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B. De Schutter and T. van den Boom

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Control Systems Engineering
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands
Current URL: https://www.dcsc.tudelft.nl

MODEL PREDICTIVE CONTROL FOR MAX-MIN-PLUS SYSTEMS

Bart De Schutter

Control Lab, Fac. Information Technology and Systems, Delft University of Technology P.O.Box 5031, 2600 GA Delft, The Netherlands

b.deschutter@its.tudelft.nl

Ton van den Boom

Control Lab, Fac. Information Technology and Systems, Delft University of Technology

t.j.j.vandenboom@its.tudelft.nl

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Abstract Model predictive control (MPC) is a widely used control design method in the process industry. Its main advantage is that it allows the inclusion of constraints on the inputs and outputs. Usually MPC uses linear discrete-time models. We extend MPC to max-min-plus discrete event systems. In general the resulting optimization problems are nonlinear and nonconvex. However, if the state equations are decoupled and if the control objective and the constraints depend monotonically on the states and outputs of system, the max-min-plus-algebraic MPC problem can be recast as problem with a convex feasible set. If in addition the objective function is convex, this leads to a convex optimization problem, which can be solved very efficiently.

Introduction

Conventional control design techniques such as pole placement, LQG, H_∞, H_2, ... yield optimal controllers or control input sequences for the entire future evolution of the system. Extending these methods to include additional constraints on the inputs and outputs is not easy. However, Model Predictive Control (MPC) easily allows the inclusion of such constraints due to the use of a receding finite horizon strategy. This advantage, in combination with the low computational requirements and the possibility to deal with slowly time-varying systems, has led to a widespread use of MPC in the process industry. Traditionally MPC uses linear discrete-time models for the process that has to be controlled. Recently we have extended the MPC framework to the class of max-plus discrete event systems (De Schutter and van den Boom, 1999). In this paper we further extend MPC to the class of max-min-plus systems.

1. MODEL PREDICTIVE CONTROL

In this section we give a short introduction to MPC for linear discrete-time systems. Since we will only consider the deterministic, i.e. noiseless, case for max-min-plus systems, we will also omit the noise terms in this introduction to MPC. More extensive information on MPC can be found in (Clarke et al., 1987; García et al., 1989) and the references therein.

Consider a plant with m inputs and l outputs that can be modeled by a linear discrete-time state space description of the following form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \ . \end{aligned}$$

In MPC we compute an optimal control strategy over a given prediction horizon N_{p} and control horizon N_{c} at each sample step k. We define a cost criterion $J = J_{\mathrm{out}} + \lambda J_{\mathrm{in}}$ that reflects the reference tracking error (J_{out}) and the control effort (J_{in}) (Clarke et al., 1987):

$$J = \sum_{j=1}^{N_{\mathrm{p}}} \Big(\hat{y}(k+j|k) - r(k+j) \Big)^T \Big(\hat{y}(k+j|k) - r(k+j) \Big) + \lambda \sum_{j=1}^{N_{\mathrm{p}}} u^T(k+j-1) u(k+j-1)$$

where $\hat{y}(k+j|k)$ is the estimate of the output at sample step $k+j$ based on the information available at step k, r is a reference signal, and λ is a nonnegative scalar. In MPC the input is taken to be constant from a certain point on: $u(k+j) = u(k+N_{\mathrm{c}}-1)$ for $j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$. The use of a control horizon leads to a reduction of the number of optimization variables. This results in a decrease of the computational burden, a smoother controller signal, and has a stabilizing effect. MPC uses a receding horizon approach: after computation of the optimal control sequence $\{u(k), \ldots, u(k+N_{\mathrm{c}}-1)\}$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one sample and the optimization is restarted with new information of the measurements.

Define $\tilde{u}(k) = [\, u^T(k) \ \ldots \ u^T(k+N_{\mathrm{p}}-1) \,]^T$ and $\tilde{y}(k) = [\, \hat{y}^T(k+1|k) \ \ldots \ \hat{y}^T(k+N_{\mathrm{p}}|k) \,]^T$. The MPC problem at each sample step k for a linear discrete-time system is defined as follows:

Find the input sequence $\{u(k), \ldots, u(k+N_\mathrm{c}-1)\}$ that minimizes the cost criterion J subject to the linear constraint

$$A_\mathrm{c}(k)\,\tilde{u}(k) + B_\mathrm{c}(k)\,\tilde{y}(k) \leqslant c_\mathrm{c}(k) \tag{1}$$

and the control horizon constraint $u(k+j) = u(k+N_\mathrm{c}-1)$ for $j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1$.

Recall that due to the receding horizon approach this problem has to be solved at each sample step k.

