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On the Equivalence of Classes of Hybrid Dynamical Models

W.P.M.H. Heemels[1] B. De Schutter[2] A. Bemporad[3]

Abstract

We establish equivalences among five classes of hybrid systems, that we have encountered in previous research: mixed logical dynamical systems, linear complementarity systems, extended linear complementarity systems, piecewise affine systems, and max-min-plus-scaling systems. These results are of paramount importance for transferring properties and tools from one class to another.

1 Introduction

Hybrid dynamical systems are systems that contain both analog (continuous) and logical (discrete) components. Recently, these systems receive a lot of attention from both the computer science and the control community. As tractable methods to analyze general hybrid systems are not available, several authors have focused on special subclasses for which analysis and control design techniques are currently being developed. We show that some of these classes are equivalent (under mild assumptions). The equivalence should be understand in the sense that the "input-state-output behavior" generated by the model classes are equal (cf. below for a more exact definition). These results enable the transfer of knowledge from one class to another, they show that more applications belong to these classes and moreover, for the study of a particular hybrid system one can choose the modeling framework that is most suitable.

2 Classes of Hybrid Models

2.1 Piecewise Affine (PWA) Systems

PWA systems [28, 29] are described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \quad \text{for} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{1}$$

where Ω_i are convex polyhedra (i.e. given by a finite number of inequalities) in the input/state space. The variables $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^l$ denote the input, state and output, respectively, at time k.

PWA systems have been studied by several authors (see [2, 18, 22, 24, 28, 29, 31–33] and the references therein) as they form the "simplest" extension of linear systems that can still model several non-linear and non-smooth processes with arbitrary accuracy and are capable of handling hybrid phenomena.

2.2 Mixed Logical Dynamical (MLD) Systems

In [4] Bemporad and Morari introduced MLD systems, a class of hybrid systems in which logic, dynamics and constraints are integrated. This led to a description of the form

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \tag{2a}$$
$$y(k) = Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) \tag{2b}$$
$$E_1 x(k) + E_2 u(k) + E_3 \delta(k) + E_4 z(k) \leqslant e_5, \tag{2c}$$

where $x(k) = [\, x_{\mathrm{r}}^\top(k) \;\; x_{\mathrm{b}}^\top(k) \,]^\top$ with $x_{\mathrm{r}}(k) \in \mathbb{R}^{n_{\mathrm{r}}}$ and $x_{\mathrm{b}}(k) \in \{0,1\}^{n_{\mathrm{b}}}$ ($y(k)$ and $u(k)$ have a similar structure), and where $z(k) \in \mathbb{R}^{r_{\mathrm{r}}}$ and $\delta(k) \in \{0,1\}^{r_{\mathrm{b}}}$ are auxiliary variables. The inequalities (2c) have to be interpreted componentwise.

In [4] it has been shown that the class of MLD systems includes piecewise affine dynamic systems, linear hybrid systems, finite state machines, (bi)linear systems with discrete inputs and so on. For MLD systems, several tools were introduced for modeling [30], control [4], state estimation and fault detection [3], verification and safety analysis [5].

2.3 Linear Complementarity (LC) Systems

LC systems are studied in e.g. [6, 17, 25–27]. In discrete time these systems are given by the equations

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 w(k) \tag{3a}$$
$$y(k) = Cx(k) + D_1 u(k) + D_2 w(k) \tag{3b}$$
$$v(k) = E_1 x(k) + E_2 u(k) + E_3 w(k) + e_4 \tag{3c}$$
$$0 \leq v(k) \perp w(k) \geq 0 \tag{3d}$$

with $v(k), w(k) \in \mathbb{R}^s$ and where $\perp$ denotes the orthogonality of vectors (i.e. $v(k) \perp w(k)$ means that $v^\top(k) w(k) = 0$). We call $v(k)$ and $w(k)$ the complementarity variables.

In [6,17,26,27] (linear) complementarity systems in *continuous* time have been studied. Applications include constrained mechanical systems, electrical networks with ideal diodes or other dynamical systems with piecewise linear relations, variable structure systems, constrained optimal control problems and so on. Issues related to modeling, well-posedness [17,26,27], simulation and discretization [6] have been of particular interest.

