

Optimization: Quadratic Programming

Quadratic programming (QP)

TYPE 1:

$$\begin{aligned}\min_x f(x) &= \min_x \frac{1}{2} x^T H x + c^T x \\ Ax &\leq b \\ x &\geq 0\end{aligned}$$

TYPE 2: Standard form

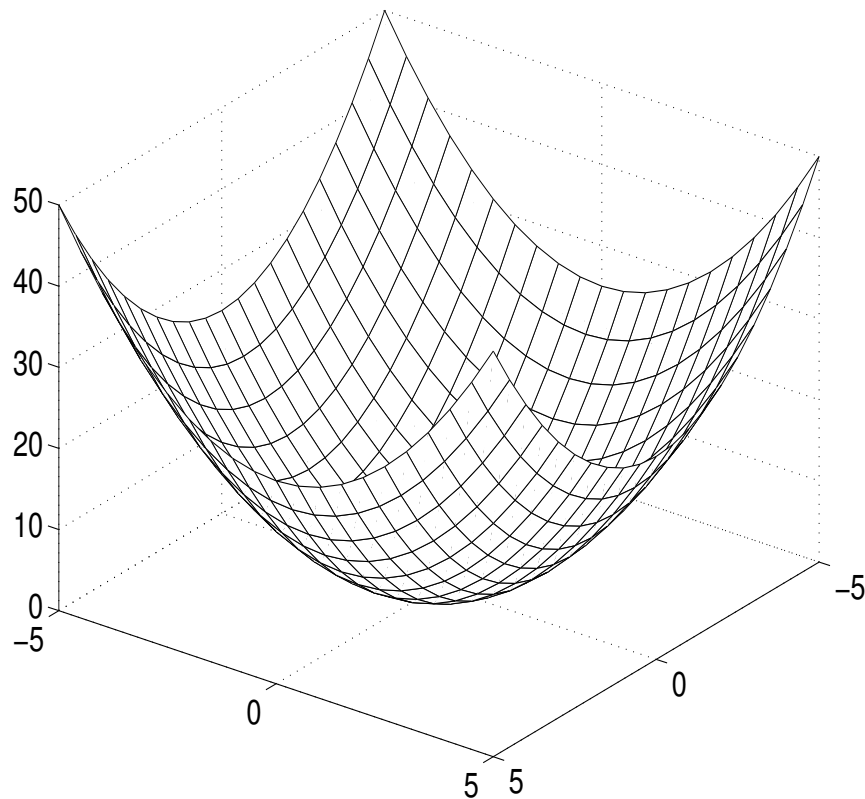
$$\begin{aligned}\min_x f(x) &= \min_x \frac{1}{2} x^T H x + c^T x \\ Ax &= b \\ x &\geq 0\end{aligned}$$

TYPE 1 $\implies$ TYPE 2 (see later)

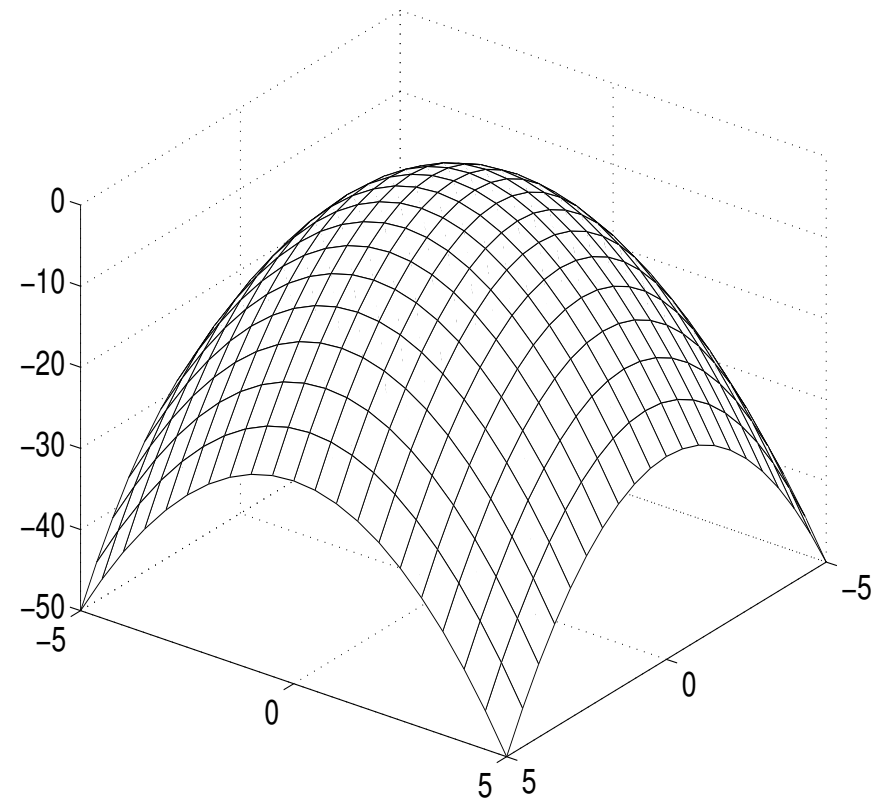
Unconstrained QP: Analytic solution: $x = -H^{-1} c$