2. MAX-MIN-PLUS SYSTEMS

We use the following notation for the basic matrix operations of the max-min-plus algebra (Baccelli et al., 1992):

$$\begin{aligned}
(A \vee B)_{ij} &= a_{ij} \vee b_{ij} = \max(a_{ij}, b_{ij}) \\
(A \otimes C)_{ij} &= \bigvee_k a_{ik} \otimes c_{kj} = \max_k(a_{ik} + c_{kj}) \\
(A \wedge B)_{ij} &= a_{ij} \wedge b_{ij} = \min(a_{ij}, b_{ij}) \\
(A \odot C)_{ij} &= \bigwedge_k a_{ik} \odot c_{kj} = \min_k(a_{ik} + c_{kj})
\end{aligned}$$

with $A, B \in \overline{\mathbb{R}}^{m\times n}$ and $C \in \overline{\mathbb{R}}^{n\times p}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Now we consider max-min-plus systems, i.e. systems that can be described by equations in which the operations maximization, minimization and addition appear. Such systems are extensions of the max-plus-algebraic discrete event systems and have been studied by several authors (Gunawardena, 1994; Jean-Marie and Olsder, 1996; Olsder, 1994). We further extend their models by adding inputs and outputs. More specifically, we consider systems that can be described by a model of the form

$$\begin{aligned}
x(k+1) &= A_{xx} \otimes x(k) \vee A_{x\bar{x}} \otimes \bar{x}(k) \vee B_x \otimes u(k) \\
\bar{x}(k+1) &= A_{\bar{x}x} \odot x(k) \wedge A_{\bar{x}\bar{x}} \odot \bar{x}(k) \wedge B_{\bar{x}} \odot u(k) \\
y(k) &= C_x \otimes x(k) \\
\bar{y}(k) &= C_{\bar{x}} \odot \bar{x}(k) \ ,
\end{aligned}$$

where the vectors $x(k)$ and $\bar{x}(k)$ correspond to the state of the system at (event) step k. The vector $u(k)$ is the input at step k and the vector $y(k) = [\, y^T(k) \quad \bar{y}^T(k)\,]^T$ is the output of the system at step k. We assume that the components of x, $\bar{x}$ and y are always finite. Note that this condition always holds for a "physical" system.

3. MODEL PREDICTIVE CONTROL FOR MAX-MIN-PLUS SYSTEMS

In this section we extend and adapt the MPC framework from linear discrete-time systems to max-min-plus systems. If possible we use analog constraints and cost criteria for both types of systems. However, in some cases different constraints and cost criteria are more appropriate.

3.1. COST CRITERION

Just as in MPC for linear discrete-time systems, we also define the MPC cost criterion for max-min-plus systems as $J = J_{\text{out}} + \lambda J_{\text{in}}$, where J_{out} is related to the output and J_{in} is related to the input. Now we discuss some possible choices for J_{out} and J_{in}.

If the due dates r for the finished products are known and if we have to pay a penalty for every delay, a possible output cost criterion is

$$J_{\text{out},1} = \sum_{j=1}^{N_{\text{p}}} \sum_{i=1}^{l} \max(\hat{y}_i(k+j|k) - r_i(k+j), 0) \ .$$

If we have perishable goods, we could minimize the differences between the due dates and the actual output time instants. This leads to

$$J_{\text{out},2} = \sum_{j=1}^{N_{\text{p}}} \sum_{i=1}^{l} |\hat{y}_i(k+j|k) - r_i(k+j)| \ .$$

If we want to balance the output rates, we could consider the following cost criterion:

$$J_{\text{out},3} = \sum_{j=2}^{N_{\text{p}}} \sum_{i=1}^{l} |\Delta^2 \hat{y}_i(k+j|k)|$$

where $\Delta^2 \hat{y}_i(k+j|k) = \hat{y}_i(k+j|k) - 2\hat{y}_i(k+j-1|k) + \hat{y}_i(k+j-2|k)$.

The conventional MPC input cost criterion $\tilde{u}^T(k)\tilde{u}(k)$ would lead to a minimization of the input time instants. Since this could result in internal buffer overflows, a better objective is to *maximize* the input time instants. For a manufacturing system, this would correspond to a just-in-time production scheme, in which raw material is fed to the system as late as possible. As a consequence, the internal buffer levels are kept as low as possible. So for max-min-plus systems a more appropriate input cost criterion is

$$J_{\text{in},0} = -\tilde{u}^T(k)\tilde{u}(k) \ .$$

Note that this input cost criterion is exactly the opposite of the conventional MPC input effort cost criterion.
Another objective function that leads to a maximization of the input time instants is

$$J_{\mathrm{in},1} = -\sum_{j=1}^{N_{\mathrm{p}}} \sum_{i=1}^{m} u_i(k+j-1) \ .$$

If we want to balance the input rates we could take

$$J_{\mathrm{in},3} = \sum_{j=1}^{N_{\mathrm{p}}-1} \sum_{i=1}^{l} |\Delta^2 u_i(k+j)| \ .$$

Note that we can also consider weighted mixtures of several cost criteria.

