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Model Predictive Control for Perturbed Max-Plus-Linear Systems: A Stochastic Approach

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Abstract

Model predictive control (MPC) is a popular controller design technique in the process industry. Conventional MPC uses linear or nonlinear discrete-time models. Recently, we have extended MPC to a class of discrete event systems that can be described by a model that is "linear" in the (max,+) algebra. In our previous work we have only considered MPC for the perturbations-free case and for the case with bounded noise and/or modeling errors. In this paper we extend our previous results on MPC for perturbed max-plus-linear systems to a stochastic setting. We show that under quite general conditions the resulting optimization problems turn out to be convex and can be solved very efficiently.

Keywords: discrete event systems, model predictive control, max-plus-linear systems, noise and modeling errors, stochastic setting.

1 Introduction

The class of the max-plus-linear (MPL) systems corresponds to the class of discrete event systems (DES) in which there is synchronization but no concurrency [1, 7]. Such systems can be modeled using the operations maximization (corresponding to synchronization: a new operation starts as soon as all preceding operations have been finished) and addition (corresponding to durations: the finishing time of an operation equals the starting time plus the duration). This leads to a description that is "linear" in the max-plus algebra [1, 7] (see also Section 2). Max-plus-linear DES usually arise in the context of manufacturing systems, telecommunication networks, railway networks, and parallel computing.

Model predictive control (MPC) [2, 4, 6, 10] is currently one of the most widely used *advanced* control design method in the process industry. MPC provides many attractive features: it is applicable to multi-input multi-output systems, it can handle constraints on inputs and outputs in a systematic way, it is capable of tracking pre-scheduled reference signals, and it is an easy-to-tune method. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to MPL systems [8, 9, 18]. In [19] we have presented some results on MPL-MPC in the presence of bounded noise and/or modeling errors. In this paper we will extend these results to cases with noise and/or modeling errors in a stochastic setting, where the noise and/or modeling errors are not bounded a priori.

In contrast to conventional linear systems, where noise and disturbances are usually modeled by including an extra term in the system equations (i.e., the noise is considered to be additive), the influence of noise and disturbances in MPL systems is not max-plus-additive, but max-plus-multiplicative. This means that the system matrices will be perturbed and as a consequence the system properties will change. Ignoring the noise can lead to a bad tracking behavior or even to an unstable closed loop. A second important feature is modeling errors. Uncertainty in the modeling or identification phase leads to errors in the system matrices. It is clear that both modeling errors, and noise and disturbances perturb the system by introducing uncertainty in the system matrices. Sometimes it is difficult to distinguish the two from one another, but usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. Similar to the results in [19], we will show that both features can be treated in one single framework and the characterization of the perturbation will determine whether it describes model mismatch or disturbance. We will also show that under quite general restrictions the resulting MPC optimization problem can be solved very efficiently.

Note that there are few results in the literature on noise and modeling errors in an MPL context. However, for other classes of DES uncertainty results can be found in [5, 12, 16, 21] and the references therein.

This paper is organized as follows. In Section 2 we give a concise introduction to MPL systems and MPC for MPL systems (without noise or modeling errors). Next, we present a noise and uncertainty model for MPL systems in a stochastic framework and we derive algorithms to make predictions in this setting. In Section 4 the MPC method for stochastic MPL systems is presented and in Section 5 we discuss the computational aspects of the algorithm.

2 Max-plus-linear systems and MPC

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows [1, 7]:

$$x \oplus y = \max(x, y) \qquad x \otimes y = x + y$$

for numbers $x, y \in \mathbb{R}_\varepsilon$ and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n}(a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$. The matrix $\mathcal{E}$ is the max-plus-algebraic zero matrix: $[\mathcal{E}]_{ij} = \varepsilon$ for all i, j.

In [1, 7] it has been shown that (time-invariant) discrete event systems (DES) in which there is synchronization but no concurrency can be described by a model of the form

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \tag{1}$$
$$y(k) = C \otimes x(k) \ . \tag{2}$$

Systems that can be described by this model will be called time-invariant max-plus-linear (MPL) systems. The index k is called the event counter. For DES the state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instants at which the input events occur for the kth time, and the output $y(k)$ contains the time instants at which the output events occur for the kth time[1].

