Sample Exam 4
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x\in\mathbb{R}^4} x_1^6 + 7x_2^2 - 8x_3x_4 + 2x_3 - 9$$
$$\text{s.t. } \exp(x_1 - x_2 - x_3 + x_4) + \cos(1 + x_1^4x_2^2 + x_4) + 2\sin(x_1x_2 - x_3x_4) + 7x_2^3x_3^2 = 12$$

P2. $$\max_{x\in\mathbb{R}^3} \frac{x_1^2 + 6x_2^2 + 5x_3^2 + 8x_1x_2 + 2x_2x_3 + 5}{1 + 4\sqrt[3]{x_1^7 + x_2^8 + x_3^9}}$$

P3. $$\min_{x\in\mathbb{R}^4} x_1^4 + x_2^2 + x_1x_2x_3x_4$$
$$\begin{aligned} \text{s.t. } & |x_1 + 2x_2 + 8x_3 - 9x_4| \leqslant 7 \\ & -x_1 + 3x_2 - x_3 + 6x_4 \geqslant 2 \\ & \|x\|_\infty \leqslant 3 \end{aligned}$$

P4. $$\min_{x\in\mathbb{R}^4} x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_3^2 + 4x_3x_4 + 6x_4^2 - x_1 - 5x_2 + x_3 + 8x_4 + 1$$
$$\begin{aligned} \text{s.t. } & 1 + (x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)^4 \leqslant 10 \\ & |x_1 + x_2 - x_3 - x_4| \leqslant 3 \end{aligned}$$

P5. $$\min_{x\in\mathbb{R}^5} \exp(|x_1| + |x_2| + |x_3| + |x_4| + |x_5|)$$
$$\begin{aligned} \text{s.t. } & (3x_1 + x_2 - x_3 + x_4 + 3x_5)^4 \leqslant 81 \\ & \min(2x_1, 3x_2, 5x_3, 6x_4, 7x_5) \geqslant 3 \end{aligned}$$

P6. $$\min_{x\in\mathbb{R}^2} (x_1^2 + 2x_2^2 - x_1 - 2x_2 + 8)^3$$
$$\begin{aligned} \text{s.t. } & \cosh(x_1 + 2x_2 - 6) \leqslant 24 \\ & \|x\|_1 \leqslant 3 \end{aligned}$$

P7. $\min\limits_{x\in\mathbb{R}^2} \dfrac{1}{x_2} - 5\log(2x_1+4x_2+7) + \max\left(5, \exp(x_1^2+x_2^2-x_1-x_2)\right)$

s.t. $x_1 \geqslant 1$

$x_2 \geqslant 1$

$|x_1| + x_2^4 \leqslant 25$

P8. $\min\limits_{x\in\mathbb{R}^4} \max(x_1, |x_2|, 3x_3, 4|x_4|)$

s.t. $\max(x_1-x_2+1, x_3-x_4+5) \leqslant 2$

$8 \leqslant (x_1+x_2-x_3-6x_4)^3 \leqslant 27$

P9. $\min\limits_{x\in\mathbb{R}^4} x_1 + |x_2| + 3x_3 + 4|x_4|$

s.t. $\min(x_1-x_2+1, x_3-x_4+5) \geqslant 3$

$(x_1^2+x_2^2-x_3-6x_4)^5 \leqslant 32$

P10. $\max\limits_{x\in\mathbb{R}^4} \min\limits_{u\in[-1,1]} \dfrac{f(u,x)+\sin(1-uf(u,x))}{2+u^2+f^2(u,x)}$

where the function $f(u,x)$ is the solution of the following parameterized nonlinear differential equation with variable u and parameter x:

$$x_1 f\frac{d^2f}{du^2} + \exp(x_1-x_2f)\frac{df}{du} + \frac{x_3u}{1+x_1^2u^2+|x_2|u^2+f^4} = u + \cos(x_4u)$$

with boundary conditions $f(-1,x)=1$ and $f(1,x)=1$.

