0$) and consider the transfer function R defined by

$$R(q) = \frac{K(q)}{1+P(q)K(q)} \quad .$$

Show that if K is parameterized as

$$K(q) = \frac{Q(q)}{1-P(q)Q(q)} \quad ,$$

then the criterion

$$\|R(q)\|_\infty < \alpha$$

is closed-loop convex in the parameter Q.