**Delft University of Technology**

**Fac. of Information Technology and Systems** **Control Systems Engineering**

Technical report CSE02-003

# On hybrid systems and closed-loop MPC systems*

A. Bemporad, W.P.M.H. Heemels, and B. De Schutter

*If you want to cite this report, please use the following reference instead:*
A. Bemporad, W.P.M.H. Heemels, and B. De Schutter, "On hybrid systems and closed-loop MPC systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 863–869, May 2002. doi:10.1109/TAC.2002.1000287

Control Systems Engineering
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands
Current URL: https://www.dcsc.tudelft.nl

# On Hybrid Systems and Closed-Loop MPC Systems

A. Bemporad*, W.P.M.H. Heemels†, B. De Schutter‡

**Abstract**

The following five classes of hybrid systems were recently proven to be equivalent: linear complementarity, extended linear complementarity, mixed logical dynamical, piecewise affine, and max-min-plus-scaling systems. Some of the equivalences were obtained under additional assumptions, such as boundedness of certain system variables. In this paper, for linear or hybrid plants in closed-loop with a model predictive control (MPC) controller based on a linear model, fulfilling linear constraints on input and state variables, and utilizing a quadratic cost criterion, we provide a simple and direct proof that the closed-loop system is a subclass of any of the former five classes of hybrid systems. This result is of extreme importance as it opens up the use of tools developed for the mentioned hybrid model classes, such as (robust) stability and safety analysis tools, to study closed-loop properties of MPC.

**Keywords**: hybrid systems, model predictive control, complementarity systems, piecewise affine systems, mixed logical dynamical systems, equivalent models.

## 1 Introduction

Hybrid dynamical models describe systems where both analog (continuous) and logical (discrete) components are relevant and interacting [1]. Recently, hybrid systems received a lot of attention from both the computer science and the control community, but general analysis and control design methods for hybrid systems are not yet available. For this reason, several authors have focused on special subclasses of hybrid systems for which analysis and synthesis techniques are currently being developed. Some examples of such subclasses are: linear complementarity (LC) systems [2,3], extended linear complementarity (ELC) systems [4], mixed logical dynamical (MLD) systems [5,6], piecewise affine (PWA) systems [7], and max-min-plus-scaling (MMPS) systems [8].

In [9] we showed that the above five subclasses of hybrid systems are equivalent. Some of the equivalences were obtained under additional assumptions related to well-posedness (i.e., existence and uniqueness of solution trajectories) and boundedness of (some) system variables. These results are extremely important, as they allow to transfer all the analysis and synthesis tools developed for one particular class to any of the other equivalent subclasses of hybrid systems.

The main result of this paper will show that all these hybrid tools can be used for the analysis of closed-loop Model Predictive Control (cl-MPC) systems as well. Indeed, as we will prove that cl-MPC systems can be written as LC and MLD systems, the transfer of the machinery is immediate.

*Alberto Bemporad is affiliated at the Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy, `bemporad@dii.unisi.it` and the Automatic Control Laboratory, ETH Zentrum, ETL I 26, 8092 Zurich, Switzerland, `bemporad@aut.ee.ethz.ch`

†Maurice Heemels is with the Dept. of Electrical Eng., Eindhoven Univ. of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, `m.heemels@tue.nl`

‡Bart De Schutter is with the Control Systems Engineering group, Fac. ITS, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands, `b.deschutter@its.tudelft.nl`

Related results were obtained in [10], where the authors showed that MPC control is equal to a piecewise affine control law that can be computed off-line by using multiparametric quadratic programming solvers (and, therefore, that the closed-loop system is a PWA system). Rather than exploiting the equivalence results of [9] in combination with [10] to convert from PWA to LC and MLD, which would require additional assumptions and would yield more complex models, we provide a simple, direct, and constructive proof to rewrite cl-MPC systems as LC and MLD systems.