Plot of $x^T H x$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ positive definite}$$



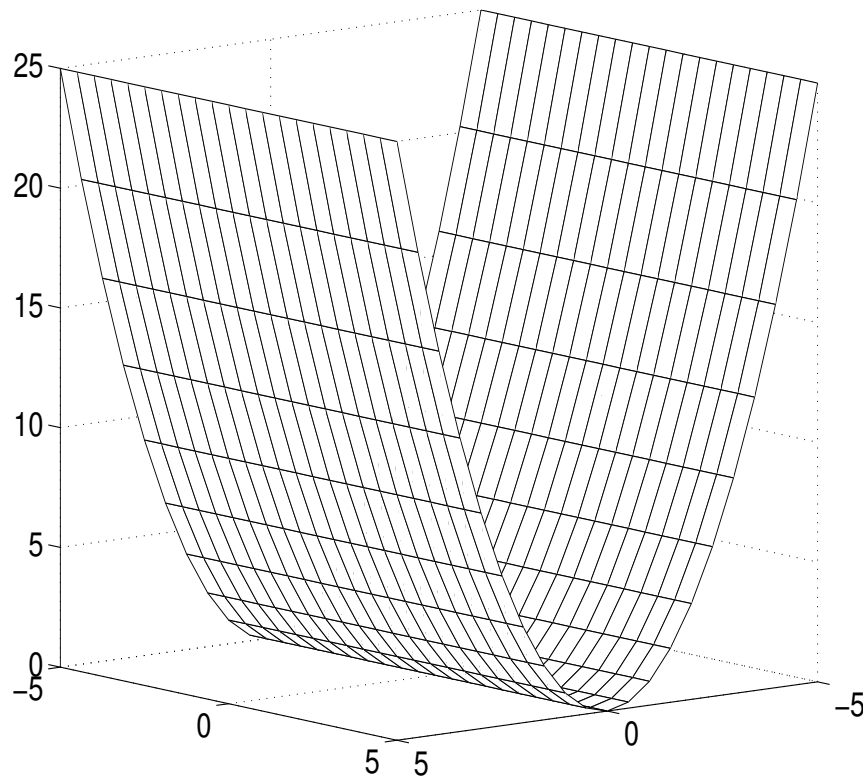
$$H = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ negative definite}$$



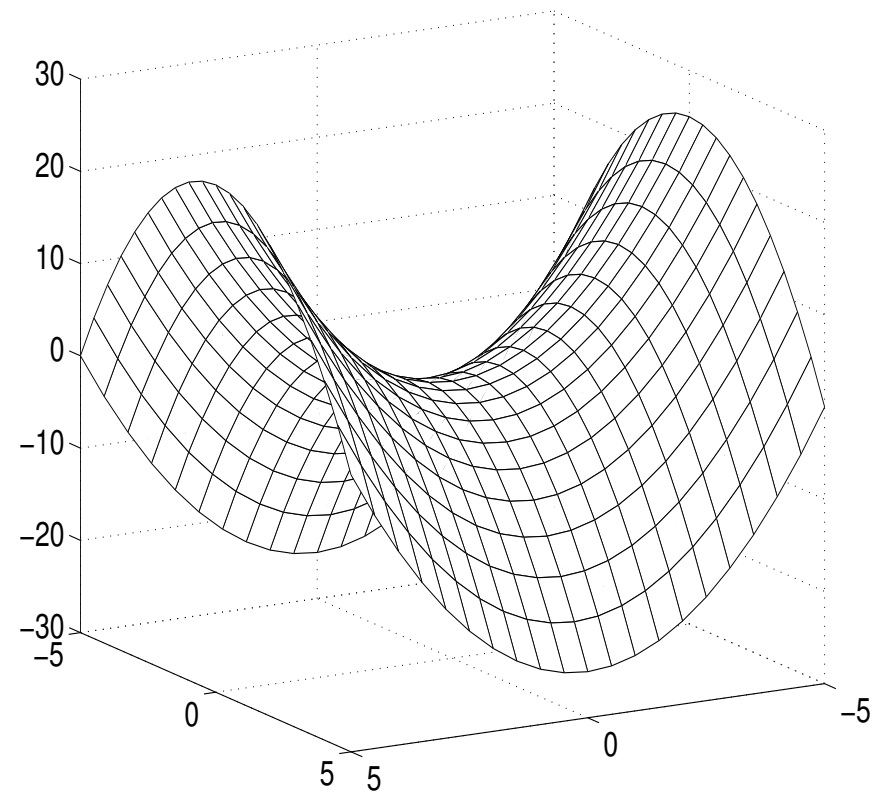
→ for QP we only consider positive (semi)-definite H