In [8, 9] we have extended the MPC framework to time-invariant MPL models (1)–(2) as follows. Just as in conventional MPC [6, 10] we define a cost criterion J that reflects the input and output cost functions ($J_{\rm in}$ and $J_{\rm out}$, respectively) in the event period $[k, k+N_{\rm p}-1]$:

$$J(k) = J_{\rm out}(k) + \lambda J_{\rm in}(k) \tag{3}$$

where $N_{\rm p}$ is the prediction horizon and λ is a weighting parameter. Possible choices for $J_{\rm out}$ and $J_{\rm in}$ are given in [8, 9] (see also Section 4). The aim is now to compute an optimal input sequence $u(k), \ldots, u(k+N_{\rm p}-1)$ that minimizes $J(k)$ subject to linear constraints on the inputs and outputs. Since the $u(k)$'s correspond to consecutive event occurrence times, we have the additional condition $\Delta u(k+j) = u(k+j) - u(k+j-1) \geq 0$ for $j = 0, \ldots, N_{\rm p}-1$. Furthermore, in order to reduce the number of decision variables and the corresponding computational complexity we introduce a control horizon $N_{\rm c}$ ($\leq N_{\rm p}$) and we impose the additional condition that the input rate should be constant from the point $k + N_{\rm c} - 1$ on: $\Delta u(k+j) = \Delta u(k+N_{\rm c}-1)$ for $j = N_{\rm c}, \ldots, N_{\rm p}-1$, or equivalently $\Delta^2 u(k+j) = \Delta u(k+j) - \Delta u(k+j-1) = 0$ for $j = N_{\rm c}, \ldots, N_{\rm p}-1$.

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k), \ldots, u(k+N_{\rm c}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one sample, and the optimization is restarted with new information of the measurements.

Define the vectors

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_{\rm p}-1) \end{bmatrix}$$
$$\tilde{y}(k) = \begin{bmatrix} \hat{y}(k) \\ \vdots \\ \hat{y}(k+N_{\rm p}-1) \end{bmatrix}$$

Now the MPL-MPC problem for event step k can be defined as:

$$\min_{\tilde{u}(k)} J_{\rm out}(k) + \lambda J_{\rm in}(k)$$

subject to

$$x(k+j) = A \otimes x(k+j-1) \oplus B \otimes u(k+j) \tag{4}$$
$$y(k+j) = C \otimes x(k+j) \tag{5}$$
$$\Delta u(k+j) \geq 0 \tag{6}$$
$$\Delta^2 u(k+\ell) = 0 \tag{7}$$
$$A_{\rm c}(k)\tilde{u}(k) + B_{\rm c}(k)\tilde{y}(k) \leq c_{\rm c}(k) \tag{8}$$

for $j = 0, \ldots, N_{\rm p}-1$, and for $\ell = N_{\rm c}, \ldots, N_{\rm p}-1$

where (8) represents the linear constraints on the inputs and the outputs.

We conclude this section with some results on a class of (max,+) functions. Let $\mathcal{S}_{\rm mpns}$ be the set of max-plus-nonnegative-scaling functions, i.e., functions f of

[1] More specifically, for a manufacturing system, $x(k)$ contains the time instants at which the processing units start working for the kth time, $u(k)$ the time instants at which the kth batch of raw material is fed to the system, and $y(k)$ the time instants at which the kth batch of finished product leaves the system.

the form $f(z) = \max_i(\alpha_{i,1}z_1 + \ldots + \alpha_{i,n}z_n + \beta_i)$ with variable $z \in \mathbb{R}_\varepsilon^n$ and constants $\alpha_{i,j} \in \mathbb{R}^+$, $\beta_i \in \mathbb{R}$, where $\mathbb{R}^+$ is the set of the nonnegative real numbers. If we want to stress that f is a function of z we will denote this by $f \in \mathcal{S}_{\mathrm{mpns}}(z)$.