Task:
For each of the problems P1 up to P10:

a) Determine the best suited optimization method for each of the given problems. Select one of the following methods:

   M1 : Simplex algorithm for linear programming

   M2 : Modified simplex algorithm for quadratic programming

   M3 : Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm

   M4 : Levenberg-Marquardt algorithm

   M5 : Steepest descent method

   M6 : Lagrange method + Nelder-Mead method

   M7 : Lagrange method + line search method with steepest descent direction and golden section line minimization

   M8 : Powell's perpendicular method

   M9 : Penalty function method + line search method with steepest descent direction and golden section line minimization

   M10 : Penalty function method + line search method with Davidon-Fletcher-Powell direction and cubic line minimization

   M11 : Cutting plane algorithm

   M12 : Simulated annealing algorithm

   - ***Explain*** why the selected algorithm is the *best suited* for the given problem.
   - Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.
   - If it is necessary to use the selected algorithm in a *multistart local minimization* procedure, you have to ***indicate*** this clearly.

b) Determine the most appropriate stopping criterion for the selected optimization method and ***explain*** why it is the most appropriate!

QUESTION 2: Optimization methods II (25 points)

Consider the optimization problem

$$\begin{aligned} &\min_{(x,y)\in\mathbb{R}^2} f(x,y) \\ &\text{with } f(x,y) = x^2 + 2y^2 - 2xy - 4x + 3 \ . \end{aligned} \tag{1}$$

This problem will be solved with an unconstrained optimization method that uses the steepest-descent direction and line search.

Tasks:

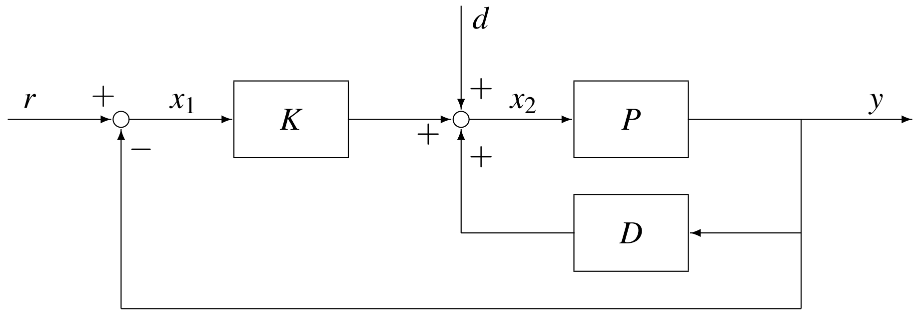
a) Briefly explain how this steepest descent method works.
Illustrate your explanation by drawing one or more pictures.
It is *not* required to explain in detail how the line minimization part of the method works, i.e., it is sufficient to briefly indicate how the line minimization part of the approach works, but you do *not* have to select and explain a *particular* line minimization method.

b) Determine the gradient of f, and next compute the point (x^*, y^*) in which the optimum of the problem (1) is reached.

c) Consider again the optimization problem (1). Compute the first two search directions for the steepest descent method with line minimization and give the values of the objective functions after each line minimization. The starting point is equal to $(x_0, y_0) = (0, 0)$ and the line minimization method is assumed to find the line minimum exactly.

QUESTION 3: Controller design (15 points)

Consider the following closed-loop system consisting of a plant with transfer function $P(q)$ and two controllers with transfer functions $K(q)$ and $D(q)$, where q denotes the forward shift operator:



There are is one external output signal (y), two external input signals (the process noise d and the reference signal r), and two internal signals (x_1 and x_2).

<u>Tasks</u>:

a) Compute the transfer function matrix M such that

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = M(q) \begin{bmatrix} r(k) \\ d(k) \end{bmatrix} \ .$$

b) Assume that the systems with transfer functions

$$V = \frac{1}{1 - DP} \quad \text{and} \quad W = \frac{P}{1 - DP}$$

are also stable, where for the sake of simplicity of notation we have omitted the argument (q) from V, W, D, and P. Now show that the set of controllers, parameterized as

$$K = \frac{(1 - DP)Q}{1 - DP - PQ}$$

with Q a rational stable transfer function, will internally stabilize the feedback system, where for the sake of simplicity of notation we have omitted the argument (q) from K, D, P, and Q.