2.4 Extended Linear Complementarity (ELC) Systems

ELC systems are described by:

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 d(k) \tag{4a}$$
$$y(k) = Cx(k) + D_1 u(k) + D_2 d(k) \tag{4b}$$
$$E_1 x(k) + E_2 u(k) + E_3 d(k) \leqslant e_4 \tag{4c}$$

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$$\sum_{i=1}^{p} \prod_{j\in\phi_i} \big(e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k)\big)_j \;=\; 0, \tag{4d}$$

where $d(k) \in \mathbb{R}^r$ is an auxiliary variable. Due to (4c), condition (4d) is equivalent to $\prod_{j\in\phi_i}(e_4 - E_1x(k) - E_2u(k) - E_3d(k))_j = 0$ for each i. This implies that (4c)–(4d) can be considered as a system of linear inequalities (i.e. (4c)), where there are p groups of linear inequalities (one group for each index set ϕ_i) such that in each group at least one inequality should hold with equality.

Remark 1 For ELC systems inequalities of the form (2c) can be incorporated directly, whereas in LC systems these inequalities have to be transformed into a (void) complementarity condition by using slack variables. For LC systems products consisting of more than 2 factors (such as e.g. $u_1(k)u_2(k)u_3(k) = 0$) are not allowed (directly) while in ELC systems products of 3 or more factors are possible. □

In [11, 12] it has been shown that the class of ELC systems encompasses max-plus-linear systems [1], first order linear hybrid systems subject to saturation [11], and *unconstrained* max-min-plus-scaling systems (see next section).

2.5 Max-Min-Plus-Scaling (MMPS) Systems

An MMPS expression f of the variables $x_1, \ldots, x_n$ is defined by the grammar

$$f := x_i|\alpha|\max(f_k, f_l)|\min(f_k, f_l)|f_k + f_l|\beta f_k$$

with $i \in \{1, \ldots, n\}$, α, $\beta \in \mathbb{R}$, and f_k, f_l again MMPS expressions. An example of an MMPS expression is $\max(\min(2x_1, -8x_2), x_2 - 3x_3)$. The symbol | stands for OR and the definition is recursive. Note that the min operation is in fact not explicitly needed since we have $\min(f_k, f_l) = -\max(-f_k, -f_l)$.

MMPS systems are now described by

$$x(k+1) = \mathcal{M}_x(x(k), u(k), d(k)) \tag{5a}$$
$$y(k) = \mathcal{M}_y(x(k), u(k), d(k)) \tag{5b}$$

together with the constraint[1]

$$\mathcal{M}_c(x(k), u(k), d(k)) \leqslant c, \tag{5c}$$

where $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_c$ are MMPS expressions in terms of the components of $x(k)$, $u(k)$ and the auxiliary variables $d(k)$. Model (5a)–(5b) is a generalized framework that encompasses several special subclasses of hybrid and discrete-event systems such as max-plus-linear discrete event systems [1], max-min-plus systems [14, 23], and max-plus-polynomial systems [12].

To each of the above models one can associate a behavior [34] consisting of all sequences $u : \mathbb{N} \mapsto \mathbb{R}^m$, $x : \mathbb{N} \mapsto \mathbb{R}^n$ and $y : \mathbb{N} \mapsto \mathbb{R}^l$ such that these sequences satisfy the model equations (e.g. (3) for LC systems) for some sequences of auxiliary variables (e.g. for an LC model (3) for some sequences $v : \mathbb{N} \mapsto \mathbb{R}^s$ and $w \in \mathbb{R}^s$). We say that every system in a model class A can be rewritten as one in a model class B, if for each system in A, there is a system in B such that the behavior of A and B are equal.

[1] If (5c) is absent, we speak of unconstrained MMPS systems.

Before proving the equivalences among the five classes of hybrid models described so far, we recall a few results on piecewise linear functions developed by the circuit and systems community.

3 Piecewise Linear (PWL) Functions

PWA systems have been around for quite some time in the systems and control community [28], but only recently the attention they receive has boosted. Also in the circuits and systems community piecewise linear (PWL)[2] *static* representations play an important role [7–9, 15, 19, 20, 22, 32] for the analysis of nonlinear circuits. These representations of PWL functions are of course immediately relevant for the dynamical systems considered here as the right-hand sides of the PWA models are multi-variable PWL mappings. As such we will give a brief overview of the work that is already available in the literature. For a more extensive survey, see [21, 22].

In the circuit theory community one has mainly focused on PWL mappings that are *continuous* and the first representations were in an explicit form [7, 9, 15, 19, 20].