Lemma 1 *The set $\mathcal{S}_{\mathrm{mpns}}$ is closed under the operations $\oplus$, $\otimes$, and scalar multiplication by a nonnegative scalar.*

Proof: This is a consequence of the fact that for $x, y, z, v \in \mathbb{R}_\varepsilon$ and $\rho \in \mathbb{R}^+$ we have $\max(x,y) \oplus \max(z,v) = \max(\max(x,y), \max(z,v)) = \max(x,y,z,v)$, $\max(x,y) \otimes \max(z,v) = \max(x,y) + \max(z,v) = \max(x+z, x+v, y+z, y+v)$ and $\rho\max(x,y) = \max(\rho x, \rho y)$. ■

3 Making predictions in the stochastic case

In this section we extend the deterministic model (1)–(2) to include uncertainty (see also [19]). So we now consider the following MPL system:

$$x(k) = A(k) \otimes x(k-1) \oplus B(k) \otimes u(k) \tag{9}$$

$$y(k) = C(k) \otimes x(k) \tag{10}$$

where $A(k)$, $B(k)$ and $C(k)$ represent uncertain system matrices due to modeling errors or disturbances. Usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. In this paper both features will be treated in one single framework. The uncertainty caused by disturbances and errors in the estimation of physical variables, is gathered in the uncertainty vector $e(k)$. In this paper we assume that the uncertainty has stochastic properties. Hence, $e(k)$ is a stochastic variable.

We assume that the uncertainty vector $e(k)$ captures the complete time-varying aspect of the system. Furthermore, the system matrices of an MPL model usually consist of sums or maximizations of internal process times, transportation times, etc. (see, e.g., [1] or [20]). Since the entries of $e(k)$ directly correspond to the uncertainties in these duration times, it follows from Lemma 1 that the entries of the uncertain system matrices belong to $\mathcal{S}_{\mathrm{mpns}}$:

$$A(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times n}(e(k)), \quad B(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times m}(e(k)), \quad C(k) \in \mathcal{S}_{\mathrm{mpns}}^{l\times n}(e(k)) \tag{11}$$

The next step is to make predictions. We collect the uncertainty over the interval $[k, k+N_\mathrm{p}]$ in one vector

$$\tilde{e}(k) = \begin{bmatrix} e(k) \\ \vdots \\ e(k+N_\mathrm{p}-1) \end{bmatrix} \in \mathbb{R}^{n_{\tilde{e}}} .$$

We assume that $\tilde{e}(k)$ is a random variable, and that all elements of $\tilde{e}(k)$ are independent, i.e. for $i \neq j$ there holds:

$$p_{ij}\Big(\tilde{e}_i(k), \tilde{e}_j(k)\Big) = p_i\Big(\tilde{e}_i(k)\Big)p_j\Big(\tilde{e}_j(k)\Big)$$

where p_ℓ is the probability density function of the ℓth element of $\tilde{e}(k)$ and p_{ij} is the joint probability density function of the ith and jth element of $\tilde{e}(k)$. The probability density function of $\tilde{e}(k)$ is denoted by $p(\tilde{e}(k))$ and satisfies:

$$p(\tilde{e}(k)) = \prod_{j=1}^{n_{\tilde{e}}} p_j\Big(\tilde{e}_j(k)\Big) \tag{12}$$

Remark: *To make sure that all elements are uncorrelated, redundant (or repeating) components should be eliminated. After elimination of the elements, there will still hold:*

$$A(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times n}(\tilde{e}(k)), \quad B(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times m}(\tilde{e}(k)), \quad C(k) \in \mathcal{S}_{\mathrm{mpns}}^{l\times n}(\tilde{e}(k)) .$$

where $\tilde{e}(k)$ now satisfies (12).

Now it is easy to verify that the prediction model, i.e., the prediction of the future outputs for the system (9)–(10), is given by

$$\tilde{y}(k) = \tilde{C}(\tilde{e}(k)) \otimes x(k-1) \oplus \tilde{D}(\tilde{e}(k)) \otimes \tilde{u}(k) \ , \tag{13}$$

in which $\tilde{C}(\tilde{e}(k))$ and $\tilde{D}(\tilde{e}(k))$ are given by

$$\tilde{C}(\tilde{e}(k)) = \begin{bmatrix} \tilde{C}_1(\tilde{e}(k)) \\ \vdots \\ \tilde{C}_{N_\mathrm{p}}(\tilde{e}(k)) \end{bmatrix}$$

$$\tilde{D}(\tilde{e}(k)) = \begin{bmatrix} \tilde{D}_{11}(\tilde{e}(k)) & \cdots & \tilde{D}_{1N_\mathrm{p}}(\tilde{e}(k)) \\ \vdots & \ddots & \vdots \\ \tilde{D}_{N_\mathrm{p}1}(\tilde{e}(k)) & \cdots & \tilde{D}_{N_\mathrm{p}N_\mathrm{p}}(\tilde{e}(k)) \end{bmatrix} \tag{14}$$