3.2. CONSTRAINTS

In the context of discrete event systems typical constraints are:

$$\begin{aligned} & a_1(k+j) \leqslant \Delta u(k+j-1) \leqslant b_1(k+j) && \text{for } j=1,\ldots,N_{\mathrm{c}} \\ & a_2(k+j) \leqslant \Delta \hat{y}(k+j|k) \leqslant b_2(k+j) && \text{for } j=1,\ldots,N_{\mathrm{p}} \\ & \hat{y}(k+j|k) \leqslant r(k+j) && \text{for } j=1,\ldots,N_{\mathrm{p}}\,, \end{aligned}$$

where $\Delta u(k+j) = u(k+j) - u(k+j-1)$. Note that all these constraints can be recast as a linear constraint of the form (1).

Since for max-min-plus systems the input sequence usually corresponds to occurrence times of consecutive events, it should always be nondecreasing. Therefore, we also have to add the condition $\Delta u(k+j) \geqslant 0$ for $j = 0, \ldots, N_{\mathrm{p}} - 1$. This is also a constraint of the form (1).

For max-min-plus systems the condition that the input should stay constant from step $k+N_{\mathrm{c}}$ on, is not very useful since the input sequences should normally be increasing. Therefore, we change this condition as follows: the feeding rate should stay constant beyond step $k+N_{\mathrm{c}}$, i.e. $\Delta u(k+j) = \Delta u(k+N_{\mathrm{c}}-1)$ for $j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$, or equivalently $\Delta^2 u(k+j) = 0$ for $j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$.

3.3. THE STANDARD MPC PROBLEM FOR MAX-MIN-PLUS SYSTEMS

If we combine the material of previous subsections, we finally obtain the *max-min-plus-algebraic MPC problem* for event step k:

Find the input sequence vector $\tilde{u}(k)$ that minimizes the cost criterion J subject to

$$x(k+j+1) = A_{xx} \otimes x(k+j) \vee A_{x\bar{x}} \otimes \bar{x}(k+j) \vee$$

$$B_x \otimes u(k+j) \qquad \text{for } j = 0, \dots, N_\mathrm{p} - 1\,, \qquad (2)$$

$$\bar{x}(k+j+1) = A_{\bar{x}x} \odot x(k+j) \wedge A_{\bar{x}\bar{x}} \odot \bar{x}(k+j) \wedge \qquad (3)$$

$$B_{\bar{x}} \odot u(k) \qquad \text{for } j = 0, \dots, N_\mathrm{p} - 1\,, \qquad (4)$$

$$y(k+j) = C_x \otimes x(k+j) \qquad \text{for } j = 1, \dots, N_\mathrm{p}\,, \qquad (5)$$

$$\bar{y}(k+j) = C_{\bar{x}} \odot \bar{x}(k+j) \qquad \text{for } j = 1, \dots, N_\mathrm{p}\,, \qquad (6)$$

$$A_\mathrm{c}(k)\,\tilde{u}(k) + B_\mathrm{c}(k)\,\tilde{y}(k) \leqslant c_\mathrm{c}(k) \qquad (7)$$

$$\Delta u(k+j) \geqslant 0 \qquad \text{for } j = 0, \dots, N_\mathrm{p} - 1\,, \qquad (8)$$

$$\Delta^2 u(k+j) = 0 \qquad \text{for } j = N_\mathrm{c}, \dots, N_\mathrm{p} - 1. \qquad (9)$$

Note that in this case we also use a receding horizon approach in which in each step we effectively apply only the first input sample.

4. ALGORITHMS TO SOLVE THE MAX-MIN-PLUS-ALGEBRAIC MPC PROBLEM

4.1. NONLINEAR OPTIMIZATION

In general the max-min-plus-algebraic MPC problem is a nonlinear nonconvex optimization problem. We could use standard multi-start nonlinear nonconvex local optimization methods to compute the optimal control policy. Using a reasoning that is an extension of the one used in (De Schutter and van den Boom, 1999) it can be shown that the set of feasible solutions defined by the constraints of the max-min-plus-algebraic MPC problem coincides with the solution set of an Extended Linear Complementarity problem (ELCP) (De Schutter and De Moor, 1995). In (De Schutter and De Moor, 1995) we have developed an algorithm to compute a compact parametric description of the solution set of an ELCP. In order to determine the optimal MPC policy we have to determine for which values of the parameters the objective function J over the solution set of the ELCP that corresponds to (2) – (9). However, the algorithm of (De Schutter and De Moor, 1995) to compute the solution set of a general ELCP requires exponential execution times. This implies that the ELCP approach is not feasible if N_c, m or l are large.

