

# Optimization: Summary

Optimization problem }  
Set of algorithms }  $\rightarrow$  most suited algorithm?

- ① Simplification of objective function and/or constraints
- ② Determination of most efficient available algorithm
- ③ Determination of stopping criterion

# Simplification of objective function or constraint

No systematic procedure.

Examples:

- $$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & \exp(x_1^2 + x_1x_2 + 2x_2^2 + x_3^2) \\ \text{s.t.} \quad & (x_1 + x_2 + x_3)^2 \leq 9 \\ & \arctan(x_1 - x_2x_3) \geq \frac{\pi}{4} \end{aligned}$$

can be recast as:

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 \\ \text{s.t.} \quad & -3 \leq x_1 + x_2 + x_3 \leq 3 \\ & x_1 - x_2x_3 \geq 1 \end{aligned}$$

# Simplification of objective function/constraint (continued)

- $\min_x \|x\|_1 = \min_x \sum_i |x_i| \quad \text{s.t.} \quad Ax \leq b$

can be recast as LP:

$$\begin{aligned} \min_{\alpha} \quad & \sum_i \alpha_i \quad \text{s.t.} \quad Ax \leq b \\ & x \leq \alpha \\ & -x \leq \alpha \end{aligned}$$

- $\min_x \|x\|_{\infty} = \min_x \max_i |x_i| \quad \text{s.t.} \quad Ax \leq b$

can be recast as LP:

$$\begin{aligned} \min_t \quad & t \quad \text{s.t.} \quad Ax \leq b \\ & x_i \leq t \\ & -x_i \leq t \end{aligned}$$

Also valid with positive weights added (see lecture notes)

## Proof

$$\begin{array}{ll} (1) & (2) \\ \min_x \sum_i |x_i| = \min_x F_1(x) & \leftrightarrow \min_{\alpha} \sum_i \alpha_i = \min F_2(\alpha) \\ \text{s.t. } Ax \leq b & \text{s.t. } Ax \leq b \\ & x, -x \leq \alpha \end{array}$$

- First prove: feasible solution of (1)  $\leftrightarrow$  feasible solution of (2)
- $x^*$ : optimal solution of (1), define  $\alpha^* = |x^*|$
- $\tilde{x}, \tilde{\alpha}$ : optimal solution of (2)

$$\left. \begin{array}{l} \tilde{x} \leq \tilde{\alpha} \\ -\tilde{x} \leq \tilde{\alpha} \end{array} \right\} \rightarrow |\tilde{x}| \leq \tilde{\alpha}$$

Suppose  $\tilde{\alpha}_j > |\tilde{x}_j|$  for some  $j$

Set  $\tilde{\alpha}_j = |\tilde{x}_j|$

$\Rightarrow$  still feasible, but lower value of  $F_2 \Rightarrow$  impossible

$\Rightarrow \tilde{\alpha}_j = |\tilde{x}_j|$  for all  $j$ , so  $\tilde{\alpha} = |\tilde{x}|$

## Proof (continued)

$$\begin{array}{ll} (1) & (2) \\ \min_x \sum_i |x_i| = \min_x F_1(x) & \Leftrightarrow \min_{\alpha} \sum_i \alpha_i = \min F_2(\alpha) \\ \text{s.t. } Ax \leq b & \text{s.t. } Ax \leq b \\ & x, -x \leq \alpha \end{array}$$

- $x^*, \alpha^*$  is feasible solution of problem (2)  
 $\Rightarrow F_1(\tilde{x}) = F_2(\tilde{\alpha}) \leq F_2(\alpha^*) = F_1(x^*)$   
 $\Rightarrow \tilde{x}$  is also optimal solution of (1)
- $\tilde{x}$  is feasible solution of problem (1)  
 $\Rightarrow F_2(\alpha^*) = F_1(x^*) \leq F_1(\tilde{x}) = F_2(\tilde{\alpha})$   
 $\Rightarrow x^*, \alpha^*$  is also optimal solution of (2)

# Examples

$$\textcircled{1} \min_x \max(x_1, x_2, x_3, x_4)$$

$$\text{s.t. } |x_1 + x_2 - 2x_3 + x_4 + 9| \leq 2$$

$$\textcircled{2} \min_x |x_1| + |x_2| + |x_3|$$

$$\text{s.t. } x_1 + x_2 + x_3 \geq 3$$

$$x_1 - 2x_2 + 4x_3 \geq 1$$

→ is LP / can be recast as LP?