Despite the fact that MPC schemes are typically designed so that they are intrinsically stable and fulfill operating constraints, stability is usually guaranteed through the introduction of stability constraints, which are often removed in practical MPC schemes as they typically deteriorate performance or complicate the optimization problem. Moreover, such guarantees only hold *when the nominal model of the plant and the prediction model coincide and the full state is available at each sample instant*. An important issue is to analyze the behavior of the closed-loop system when the nominal model and the plant model differ, e.g., because of the presence of nonlinearities, or when an observer is used to estimate the state. Robust MPC techniques [11] partially solve this issue, by taking into account a class of linear uncertain models rather than one single prediction model, although this typically requires increased computation effort and, again, leads to deterioration of performance. Now, based on the results of this paper, the (robust) stability analysis, well-posedness results, and safety analysis tools available for any of the five mentioned classes of hybrid systems (PWA, MLD, LC, ELC, MMPS) can be applied to any combination of a linear MPC controller and a linear plant (possibly including disturbances and model uncertainties). The results can be easily extended to arbitrary combinations of linear MPC controllers and *hybrid* plants, such as hybrid approximations of complex nonlinear dynamic models of the process to be controlled. An example will demonstrate the use of hybrid tools for stability analysis and verification in the setting of cl-MPC systems.

## 2 Classes of Hybrid Dynamical Models

In this paper we consider discrete-time models of the form

$$\begin{aligned} x(k+1) &= f(x(k),u(k),w(k)) && \text{(1a)}\\ y(k) &= g(x(k),u(k),w(k)) && \text{(1b)}\\ 0 &\le h(x(k),u(k),w(k)), && \text{(1c)} \end{aligned}$$

where the variables $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^l$ denote the input, state and output, respectively, at time $k$, and $w(k) \in \mathbb{R}^r$ is a vector of auxiliary variables (this notation also holds for all the hybrid models introduced later), $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^p$, $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^q$, and the inequality (1c) should be interpreted componentwise. The evolution of system (1) is determined as follows. Given the current state $x(k)$ and input $u(k)$ the collection of inequalities (1c) is solved for $w(k)$. By substitution of $w(k)$ in (1a)-(1b), the state update $x(k+1)$ and the current output $y(k)$ are obtained. Specific choices of the form of the functions $f$, $g$, $h$ will determine different classes of hybrid systems, as we will detail in the rest of this section.

**Definition 1** *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a set of input+state pairs. A hybrid system of the form (1) is called* well-posed *on $\Omega$, if for all pairs $(x(k),u(k)) \in \Omega$ the equations (1) have a solution $(x(k+1), y(k), w(k))$ and moreover, $(x(k+1), y(k))$ are uniquely determined.*

Definition 1 implies that $x(k+1)$, $y(k)$ are unique functions of $(x(k),u(k))$, and therefore that the possible non-uniqueness of $w(k)$ is removed through the mappings $f$, and $g$.

**Remark 1** The general formulation (1) allows some of the state, input, output, or auxiliary variables to attain only discrete values, e.g., $w_i(k) \in \{0, 1\}$ can be represented by the two inequalities $\max(w_i(k) - 1, -w_i(k)) \geq 0$, $-\max(w_i(k) - 1, -w_i(k)) \geq 0$, or by $w_i(k)(1 - w_i(k)) \leq 0$, $w_i(k) \geq 0$, $1 - w_i(k) \geq 0$. □

**Remark 2** As will also be clarified later for PWA systems, for well-posedness of several instances of (1) over compact sets of $\mathbb{R}^n \times \mathbb{R}^m$, the inequalities in (1) should be split as strict inequalities $h_i(x(k), u(k), w(k)) > 0$, $i \in I$, and nonstrict inequalities $h_j(x(k), u(k), w(k)) \geq 0$, $j \in J$, $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, q\}$. Although this would be important from a system theoretical point of view, it is not of practical interest from a numerical point of view, as "$>$" cannot be represented in numerical algorithms working in finite precision. Indeed, $h > 0$ can be only represented as $h \geq \epsilon$, and $\epsilon$ is some pre-specified tolerance, e.g., the machine precision. □

## 2.1 Piecewise Affine (PWA) Systems

Piecewise affine (PWA) systems are described by

$$\begin{array}{rcl} x(k+1) & = & A^i x(k) + B^i u(k) + f^i \\ y(k) & = & C^i x(k) + D^i u(k) + g^i \end{array} \quad \text{for} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{2}$$

where $\Omega_i \triangleq \{[\begin{smallmatrix} x \\ u \end{smallmatrix}] : H_x^i x + H_u^i u \leq K_i\}$, $i = 1, \ldots, s$, are convex polyhedra in the input+state space. $A^i$, $B^i$, $C^i$, $D^i$, $H_x^i$ and $H_u^i$ are real matrices of appropriate dimensions and $f^i$ and $g^i$ are real vectors for all $i = 1, \ldots, s$. PWA systems have been studied by several authors (see [6, 7, 12] and the references therein) as they form the "simplest" extension of linear systems that can still model non-linear and non-smooth processes with arbitrary accuracy and are capable of handling hybrid phenomena.