where

$$\tilde{C}_m(\tilde{e}(k)) = C(k+m-1) \otimes A(k+m-1) \otimes \ldots \otimes A(k) \tag{15}$$

and

$$\tilde{D}_{mn}(\tilde{e}(k)) = \begin{cases} C(k+m-1) \otimes A(k+m-1) \otimes \ldots \\ \qquad \otimes A(k+n) \otimes B(k+n-1) & \text{if } m > n \\ C(k+m-1) \otimes B(k+m-1) & \text{if } m = n \\ \mathcal{E} & \text{if } m < n\,. \end{cases} \tag{16}$$

4 MPC for stochastic MPL systems

Recall that in MPL-MPC the cost function is given by (3). In this paper $J_{\rm out}$ and $J_{\rm in}$ are chosen as follows:

$$\begin{aligned} J_{\rm out}(k) &= \sum_i \mathbb{E}[\tilde{\eta}_i(k)] \\ J_{\rm in}(k) &= \sum_j \tilde{u}_j(k) \end{aligned}$$

where $\mathbb{E}[\tilde{\eta}_i(k)]$ denotes the expectation of the ith "tardiness" error $\tilde{\eta}_i(k)$, which is given by

$$\tilde{\eta}_i(k) = \max(\ \tilde{y}_i(k) - \tilde{r}_i(k)\ ,\ 0\ )\ , \tag{17}$$

the due date signal $r(k)$ is stacked in the vector

$$\tilde{r}(k) = \begin{bmatrix} r(k+1) \\ \vdots \\ r(k+N_{\rm p}) \end{bmatrix}$$

and $\tilde{y}_i(k)$, $\tilde{u}_i(k)$ and $\tilde{r}_i(k)$ denote the ith element of $\tilde{y}(k)$, $\tilde{u}(k)$ and $\tilde{r}(k)$, respectively. Other choices for $J_{\rm out}$ and $J_{\rm in}$ are given in [8, 9].

We combine the material of previous subsections, and obtain

$$\begin{aligned} J_{\rm out}(k) &= \sum_i \mathbb{E}\Big[\max\Big(\ \Big\{[\tilde{C}(k)]_i \otimes x(k)\ \oplus \\ &\qquad [\tilde{D}(k)]_i \otimes \tilde{u}(k)\Big\} - \tilde{r}_i(k)\ ,\ 0\ \Big)\Big] \qquad (18) \\ J_{\rm in}(k) &= \sum_j \tilde{u}_j(k) \qquad (19) \end{aligned}$$

where $[\tilde{C}(k)]_i$ and $[\tilde{D}(k)]_i$ denote the ith row of $\tilde{C}(k)$ and $\tilde{D}(k)$, respectively. Finally the following problem is obtained:

$$\min_{\tilde{u}(k)}\ J_{\rm out}(k) + \lambda J_{\rm in}(k) \tag{20}$$

subject to

$$\begin{aligned} & A_{\rm c}(k)\tilde{u}(k) + B_{\rm c}(k)\mathbb{E}[\tilde{y}(k)] \leq c_{\rm c}(k) && (21) \\ & \Delta u(k+j) \geq 0 \qquad \text{for } j = 0,\ldots,N_{\rm p}-1 && (22) \\ & \Delta^2 u(k+j) = 0 \qquad \text{for } j = N_{\rm c},\ldots,N_{\rm p}-1 && (23) \end{aligned}$$

where $\mathbb{E}[\cdot]$ denotes the expectation of a signal. This problem will be called the stochastic MPL-MPC problem for event step k.

Recall that MPC uses a receding horizon principle. So this means that after computation of the optimal control sequence $u(k),\ldots,u(k+N_{\rm c}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one sample, and the optimization is restarted with new information of the measurements.