# Selection of algorithm

- ① Linear objective + linear constraints:  
simplex (or interior-point, if  $>1000$  variables or constraints)
- ② (Convex) quadratic objective + linear constraints:  
modified simplex (or interior-point)
- ③ Convex objective + convex constraints:  
cutting-plane, ellipsoid, or interior-point
- ④ Multiple local minima:  
multi-start local optimization, multi-run simulated annealing, or  
multi-run genetic algorithm
- ⑤ Nonlinear non-convex problems  
→ use gradient and Hessian if possible!  
analytic/numerical computation

# Selection of algorithm

- Nonlinear non-convex, unconstrained:
  - 1 Levenberg-Marquardt or Newton
  - 2 quasi-Newton algorithm
  - 3 steepest descent
  - 4 Powell's perpendicular method (or Nelder-Mead)
- Nonlinear non-convex constrained:
  - 1 equality constraints:  
**elimination**, Lagrange method
  - 2 linear inequality constraints:  
gradient projection or SQP
  - 3 nonlinear inequality constraints:  
SQP, or penalty or barrier function



# Stopping criterion

- Simplex & modified simplex: **finite** number of steps
- Convex optimization algorithms:

$$\begin{aligned} |f(x^*) - f(x_k)| &\leq \varepsilon_f, & g(x_k) &\leq \varepsilon_g \\ \|x^* - x_k\|_2 &\leq \varepsilon_x & & \text{(for ellipsoid)} \end{aligned}$$

- Unconstrained nonlinear optimization:  $\|\nabla f(x_k)\|_2 \leq \varepsilon_\nabla$
- Constrained nonlinear optimization:

$$\begin{aligned} \|\nabla f(x_k) + \nabla g(x_k) \mu + \nabla h(x_k) \lambda\|_2 &\leq \varepsilon_{\text{KT},1} \\ |\mu^T g(x_k)| &\leq \varepsilon_{\text{KT},2} \\ \mu &\geq -\varepsilon_{\text{KT},3} \\ \|h(x_k)\|_2 &\leq \varepsilon_{\text{KT},4} \\ g(x_k) &\leq \varepsilon_{\text{KT},5} \end{aligned}$$

# Stopping criterion (continued)

- Simulated annealing, genetic algorithm, ... :  
maximum number of iterations

In general, it is recommended to always include this (provided maximum number is taken large enough)

- *Last* resort:

$$\|x_{k+1} - x_k\|_2 \leq \varepsilon_x, \quad |f(x_{k+1}) - f(x_k)| \leq \varepsilon_f$$

- Combinations

# Examples

- $\max_{x \in \mathbb{R}^3} 4x_1 + 5x_2 - 6x_3$

s.t.  $\log |2x_1 + 7x_2 + 5x_3| \leq 1$

$x_1, x_2, x_3 \geq 0.01$

- $\min_{x \in \mathbb{R}^3} \max (\cosh(x_1 + x_2 + x_3), (5x_1 - 6x_2 + 7x_3 + 6)^2)$

s.t.  $\|x\|_2 \leq 10$

Remark:  $\cosh x = \frac{e^x + e^{-x}}{2}$

- $\max_{x \in \mathbb{R}^2} e^{-x_1^2 - x_2^2} (x_1^2 + x_1x_2 + 6x_1)$

- $\max_{x \in \mathbb{R}^3} \frac{x_1x_2x_3}{1 + x_1^6 + x_2^4 + x_3^2}$

s.t.  $x_1 + x_2 + x_3 = 1$

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