System (2) belongs to the general class (1) by letting $f$, $g$ be PWA functions defined over $\Omega \triangleq \cup_{i=1}^s \Omega_i$, and $r = q = 0$ (i.e., the auxiliary variable $w(k)$ and the mapping $h$ are not required). A necessary and sufficient condition for the PWA system (2) to be well-posed over $\Omega$ is therefore that $f$, $g$ are single-valued PWA functions. Therefore, typically the sets $\Omega_i$ have mutually disjoint interiors, and are often defined as the partition of a convex polyhedral set $\Omega$. In case of discontinuities of $f$, $g$ over overlapping boundaries of the regions $\Omega_i$, to ensure well-posedness we should write some of the inequalities in the form $(H_i^x)^j x + (H_i^u)^j u < K_i^j$ (see Remark 2). In the following we shall neglect this issue for the sake of compactness of notation and the fact that we will actually deal with *continuous* piecewise affine systems, as we will see.

## 2.2 Mixed Logical Dynamical (MLD) Systems

In [5] a class of hybrid systems has been introduced in which logic, dynamics and constraints are integrated. This lead to a description of the form

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \tag{3a}$$

$$y(k) = Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) \tag{3b}$$

$$E_1 x(k) + E_2 u(k) + E_3 \delta(k) + E_4 z(k) \leqslant g_5, \tag{3c}$$

where $x(k) = [\, x_\mathrm{r}'(k) \; x_\mathrm{b}'(k) \,]'$, $x_\mathrm{r}(k) \in \mathbb{R}^{n_\mathrm{r}}$ and $x_\mathrm{b}(k) \in \{0, 1\}^{n_\mathrm{b}}$ ($y(k)$ and $u(k)$ have a similar structure), and where $z(k) \in \mathbb{R}^{r_\mathrm{r}}$ and $\delta(k) \in \{0, 1\}^{r_\mathrm{b}}$ are auxiliary variables. $A$, $B_i$, $C$, $D_i$ and $E_i$ denote real constant matrices and $g_5$ is a real vector. The inequalities (3c) have to be interpreted componentwise. Systems that can be described by model (3) are called Mixed Logical Dynamical

(MLD) systems. By letting $w(k) \triangleq [z'(k)\ \delta'(k)]'$, clearly (3) together with the integrality conditions over $\delta$, $x_{\mathrm{b}}$, $y_{\mathrm{b}}$, and $u_{\mathrm{b}}$ (expressed as inequalities, see Remark 1), forms a subclass of (1).

The MLD formalism allows specifying the evolution of continuous variables through linear dynamic equations, of discrete variables through propositional logic statements and automata, and the mutual interaction between the two. The key idea of the approach consists of embedding the logic part in the state equations by transforming Boolean variables into 0-1 integers, and by expressing the relations as mixed-integer linear inequalities (see [5] and references therein). MLD systems are therefore capable of modeling a broad class of systems, in particular those systems that can be modeled through the hybrid system description language HYSDEL [13].

### 2.3 Linear Complementarity (LC) Systems

Linear complementarity (LC) systems are given in discrete-time by the equations

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2w(k) && \text{(4a)}\\ y(k) &= Cx(k) + D_1u(k) + D_2w(k) && \text{(4b)}\\ v(k) &= E_1x(k) + E_2u(k) + E_3w(k) + g_4 && \text{(4c)}\\ 0 \le v(k) &\perp w(k) \ge 0 && \text{(4d)} \end{aligned}$$

with $v(k), w(k) \in \mathbb{R}^s$ and where $\perp$ denotes the orthogonality of vectors (i.e. $v(k) \perp w(k)$ means that $v'(k)w(k) = 0$). We call $v(k)$ and $w(k)$ the complementarity variables. $A$, $B_i$, $C$, $D_i$ and $E_i$ are real matrices and $g_4$ is a real vector. Clearly, (4) is a subclass of (1).

In [2,3,14–16] (linear) complementarity systems in *continuous* time have been studied. Applications include constrained mechanical systems, electrical networks with ideal diodes or other dynamical systems with piecewise affine relations, variable structure systems, constrained optimal control problems, projected dynamical systems, and so on [15, Ch. 2].