5 Convexity of stochastic MPL-MPC

In the previous section we found that we need the expectation of the signals $\tilde{\eta}(k)$ and $\tilde{y}(k)$. We now derive the following property:

Lemma 2 *Define the vector $z(k)$ as*

$$z(k) = \begin{bmatrix} -\tilde{r}(k) \\ x(k-1) \\ \tilde{u}(k) \\ \tilde{e}(k) \end{bmatrix}$$

Then, the future tardiness error $\tilde{\eta}(k)$ and the future output signal $\tilde{y}(k)$ belong to $\mathcal{S}_{\rm mpns}(z(k))$.

Proof: Equations (14)-(16) in combination with (11) and Lemma 1 show that the entries of $\tilde{C}(\tilde{e}(k))$ and $\tilde{D}(\tilde{e}(k))$ belong to $\mathcal{S}_{\rm mpns}(\tilde{e}(k))$. Then, using (13), (17) and again Lemma 1 we find that both $\tilde{\eta}(k)$ and $\tilde{y}(k)$ belong to $\mathcal{S}_{\rm mpns}(z(k))$. ■

Let $v(k) \in \mathcal{S}_{\rm mpns}(z(k))$, where $z(k)$ is as defined in Lemma 2. In the sequel of this section we will derive how to compute the expectation $\mathbb{E}[v(k)]$, and show that $\mathbb{E}[v(k)]$ has some nice convexity properties.

Define $w(k) = \begin{bmatrix} -\tilde{r}^T(k) & x^T(k-1) & \tilde{u}^T(k) \end{bmatrix}^T$ to be the non-stochastic part of $z(k)$. Then, because of the above lemma and the definition of max-plus-nonnegative-scaling functions, there exist scalars α_j and nonnegative vectors β_j and γ_j, such that

$$v(k) = \max_{j=1,\ldots,n_v}\Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)\Big)$$

Define the sets $\Phi_j(w(k))$, $j = 1,\ldots,n_v$ such that for all $\tilde{e}(k) \in \Phi_j(w(k))$ there holds:

$$v(k) = \alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)$$

and

$$\bigcup_{j=1}^{n_v} \Phi_j(w(k)) = \mathbb{R}^{n_{\tilde{e}}}$$

Denote, for a given $w(k)$, the expectation of $v(k)$ by $\hat{v}(w(k)) = \mathbb{E}[v(k)]$, then

$$\begin{aligned} \hat{v}(w(k)) &= \mathbb{E}[v(k)] && (24) \\ &= \int_{-\infty}^{+\infty}\!\!\ldots\int_{-\infty}^{+\infty} v(k)\, p(\tilde{e})\, d\tilde{e} && (25) \\ &= \int_{-\infty}^{+\infty}\!\!\ldots\int_{-\infty}^{+\infty} \max_{j=1,\ldots,n_v}\Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} && (26) \\ &= \sum_{j=1}^{n_v} \int\!\!\ldots\int_{\tilde{e}\in\Phi_j(w(k))} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} && (27) \end{aligned}$$

where $d\tilde{e} = d\tilde{e}_1\, d\tilde{e}_2 \dots d\tilde{e}_{n_{\tilde{e}}}$.

The following lemma is on the convexity of $\hat{v}(w(k))$ in the vector $w(k)$

Lemma 3 *The function $\hat{v}(w(k))$ as defined in (27) is convex in $w(k)$ and a subgradient $g_v(w(k))$ is given by*

$$g_v(w(k)) = \sum_{\ell=1}^{n_v} \beta_\ell^T \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w(k))} p(\tilde{e})\, d\tilde{e} \qquad (28)$$

Proof: Recall that (cf. (27))

$$\hat{v}(w_o(k))) = \sum_{j=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_j(w_o(k))} \Big(\alpha_j + \beta_j^T w_o(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e}$$

Then, using the fact that $\bigcup_{j=1}^{n_v} \Phi_j(w_o(k)) = \mathbb{R}^{n_{\tilde{e}}}$, there holds for any $w(k)$:

$$\begin{aligned}
\hat{v}(w(k)) &= \\
&= \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} \max_{j=1,\dots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (26))} \\
&= \sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \max_{j=1,\dots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\geq \sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w(k) + \gamma_\ell^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e}
\end{aligned}$$