A (continuous) PWL function is a function $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfying the following conditions [7]:

1. The domain space $\mathbb{R}^n$ is divided into polyhedral regions Ω_i, $i = 1, \ldots, N$ by a finite number of boundaries such that each boundary is (a subset of) an $(n-1)$ dimensional hyperplane $\alpha_i^\top x - \beta_i = 0$ with $\alpha_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, and cannot be covered[3] by any $(n-2)$-dimensional hyperplane.

2. For any region Ω_i, f can be expressed by an affine representation $f(x) = J_i x + w_i$ for all $x \in \Omega_i$.

3. f is continuous over the boundary between two regions, i.e. $J_i x + w_i = J_j x + w_j$ for all $x \in \Omega_i \cap \Omega_j$.

The first canonical representation of PWL functions proposed in [8, 9, 20] is of the form $f : \mathbb{R}^n \to \mathbb{R}^m$ with

$$f(x) = a + Bx + \sum_{i=1}^{p} c_i|\alpha_i^\top x - \beta_i| \tag{6}$$

The notation $|\cdot|$ indicates the absolute value (or "vee")-function. Any one-dimensional PWL function $f : \mathbb{R} \to \mathbb{R}$ can be written in this form. A drawback of this representation is that it cannot capture all PWL models (see [7]).

To overcome the limitations of (6) Güzelis came up with a more general canonical form (see also [22, Ch. 2]) based on 2-nested "vee" functions of the form

$$f(x) = a + Bx + \sum_{i=1}^{p} b_i|\alpha_i^\top x - \beta_i| +$$

[2] Strictly speaking "piecewise affine" might be a more appropriate terminology (and therefore we have used it in Section 2.1). For historical reasons we will use PWL in the context of circuit theory.

[3] A boundary B is said to be covered by a hyperplane H, if $B \subseteq H$.

$$+\sum_{j=1}^{q} c_j \left|\delta_j + \gamma_j^\top x + \sum_{i=1}^{r} d_{ji} |\alpha_i^\top x + \beta_i|\right| \quad (7)$$

This representation allows boundaries that are PWL themselves. However, the example in [22, p. 40] demonstrates that still not all continuous PWL mappings can be described using this model.

Yet another extension was formulated by Kahlert and Chua [19] that could represent all two-dimensional (continuous) PWL functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. Instead of presenting the details of this representation, which can be found in [19] or in one of the overviews [21,22], we will now go from the *explicit* models above to the more general *implicit* model as proposed by Van Bokhoven [31] and based on the linear complementarity problem (LCP) [10]. In [31] a PWL function $f : \mathbb{R}^n \to \mathbb{R}^m$ has been recast in the form

$$y = Ax + Bw + g \quad (8a)$$
$$v = Cx + Dw + h \quad (8b)$$
$$0 \leq v \perp w \geq 0 \quad (8c)$$

with x the argument of f and y its value. Given x one has to solve (8b)–(8c) for w and v after which v can be substituted in (8a) to obtain y. By this implicit modeling one can even include certain "one-to-many" or "set-valued" mappings. However, for some x the above representation may not define any function value y as the LCP (8b)–(8c) may have no solutions at all.

In [21, 22] it has been shown that the model description (8) includes all the previously mentioned canonical representations introduced by Chua and Kang [8], Güzelis and Göknar [15], and Kahlert and Chua [19]. The only issue left is related to the question if any *continuous* PWL mapping can be cast into the formulation (8).

Theorem 1 *Any continuous PWL mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ can be written in terms of the representation (8).*

Proof: Combining Theorem 5.2 and the second remark in Section 6 of [13] proves the result. □

4 Relations Inherited from Circuit Theory

The results of the previous section yield immediately specific relations between certain classes of *unconstrained* MMPS (systems with right-hand sides being explicit canonical representations based on "vee" functions), PWA (with right-hand sides being *continuous* PWL functions) and LC systems (via the explicit model based on LCPs):

Corollary 1 *The classes of* unconstrained *MMPS systems with right-hand sides given by (6), (7) or as in [19] can be written as LC systems [21, 22].*

Corollary 2 *Every* continuous *PWA system can be written as an LC system (Theorem 1).*