### 2.4 Equivalence of hybrid model classes

In [9] we discussed the relationships between the model classes mentioned above and two others: *min-max-plus-scaling* (MMPS) and *extended linear complementarity* (ELC) systems. As ELC systems are of similar nature as LC systems, we will not define them here, but refer to [4, 9]. MMPS systems are obtained by choosing $f$, $g$, $h$ in (1) as (nested) combinations of the operations maximization, minimization, addition and scalar multiplication. More details on this class can be found in [8, 9].

**Fact 1** *PWA systems, MLD systems, LC systems, and MMPS systems are equivalent (certain equivalences require assumptions on the boundedness of input, state, and auxiliary variables or on well-posedness), and form subsets of the general class of hybrid systems (1).*

**Proof:** See [9] for full details on assumptions, relationships, and a constructive proof. □

## 3 Closed-Loop Model Predictive Control (cl-MPC) Systems and Hybrid Systems

Model Predictive Control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. Only the first computed control value in the sequence is implemented. At the next time step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy [17].

For the discrete-time linear time-invariant system

$$\begin{cases} x(k+1) &=& Ax(k)+Bu(k) \\ y(k) &=& Cx(k), \end{cases} \tag{5}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^p$ are the state, input, and output vector, respectively, consider the problem of tracking the output reference signal $r(k) \in \mathbb{R}^p$ while fulfilling the constraints

$$D_1 x(k) + D_2 u(k) + D_3 \Delta u(k) \leq d_4 \tag{6}$$

at all time instants $k \geq 0$, where $\Delta u(k) \triangleq u(k) - u(k-1)$ are the increments of the input.

Assume for the moment that a full measurement of the state $x(k)$ and the previously implemented control value $x_u(k) \triangleq u(k-1)$ (which might be considered as an additional state) are available at the current time $k$. Then, the optimization problem

$$\begin{array}{ll} \min\limits_{U} & \left( \sum\limits_{t=1}^{N_y} \epsilon'_{k+t|k} Q \epsilon_{k+t|k} + \sum\limits_{t=0}^{N_u-1} \Delta u'_{k+t} R \Delta u_{k+t} \right) \\ \\ \text{subj. to} & D_1 x_{k+t|k} + D_2 u_{k+t} + D_3 \Delta u_{k+t} \leq d_4,\ t = 0,1,\ldots,N_c \\ & x_{k+t+1|k} = Ax_{k+t|k} + Bu_{k+t},\ t \geq 0 \\ & y_{k+t|k} = Cx_{k+t|k},\ t \geq 0 \\ & u_{k+t} = u_{k+t-1} + \Delta u_{k+t},\ t \geq 1 \\ & \Delta u_{k+t} = 0,\ N_u \leq t < N_y \\ & x_{k|k} = x(k),\ u_k = u(k-1) + \Delta u_k \end{array} \tag{7}$$

is solved with respect to the column vector $U \triangleq [\Delta u'_k, \ldots, \Delta u'_{k+N_u-1}]' \in \mathbb{R}^s$, $s \triangleq mN_u$, at each time $k$, where $x_{k+t|k}$ denotes the predicted state vector at time $k+t$, obtained by applying the input sequence $u_k, \ldots, u_{k+t-1}$ to model (5) starting from the state $x(k)$, and $\epsilon_{k+t|k} \triangleq y_{k+t|k} - r(k)$ is the predicted tracking error[1]. In (7), we assume that $Q = Q' \succeq 0$, $R = R' \succ 0$ ("$\succ$" denotes matrix positive definiteness), $N_y$, $N_u$, $N_c$ are the output, input, and constraint horizons, respectively, with $N_u \leq N_y$ and $N_c \leq N_y - 1$.

The MPC control law is based on the following idea: At time $k$ compute the optimal solution $U^*(k) = [\Delta u^{*\prime}_k, \ldots, \Delta u^{*\prime}_{k+N_u-1}]'$ to problem (7), apply

$$u(k) = x_u(k) + \Delta u^*_k \tag{8}$$

as input to system (5), and repeat the optimization (7) at the next time step $k+1$, based on the new measured (or estimated) state $x(k+1)$. Note that

$$\Delta u^*_k = I_1 U^*(k), \tag{9}$$

[1] If the reference is known in advance, one can replace $r(k)$ with $r(k+t)$, with a consequent anticipative action of the resulting MPC controller. Otherwise, we set $r(k+t) = r(k)$ for $t \geq 0$.