Note that the sets $\Phi_\ell(w_o(k))$ are computed for $w_o(k)$, whereas $\hat{v}(w(k))$ is computed for $w(k)$. Now we derive:

$$\begin{aligned}
&\sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w(k) + \gamma_\ell^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\quad = \sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w_o(k) + \gamma_\ell^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\qquad + \sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \Big(\beta_\ell^T (w(k) - w_o(k))\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\quad = \sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w_o(k) + \gamma_\ell^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \\
&\qquad + \left(\sum_{\ell=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,\Phi_\ell(w_o(k))} (\beta_\ell^T)\, p(\tilde{e})\, d\tilde{e}\right) \Big(w(k) - w_o(k)\Big) \\
&\quad = \hat{v}(w_o(k)) + g_v(w_o(k)) \Big(w(k) - w_o(k)\Big)
\end{aligned}$$

and we conclude:

$$\hat{v}(w(k)) \geq \hat{v}(w_o(k)) + g_v(w_o(k)) \Big(w(k) - w_o(k)\Big) \qquad (29)$$

Equation (29) proves that $\hat{v}$ is convex in $w(k)$ and g_v is a sub-gradient of $\hat{v}$ ([17]). ∎

Now consider the MPC problem (20) – (23). First note that because of Lemma 3, $\mathbb{E}[\tilde{\eta}_i(k)]$ and $\tilde{y}(k)$ are convex in $w(k)$. This means that $J_{\text{out}}(k)$ and $J(k)$ are convex in $\tilde{u}(k)$. It is easy to verify that if the linear MPC constraints are monotonically non-decreasing as a function of $\mathbb{E}[\tilde{y}(k)]$ (in other words, if $[B_{\text{c}}]_{ij} \geq 0$ for all i, j), the constraint (21) becomes convex in $\tilde{u}(k)$. In that way, the MPL-MPC problem turns out to be a convex problem and both a subgradient of the constraints and a subgradient of the cost criterion can easily be derived using Lemma 3. More on convex optimization algorithms can be found in [15].

So far, we did not make any assumption on the characterization of probability function $p_i(\tilde{e}_i)$. For the computation of the cost criterion and the constraints we need the values of $\mathbb{E}[\tilde{y}(k)]$ and $\mathbb{E}[\tilde{\eta}(k)]$. If we choose for example a Gaussian distribution, they can be calculated from equation (27) using numerical integration. Numerical integration is usually time-consuming and cumbersome, but can be avoided by choosing piecewise affine probability density functions.

Piecewise affine probability density functions
Let $p_i(\tilde{e}_i)$ for all $i = 1, \dots, n_{\tilde{e}}$ be piecewise affine functions. So there exist sets P_ℓ, $\ell = 1, \dots, n_p$, such that for $\tilde{e} \in P_\ell$ the probability density functions are given by:

$$p_i(\tilde{e}_i) = \mu_{i,\ell} + \eta_{i,\ell}^T \tilde{e}_i \qquad \text{for } i = 1, \dots, n_{\tilde{e}}\,.$$

Consider a signal $v(k) \in \mathcal{S}_{\text{mpns}}(z(k))$. Let $E_{j\ell}(w(k)) = \Phi_j(w(k)) \cap P_\ell$, then $\hat{v}(w(k))$ is given by

$$\begin{aligned}
\hat{v}(w(k)) \;=\; &\sum_{\ell=1}^{n_p} \sum_{j=1}^{n_v} \int\dots\int_{\tilde{e}\,\in\,E_{j\ell}(w(k))} \Big(\alpha_j + \beta_j^T w(k) + \\
&\qquad + \gamma_j^T \tilde{e}\Big) \left(\prod_{i=1}^{n_{\tilde{e}}} (\mu_{i,\ell} + \eta_{i,\ell}^T \tilde{e}_i)\right) d\tilde{e}
\end{aligned}$$

This is an integral of polynomial functions and can be solved analytically for all regions $E_{j\ell}$. Methods for integration on convex polytopes are given by Lasserre [11] and Büeler *et al.* [3].
If piecewise affine probability density functions are used as an approximation of 'true' non-affine probability functions, the quality of the approximation can be improved by increasing the number of sets n_p.

6 Discussion

We have further extended the MPC framework to include max-plus-linear discrete event systems with

stochastic uncertainties. We have shown that, if the constraints are a non-decreasing function of the output, the resulting optimization problem is a convex optimization problem. In general, the computation of the predictions requires a numerical integration. However, in the case of piecewise affine probability density functions, this numerical integration can be prevented.
Topics for future are: determination of rules of thumb for appropriate values for the tuning parameters (control horizon N_c, prediction horizon N_p, and performance weighting parameter λ) in the stochastic case, complexity reduction and approximation to further improve the efficiency of our approach.