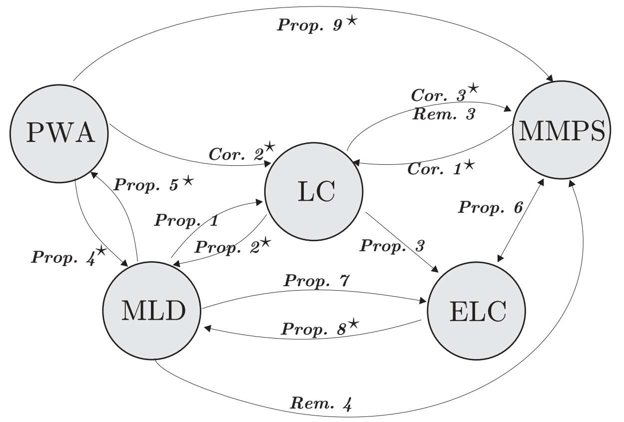

Figure 1: Graphical representation of the links between the classes of hybrid systems considered in this paper. An arrow going from class A to class B means that A is a subset of B. The label next to each arrow corresponds to the result that states this relation. Moreover, arrows with a star ($\star$) require conditions to establish the indicated inclusion.

5 The Equivalence of MLD, LC, ELC, PWA and MMPS Systems

The relations in Section 4 are far from complete. Now we will actually show that MLD, LC, ELC, PWA and MMPS systems are equivalent (although in some cases additional assumptions are required). The relations between the different models proved in this paper are depicted in Figure 1. Unless specified otherwise, the proofs of the propositions can be found in [16]. The examples in Section 7 will illustrate some of the ideas used in the proofs.

Proposition 1 *Every MLD system can be written as an LC system.*

Proposition 2 *Every LC system can be written as an MLD system, provided that the variables $w(k)$ and $v(k)$ are (componentwise) bounded.*

Proposition 2 assumes that upper bounds on w and v are known. This hypothesis is not restrictive in practice, as these quantities are related to continuous inputs and states of the system, which are usually bounded for physical reasons.

Proposition 3 *Every LC system can be written as an ELC system.*

A PWA system of the form (1) is called *well-posed*, if (1) is uniquely solvable in $x(k+1)$ and $y(k)$ once $x(k)$ and $u(k)$ are specified. Similar definitions apply to the MLD, LC, ELC and MMPS systems.

Proposition 4 [4] *Every well-posed PWA system can be rewritten as an MLD system assuming that the set of feasible states and inputs is bounded.*

Remark 2 As MLD models only allow non-strict inequalities in (2c), in rewriting a discontinuous PWA system as an MLD model strict inequalities like $x(k) < 0$ must be approximated by $x(k) \leq -\varepsilon$ for some $\varepsilon > 0$ (typically the machine precision) and the condition $-\varepsilon < x(k) < 0$ is included implicitly. It can be argued that the situation $-\varepsilon < x(k) < 0$ cannot occur due to the finite number of bits used for representing real numbers (no problem exists when the PWA system is continuous, where the strict inequality can be equivalently rewritten as non-strict, i.e. $\varepsilon = 0$). See [4] for more details and Section 7 for a discontinuous example. From a strictly theoretical point of view, the inclusion stated in Proposition 4 is therefore not exact for discontinuous PWA systems, and the same clearly holds for an LC, ELC or MMPS reformulation of a discontinuous PWA system when the route via MLD systems is taken. One way to circumvent such an inexactness is to allow a part of the inequalities in (2c) to be strict. On the other hand, from a numerical point of view this issue is not relevant. The equivalence of LC and MLD systems implies that all continuous PWA systems can be exactly written as LC systems as well (see also Corollary 2). □

Proposition 5 [2] *A completely well-posed[4] MLD system can be rewritten as a PWA system.*

Proposition 6 *The classes of (constrained) MMPS and ELC systems coincide.*

Remark 3 As a consequence of the above result and Proposition 3 it is obvious that every LC system can be recast as an MMPS system. A more direct route rewrites an LC system (3) as the constrained MMPS system (3a)–(3b) and $\min((E_1x(k) + E_2u(k) + E_3w(k) + e_4)_j, w_j(k)) = 0$ for all j. □

Proposition 7 *Every MLD system can be rewritten as an ELC system.*

Remark 4 To prove that an MLD model can be written as an MMPS system we can use ELC systems as an intermediate (Theorem 7). However, $\delta_i(k) \in \{0, 1\}$ is also equivalent to the MMPS constraint $\min(\delta_i(k), 1 - \delta_i(k)) = 0$. □

Proposition 8 *Every ELC system can be written as an MLD system, provided that the quantity $e_4 - E_1x(k) - E_2u(k) - E_3d(k)$ is (componentwise) bounded.*

In the next section we present a direct equivalence between PWA and MMPS systems, which is a new and stronger result than in [16].