Plot of $x^T H x$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ positive semi-definite}$$



$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ indefinite}$$



→ for QP we only consider positive (semi)-definite H

Type 1 $\rightarrow$ Type 2 (= standard form)

- Type 1: $Ax \leq b, \quad x \geq 0$

$$\Rightarrow Ax + Is = b, \quad x, s \geq 0$$

$$\Rightarrow \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b, \quad \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \quad \text{Type 2!}$$

- Type 2: $Ax = b, \quad x \geq 0$

- Type 3: $Ax \leq b, \quad x \in \mathbb{R}^n$

Define $x = x^+ - x^-$ with $x^+, x^- \geq 0$

$$\Rightarrow Ax^+ - Ax^- + Is = b, \quad x^+, x^-, s \geq 0$$

$$\Rightarrow \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b, \quad \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \geq 0 \quad \text{Type 2!}$$

In general: for constrained problem:

$$\min_x f(x) \quad , \quad h(x) = 0 \quad , \quad g(x) \leq 0$$

Karush-Kuhn-Tucker conditions:

$$\nabla f(x) + \nabla g(x) \mu + \nabla h(x) \lambda = 0$$

$$h(x) = 0 \quad , \quad g(x) \leq 0$$

$$\mu^T g(x) = 0 \quad , \quad \mu \geq 0$$

QP-problem:

$$f(x) = \frac{1}{2} x^T H x + c^T x$$

$$h(x) = A x - b \quad , \quad g(x) = -x$$

Karush-Kuhn-Tucker conditions:

$$A x = b \quad , \quad x \geq 0$$

$$H x + A^T \lambda - \mu = -c$$

$$\mu^T x = 0 \quad , \quad \mu \geq 0$$

Note:

$$\nabla(P^T x) = P$$

Modified simplex method

$$\begin{aligned}Ax + u_1 &= b \\ Hx + A^T \lambda - \mu + u_2 &= -c \\ x, \mu, u_1, u_2 &\geq 0 \\ \mu^T x &= 0\end{aligned}$$

Extended linear programming problem:

$$\min_{u_1, u_2} \sum_i (u_1)_i + \sum_j (u_2)_j$$

$$\begin{bmatrix} A & 0 & 0 & I & 0 \\ H & A^T & -I & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \mu \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix}, \quad x, \mu, u_1, u_2 \geq 0$$

and so:

$$\min_{x_0} c_0^T x_0, \quad A_0 x_0 = b_0, \quad x_0 \geq 0$$

but with additional nonlinear constraint $\mu^T x = 0$

Modified simplex method

$$\begin{aligned}x^T \mu &= 0 & x, \mu &\geq 0 \\ \Rightarrow \sum_i x_i \mu_i &= 0 & x_i, \mu_i &\geq 0 \\ \Rightarrow x_i \mu_i &= 0 & \text{for all } i \\ \Rightarrow x_i = 0 \text{ or } \mu_i &= 0 & \text{for all } i\end{aligned}$$

- extra feasibility constraint for basic solutions
- modified simplex method (Wolfe, Lemke)

Finite number of steps!!!

Example

ARX (Auto Regressive eXogenous input) model:

$$y(n+1) + ay(n) = bu(n) + e(n)$$

e : zero-mean white noise

Given: $y(n)$, $u(n)$, $n = 1, 2, \dots, N$, and $y(N+1)$

$\Rightarrow a, b$?

$$y(n+1) - \begin{bmatrix} -y(n) & +u(n) \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \hat{e}(n)$$
$$Y - \Phi x = E$$

Best estimate for a and $b \rightarrow$ minimize energy of $\hat{e}$

Example (continued)

$$Y - \Phi x = E$$

$$\begin{aligned}\min_x \sum_{n=1}^N \hat{e}^2(n) &= \min_x E^T E \\ &= \min_x (Y - \Phi x)^T (Y - \Phi x) \\ &= \min_x Y^T Y + x^T \Phi^T \Phi x - x^T \Phi^T Y - Y^T \Phi x \\ &= \min_x d + \frac{1}{2} x^T H x + c^T x\end{aligned}$$

→ quadratic objective function

Stable model → $-0.99 \leq \hat{a} \leq 0.99$

→ linear constraint

⇒ Quadratic Programming problem

Summary

- Quadratic programming: Standard form

$$\begin{aligned} \min_x \quad & \frac{1}{2}x^T Hx + c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Alternative standard forms & conversion
- Modified simplex method
 - finds exact solution in finite number of steps