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References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity.* New York: John Wiley & Sons, 1992.

[2] L. Biegler, "Efficient solution of dynamic optimization and NMPC problems," in *Nonlinear Model Predictive Control* (F. Allgöwer and A. Zheng, eds.), vol. 26 of *Progress in Systems and Control Theory*, Basel, Switzerland: Birkhäuser Verlag, 2000.

[3] B. Bueler and A. Enge and K. Fukuda, "Exact Volume Computation for Polytopes: A Practical Study, " In *Proceedings of the conference 'Computational Geometry'*, Münster (Germany), March 1996.

[4] E.F. Camacho and C. Bordons, *Model Predictive Control in the Process Industry.* Berlin, Germany: Springer-Verlag, 1995.

[5] J. Cardoso, R. Valette, and D. Dubois, "Possibilistic Petri nets," *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, vol. 29, no. 5, pp. 573–582, 1999.

[6] D.W. Clarke, C. Mohtadi, and P.S. Tuffs, "Generalized predictive control – Part I. The basic algorithm," *Automatica*, vol. 23, no. 2, pp. 137–148, Mar. 1987.

[7] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems.* Berlin, Germany: Springer-Verlag, 1979.

[8] B. De Schutter and T. van den Boom, "Model predictive control for max-plus-linear discrete event systems," Automatica, vol. 37, no. 7, pp. 1049–1056 ,July 2001.

[9] B. De Schutter and T. van den Boom, "Model predictive control for max-plus-linear systems," in *Proceedings of the 2000 American Control Conference*, Chicago, Illinois, pp. 4046–4050, June 2000.

[10] C.E. García, D.M. Prett, and M. Morari, "Model predictive control: Theory and practice — A survey," *Automatica*, vol. 25, no. 3, pp. 335–348, May 1989.

[11] J.B. Lasserre, "Integration on a convex polytope," In *Proceedings of the 1998 American Mathematical Society*, vol.126, no. 8 pp. 2433–2441, August 1998.

[12] F. Lin, "Robust and adaptive supervisory control of discrete event systems," *IEEE Transactions on Automatic Control*, vol. 38, no. 12, pp. 1848–1852, Dec. 1993.

[13] T.H. Mattheiss and D.S. Rubin, "A survey and comparison of methods for finding all vertices of convex polyhedral sets," *Mathematics of Operations Research*, vol. 5, no. 2, pp. 167–185, May 1980.

[14] T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall, "The double description method," in *Contributions to the Theory of Games* (H.W. Kuhn and A.W. Tucker, eds.), no. 28 in Annals of Mathematics Studies, pp. 51–73, Princeton, New Jersey: Princeton University Press, 1953.

[15] Y. Nesterov and A. Nemirovskii, "Interior-Point Polynomial Algorithms in Convex Programming", SIAM, Philadelphia, Pennsylvania, 1994.

[16] S.J. Park and J.T. Lim, "Fault-tolerant robust supervisor for discrete event systems with model uncertainty and its application to a workcell," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 2, pp. 386–391, Apr. 1999.

[17] R.T. Rockafellar, "Convex Analysis", Princeton University Press, Princeton, New Jersey, 1970.

[18] T. van den Boom and B. De Schutter, "MPC for max-plus-linear systems: Closed-loop behavior and tuning," In *Proceedings of the 2001 American Control Conference*, Arlington, Virginia, pp. 325–330, June 2001.

[19] T. van den Boom and B. De Schutter, "MPC for Perturbed Max-Plus-Linear Systems," Tech. rep., Control In *Proceedings of the 2001 European Control Conference*, Porto, Portugal, September 2001.

[20] T. van den Boom and B. De Schutter, "Model Predictive Control for Perturbed Max-Plus-Linear Systems: A Stochastic Approach," Tech. rep., Control, Laboratory, Fac. of Information Technology and Systems, Delft University of Technology, Delft, The Netherlands, March 2001.

[21] S. Young and V.K. Garg, "Model uncertainty in discrete event systems," *SIAM Journal on Control and Optimization*, vol. 33, no. 1, pp. 208–226, Jan. 1995.