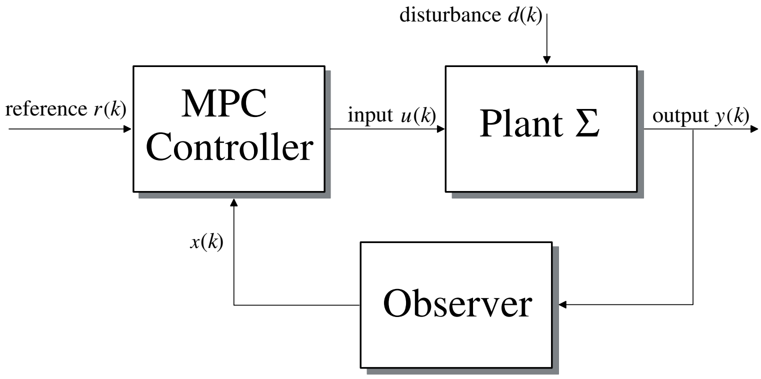



Figure 1: Closed-loop model predictive control system

where $I_1 \triangleq [I_m \; 0 \; \dots \; 0]$. By substituting $x_{k+t|k} = A^t x(k) + \sum_{j=0}^{t-1} A^j B u_{k+t-1-j}$ in (7), this can be written as

$$\begin{aligned} \min_U \quad & \tfrac{1}{2}U'HU + \xi'(k)FU + \tfrac{1}{2}\xi'(k)Y\xi(k) \\ \text{subj. to} \quad & GU \le W + S\xi(k), \end{aligned} \tag{10}$$

where $\xi(k) \triangleq [x'(k) \; x'_u(k) \; r'(k)]'$, $H = H' \succ 0$, and $H$, $F$, $Y$, $G$, $W$, $S$ are easily obtained from (7).

The optimization problem (10) is a quadratic program (QP), which depends on the current state $x(k)$, past input $x_u(k) = u(k-1)$, and reference $r(k)$.

Consider the closed-loop model predictive control system depicted in Fig. 1. The plant $\Sigma$ is described by the difference equations

$$\Sigma : \begin{cases} \chi(k+1) &= \mathcal{A}\chi(k) + \mathcal{B}u(k) + \mathcal{H}d(k) \\ y(k) &= \mathcal{C}\chi(k) + \mathcal{D}d(k), \end{cases} \tag{11}$$

where $\chi(k) \in \mathbb{R}^{\bar{n}}$ is the state vector, and $d(k) \in \mathbb{R}^{\ell}$ is a vector of unmeasured disturbances. We distinguish between model $\Sigma$ in (11), which is the actual plant, and model (5), which is the linear model used for designing the MPC controller. Typically (5) is an approximation of (11), e.g., a low-order approximation where only the relevant dynamics are kept. As the MPC optimization problem (7) is based on model (5), it requires a state $x(k)$ that is coherent with the same model (5). A common solution consists of generating $x(k)$ via the state observer

$$x(k+1) = Ax(k) + Bu(k) + K_e(y(k) - Cx(k)). \tag{12}$$

Now we prove that cl-MPC systems are a subclass of LC systems.

**Theorem 1** *Every cl-MPC system (7), (9)–(12) can be written as an LC system.*

**Proof:** The proof follows from the first-order Karush-Kuhn-Tucker (KKT) conditions for QP (10) [18, Ch. 10.6], which are necessary and sufficient for optimality of $U^*(k)$:

$$HU^*(k) + F'\xi(k) + G'\lambda(k) = 0, \;\; \lambda(k) \in \mathbb{R}^q \tag{13a}$$

$$\lambda'(k)(GU^*(k) - W - S\xi(k)) = 0 \tag{13b}$$

$$\lambda(k) \ge 0 \tag{13c}$$

$$W + S\xi(k) - GU^*(k) \ge 0. \tag{13d}$$

From (13a), it follows that

$$\begin{aligned} U^*(k) &= -H^{-1}F'\xi(k) - H^{-1}G'\lambda(k) \\ &\triangleq Tx(k) + Vx_u(k) + Zr(k) + \Lambda\lambda(k). \end{aligned} \tag{14}$$