6 Direct Equivalence between PWA and MMPS Systems

Well-posed PWA systems form a subset of MMPS systems by applying Proposition 4, 7 and 6, respectively. However, the intermediate equivalence through MLD systems requires that bounds on input and state variables are specified. We provide here a direct proof which does not require any boundedness conditions. Moreover, this constructive proof leads in general to “smaller” models.

[4]The MLD system (2a) is called completely well-posed, if $x(k + 1)$, $y(k)$, $\delta(k)$ and $z(k)$ are uniquely defined in their domain, once $x(k)$ and $u(k)$ are assigned [4].

Proposition 9 *Every well-posed PWA system can be written as an MMPS system.*

Proof Consider for simplicity autonomous well-posed PWA systems of the form

$$\begin{aligned} x(k+1) &= A_i x(k) \\ y(k) &= C_i x(k) \end{aligned} \quad \text{for } x(k) \in \Omega_i, \tag{9}$$

where $\Omega_i = \{x : H_i x \leq K_i\} \subseteq \mathbb{R}^n$ are convex polyhedra with $H_i \in \mathbb{R}^{q_i \times n}$ and $K_i \in \mathbb{R}^{q_i}$, $i = 1, \ldots, m$, which form a partition of the state-space[5].

The equivalent MMPS of (9) is

$$x(k+1) = \sum_{i=1}^{m} A_i d_i(k) \tag{10a}$$

$$y(k) = \sum_{i=1}^{m} C_i d_i(k) \tag{10b}$$

$$H_i x(k) - w_i(k) \leqslant K_i \tag{10c}$$

$$w_i(k) \geqslant 0 \tag{10d}$$

$$\min_{i=1,\ldots,m} \left(\max_{j=1,\ldots,n} (|(x(k) - d_i(k))_j|) \right) = 0 \tag{10e}$$

$$\min \left(\sum_{j=1}^{n} |d_i(k)_j|, \sum_{j=1}^{q_i} w_i(k)_j \right) = 0, \; i = 1, \ldots, m \tag{10f}$$

where $d_i(k) \in \mathbb{R}^n$, $w_i(k)$ a real vector of the same dimension as K^i. Note that $|(x(k) - d(k))_j|$ is equivalent to $\max((x(k) - d(k))_j, (d(k) - x(k))_j))$. Given $x(k)$, (10e) imposes that at least for one i the corresponding $d_i(k)$ equals $x(k)$, and (10f) imposes the logic condition

$$[\exists\, j : \; w_i(k)_j > 0] \rightarrow [d_i(k) = 0], \; \forall i = 1, \ldots, m$$

i.e., that if the constraints (10c) can be satisfied only with the aid of nonzero $w_i(k)_j$ slack variables, then the vector $d_i(k)$ must be 0. Because of (10c) and the fact that the polyhedra are disjoint, $d_i(k)$ can only be nonzero for the index i of the region Ω_i where $x(k)$ lies. The extension to non-autonomous systems where $\Omega_i = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : H_i^x x + H_i^u u \leq K_i\}$ can be easily proved by replacing A_i with $[A_i \; B_i \; f_i]$, C_i with $[C_i \; D_i \; g_i]$, $H_i(d_i(k) - w_i(k)) \leqslant K_i$ with $[H_i^x \; H_i^u \; - K_i](d_i(k) - w_i(k)) \leqslant 0$. □

7 Examples

To demonstrate the equivalences given above and to give some idea on the proofs in [16], we will consider two examples of PWA systems: one for which the right-hand side is continuous and one for which it is discontinuous.