4.2. MONOTONIC OBJECTIVE FUNCTIONS AND CONSTRAINTS

Now we consider the *relaxed MPC problem* which is also defined by (2) – (9) but with the =-sign in (2) and (5) replaced by a $\geqslant$-sign, and the =-sign in (4) and (6) replaced by a $\leqslant$-sign. As a consequence, the set of feasible solutions of the relaxed MPC problem is convex. Hence, the relaxed problem is much easier to solve numerically.

We say that a function F is a monotonically nondecreasing (nonincreasing) function of y if $y^* \leqslant y^\sharp$ implies that $F(y^*) \leqslant F(y^\sharp)$ $\big(F(y^*) \geqslant F(y^\sharp)\big)$. Now consider the case in which (2) and (4) are decoupled, i.e. $(A_{x\bar{x}})_{ij} = -\infty$ and $(A_{\bar{x}x})_{ij} = +\infty$ for all i, j. Using a reasoning that is an extension of that used in (De Schutter and van den Boom, 1999) for the max-plus-algebraic MPC, it can be shown that if the objective function J and the linear constraints are monotonically nondecreasing functions of x and y and monotonically nonincreasing functions of $\bar{x}$ and $\bar{y}$, then the optimal solution of the relaxed MPC problem can be transformed into a solution of the original MPC problem[1]:

Theorem 1 *Let $(A_{x\bar{x}})_{ij} = -\infty$ and $(A_{\bar{x}x})_{ij} = +\infty$ for all i, j. Let the objective function J and the mapping $\tilde{y} \to B_{\mathrm{c}}(k)\tilde{y}$ be monotonically nondecreasing functions of y (and x) and monotonically nonincreasing functions of $\bar{y}$ (and $\bar{x}$), Let $(\tilde{u}^*, \tilde{y}^*)$ be an optimal solution of the relaxed MPC problem. If we define $\tilde{y}^\sharp$ by*

$$
\begin{aligned}
x^\sharp(k+j+1) &= A_{xx} \otimes x^\sharp(k+j) \vee B_x \otimes u^*(k+j) \\
\bar{x}^\sharp(k+j+1) &= A_{\bar{x}\bar{x}} \odot \bar{x}^\sharp(k+j) \wedge B_{\bar{x}} \odot u^*(k+j) \\
y^\sharp(k+j+1) &= C_x \otimes x^\sharp(k+j+1) \\
\bar{y}^\sharp(k+j+1) &= C_{\bar{x}} \odot \bar{x}^\sharp(k+j+1)
\end{aligned}
$$

for $j = 0, \ldots, N_{\mathrm{p}} - 1$ and with $x^\sharp(k) = x(k)$ and $\bar{x}^\sharp(k) = \bar{x}(k)$, then $(\tilde{u}^*, \tilde{y}^\sharp)$ is an optimal solution of the original max-min-plus-algebraic MPC problem.

So if the theorem holds[2], then the optimal MPC policy can be computed very efficiently. If in addition the objective function is convex (e.g., if $J = J_{\mathrm{out},1}$ or $J_{\mathrm{in},1}$), we finally get a convex optimization problem, which can be solved very efficiently. Since $J_{\mathrm{in},1}$ is a linear function, the problem even reduces to a linear programming problem for $J = J_{\mathrm{in},1}$.

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[1] The proof of this theorem is similar to the proof of the fact that a feasible linear programming problem with a finite optimal solution always has an optimal solution in which at least one of the constraints is active.

[2] Note that we can always obtain an objective function that is a monotonically nondecreasing function of y and a monotonically nonincreasing function of $\bar{y}$ by eliminating $\tilde{y}(k)$ from the expression for J using the evolution equations (2) – (6) before relaxing the problem. However, some of the properties (such as convexity or linearity) of the original objective function may be lost in that way.

5. CONCLUSIONS

We have extended the popular MPC framework to max-min-plus discrete event systems. The reason for using an MPC approach for max-min-plus systems is the same as for conventional linear systems: MPC allows the inclusion of constraints on inputs and outputs, it is easy to tune and flexible for structure changes (since the optimal strategy is recomputed regularly so that model changes can be taken into account as soon as they are identified). In general the max-min-plus-algebraic MPC problem leads to a nonlinear nonconvex optimization problem. However, if the state equations are decoupled and if the objective function and the constraints are monotonic functions of the states and the outputs, we can relax the MPC problem to a problem with a convex feasible set. If in addition the objective function is convex or linear, this leads to a problem that can be solved very efficiently.

Topics for future research include: extension of the MPC framework to nondeterministic max-min-plus-algebraic models, thorough investigation of the effects of the tuning parameters (input cost weight, the prediction horizon, and the control horizon), and determination of appropriate values for the tuning parameters.

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