Example 1 Consider the following hybrid system:

$$x(k+1) = \begin{cases} x(k) + u(k) & \text{if } x(k) + u(k) \leq 1 \\ 1 & \text{if } x(k) + u(k) > 1 \end{cases} \tag{11}$$

[5]As observed in Remark 2, in case of discontinuities of the PWA functions in the right-hand side of (9), we should replace some of the strict inequalities by non-strict inequalities. To keep the proof compact this will be avoided here.

representing an integrator with upper saturation, within the range $-10 \leq x(k) \leq 10$, $-1 \leq u(k) \leq 1$. System (11) is in PWA form with the two-dimensional input/state space partitioned by the hyperplane $x(k) + u(k) = 1$. In order to get the MLD form of (11), we introduce a binary variable $\delta(k) \in \{0, 1\}$ and a continuous variable $z(k)$, to obtain

$$x(k+1) = z(k) \tag{12a}$$

together with the linear inequalities

$$x(k) + u(k) + 10\delta(k) \leq 11 \tag{12b}$$
$$-x(k) - u(k) - (12+\varepsilon)\delta(k) \leq -1 - \varepsilon \tag{12c}$$
$$-10\delta(k) + z(k) \leq 1 \tag{12d}$$
$$-12\delta(k) - z(k) \leq -1 \tag{12e}$$
$$-x(k) - u(k) + 12\delta(k) + z(k) \leq 12 \tag{12f}$$
$$x(k) + u(k) + 10\delta(k) - z(k) \leq 10 \tag{12g}$$

where, using the techniques of [4], (12b)–(12c) translate the relation $[\delta(k) = 1] \leftrightarrow [x(k) + u(k) \leq 1]$, (12d)–(12g) the relation $z(k) = (x(k)+u(k))\delta(k)+(1-\delta(k))$, and $\varepsilon > 0$ is a small number (e.g. the machine precision) used to replace the strict inequality $x(k) + u(k) > 1$ by $x(k) + u(k) \geq 1 + \varepsilon$. In view of Remark 2 observe that $\varepsilon = 0$ results in a mathematically exact MLD model, which is well-posed as $x(k+1)$ is uniquely determined given $x(k)$ and $u(k)$, but not completely well-posed as $x(k) + u(k) = 1$ allows both $\delta(k) = 0$ and $\delta(k) = 1$.

One can easily verify that (11) can be rewritten as the (unconstrained) MMPS model

$$x(k+1) = x(k)+u(k)-\max(0, x(k)+u(k)-1) \ , \tag{13}$$

as the LC formulation

$$\begin{aligned} x(k+1) &= x(k) + u(k) - w(k) && (14a)\\ v(k) &= -x(k) - u(k) + w(k) + 1 && (14b)\\ 0 &\leq v(k) \perp w(k) \geq 0 \ , && (14c) \end{aligned}$$

and as the ELC representation

$$x(k+1) = x(k) + u(k) - d(k) \tag{15a}$$
$$-d(k) \leq 0, \ \ x(k) + u(k) - d(k) \leq 1 \tag{15b}$$
$$d(k)\big(1 - x(k) - u(k) + d(k)\big) = 0 \ . \tag{15c}$$

While the MLD representation (12) requires bounds on $x(k)$, $u(k)$ to be specified (although such bounds can be arbitrarily large), the PWA, MMPS, LC, and ELC expressions do not require such a specification. $\square$

Note that we only need one max-operator in (13) and one complementarity pair in (14). If we would transform the MLD system (12) into e.g. the LC model as indicated by the equivalence proof in [16], this would require nine complementarity pairs (one for each inequality in (12), one for the binary variable $\delta(k)$ and two for the auxiliary variable $z(k)$). Hence, it is clear that the proofs only show that the system representations can be transferred into each other, but do not result in the most efficient models.

The following example illustrates some of the issues related to discontinuous PWA systems (cf. Remark 2).

Example 2 Consider the PWA system

$$x(k+1) = \begin{cases} 0, & u(k) > 0 \\ 1, & u(k) \leq 0 \end{cases} \tag{16}$$

which represents a discrete-time relay system with a discontinuity on the plane $u(k) = 0$. Similarly as above we can rewrite (16) as the MLD (17) by assuming that $u(k)$ is restricted to $[m, M]$ and $\varepsilon > 0$ is a small constant.

$$\begin{aligned} x(k+1) &= \delta(k) && (17a)\\ u(k) &\leq M(1 - \delta(k)) && (17b)\\ u(k) &\geq \varepsilon + (m - \varepsilon)\delta(k) && (17c)\\ \delta(k) &\in \{0, 1\} && (17d) \end{aligned}$$

Note that $u(k) > 0$ has been replaced by $u(k) \geq \varepsilon$. Moreover, the relations in (17) contain implicitly the condition $u(k) \in [m, 0] \cup [\varepsilon, M]$ meaning that $u(k)$ is not allowed to be situated in the interval $(0, \varepsilon)$. Of course, the MLD model can be written as an ELC or (constrained) MMPS system by replacing the condition (17d) by $-\delta(k) \leq 0$, $\delta(k) \leq 1$, and $\delta(k)(1 - \delta(k)) = 0$ or by $\min(1 - \delta(k), \delta(k)) \leq 0$ and $-\min(1 - \delta(k), \delta(k)) \leq 0$, respectively. An explicit (unconstrained) MMPS may be of the form

$$x(k+1) = 1 - \frac{1}{\varepsilon}\max(u(k), 0) + \frac{1}{\varepsilon}\max(u(k) - \varepsilon, 0), \tag{18}$$

where in the interval $(0, \varepsilon)$ a linear interpolation is used between the discontinuous pieces. As mentioned, the MLD formulation includes the condition $u(k) \in [m, 0] \cup [\varepsilon, M]$ implicitly. Here we have to add this restriction to (18) to prevent the state from lying in the region $(0, \varepsilon)$ (where the model (18) does not comply with (16)) or assume that this is implied by the computer implementation of the model.

Under the condition that $0 < u(k) < \varepsilon$ will not happen, an LC model can be obtained by rewriting a relay characteristic in complementarity terms as in [27]:

$$\begin{aligned} x(k+1) &= 1 - w_2(k) && (19a)\\ v_1(k) &= 1 - w_2(k) && (19b)\\ v_2(k) &= \frac{\varepsilon}{2} - u(k) + w_1(k) && (19c)\\ 0 \leq v_i(k) &\perp w_i(k) \geq 0 \text{ for } i = 1, 2. && (19d) \end{aligned}$$

Observe that the discontinuity is now placed at $\frac{\varepsilon}{2}$, which lies in the "forbidden region." Also the method in the proof of Proposition 1 may be used to obtain another LC model, which is exactly equivalent to the MLD model (i.e. including the condition $u(k) \in [m, 0] \cup [\varepsilon, M]$) given by

$$\begin{aligned} x(k+1) &= w_1(k) && (20a)\\ v_1(k) &= 1 - w_1(k) && (20b)\\ v_2(k) &= M - Mw_1(k) - u(k) && (20c)\\ v_3(k) &= -\varepsilon - (m - \varepsilon)w_1(k) + u(k) && (20d)\\ 0 \leq v_i(k) &\perp w_i(k) \geq 0 \text{ for } i = 1, 2, 3. && (20e) \end{aligned}$$

Note that $w_2(k)$ and $w_3(k)$ do not influence any of the equations and can be taken equal to zero to satisfy $0 \leq$

$v_i(k) \perp w_i(k) \geq 0$, $i = 2, 3$. In fact, the "dummy" complementarity conditions $0 \leq v_i(k) \perp w_i(k) \geq 0$, $i = 2, 3$ and (20c)-(20d) are equivalent to (17b)-(17c). The complementarity between $w_1(k)$ and $v_1(k)$ implies that $w_1(k) \in \{0, 1\}$ as in (17d) and is actually equal to $\delta(k)$ in (17). $\square$

8 Conclusions and Topics for Future Research

In this paper we have shown the equivalence of five classes of hybrid systems: MLD, LC, ELC, PWA, and MMPS systems. For some of the transformations additional conditions like boundedness of the state and input variables or well-posedness had to be made.

An important topic for future research is to use the equivalences to transfer techniques for analysis and synthesis from one class of hybrid systems to another. By doing so, a combined effort will be realized for researching systems with a behavior that can be modeled by any of the hybrid model descriptions as presented in this paper. Moreover, it is interesting to study which modeling framework is most appropriate for solving specific control problems related to e.g. well-posedness, safety analysis, and stability. Also the computational side is crucial; one might pose the question which representation leads to the most efficient numerical algorithms for synthesizing and analyzing control strategies. A related question is suggested by Example 1, which demonstrated that certain hybrid models are more compact ("economical") than others if one considers a specific application. The constructive proofs of the equivalences will not always yield the most efficient models in going from one class to another. Hence, it deserves more attention which model class should be chosen for a particular kind of application and how to obtain a model within the class of "smallest size," which will lead to computational advantages.

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