By letting $Mx(k) + Nx_u(k) + Lr(k) \triangleq S\xi(k)$, $v(k) \triangleq W + Mx(k) + Nx_u(k) + Lr(k) - GU^*(k)$, $w(k) \triangleq \lambda(k)$, and recalling (8) and (9) we can rewrite the closed-loop MPC system in the LC form

$$\begin{bmatrix} \chi(k+1) \\ x(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}I_1T & \mathcal{B}(I_m + I_1V) \\ K_e\mathcal{C} & A - K_eC + BI_1T & B(I_m + I_1V) \\ 0 & I_1T & (I_m + I_1V) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1Z & \mathcal{H} \\ BI_1Z & K_e\mathcal{D} \\ I_1Z & 0 \end{bmatrix} \begin{bmatrix} r(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1\Lambda \\ BI_1\Lambda \\ \Lambda \end{bmatrix} w(k) \tag{15a}$$

$$y(k) = \mathcal{C}\chi(k) + \mathcal{D}d(k) \tag{15b}$$

$$v(k) = \begin{bmatrix} 0 & M - GT & N - GV \end{bmatrix} \begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} +$$

$$+ (L - GZ)r(k) - G\Lambda w(k) + W \tag{15c}$$

$$0 \leq v(k) \perp w(k) \geq 0, \tag{15d}$$

where $\left[\begin{smallmatrix} \chi \\ x \\ x_u \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} r \\ d \end{smallmatrix}\right]$ are the state and input vectors, respectively, of the LC system. □

**Remark 3** In [10], by exploiting the fact that the coefficients of the linear term in the cost function and the right-hand side of the constraints in (10) depend linearly on a vector $\xi(k)$ of parameters, the quadratic program (10) has been tackled as a *multi-parametric quadratic program* (mp-QP), and it has been shown that the optimal solution is a continuous piecewise affine function of the state. Consequently, the MPC controller admits the explicit continuous PWA form

$$u(k) = F_i\xi(k) + g_i \text{ if } \xi(k) \in \Omega_i^\xi,\ i = 1, \ldots, N,$$

where $\Omega_i^\xi \triangleq \{\xi :\ H_\xi^i\xi(k) \leq K_\xi^i\}$, and $\{\Omega_i\}_{i=1}^N$ is a partition of a given state+input+reference set $\Xi$. Or stated differently, every cl-MPC system (7), (9)–(12) can be written as a continuous PWA system. By applying Fact 1, one can now also show that cl-MPC systems can be equivalently rewritten as LC systems. However, this requires boundedness assumptions over some of the variables, as the transformation through MLD is involved, plus a large number of complementarity pairs. An alternative could be based on [19] in which continuous PWA functions are transferred into linear complementarity problems [20] of the form (4c)-(4d). However, the proof presented above is more direct, does not require any assumptions, and limits the number of required complementarity pairs. □

In order to show directly that cl-MPC systems are also a subclass of MLD systems, we prove the following lemma.

**Lemma 1** *Let* $\xi \triangleq [x'\ x_u'\ r']'$ *belong to a bounded set* $\Xi$. *Then, there exists an upper-bound* $\lambda^+ \geq 0$ *such that at least one vector of Lagrange multipliers* $\lambda$ *satisfies the Karush-Kuhn-Tucker conditions (13) and* $0 \leq \lambda \leq \lambda^+$.

**Proof:** Consider the combination $I \subseteq \{1, \ldots, k\}$ of active constraints $G_I U^* = W_I + S_I \xi$ at the optimum, where $G_I$ denotes the submatrix of $G$ obtained by collecting the rows indexed by the elements of $I$ (similar for $W_I$ and $S_I$), and assume that $G_I$ has full row rank. From the KKT conditions (13), $U^* = -H^{-1}(F'\xi + G_I' \lambda_I(\xi))$, where $\lambda_I(\xi) \geq 0$ is a vector collecting the subset of Lagrange multipliers relative to the active constraints (the remaining multipliers are zero due to (13b)-(13d)). Substituting $U^*$, we obtain $\lambda_I(\xi) = -(G_I H^{-1} G_I')^{-1}[W_I + S_I \xi + G_I H^{-1} F' \xi]$, which admits an upper-bound $\lambda_I^+ \triangleq \max_{\xi \in \Xi} \lambda_i(\xi) \geq 0$. Take $\lambda^+ \triangleq \max \lambda_I^+$ over all combinations $I$ of linearly independent active constraints. If for some $\xi$ a linearly dependent combination of constraints is active at the optimum, (i.e., the QP is primal degenerate, and $\lambda$ is not unique), then a subset of linearly independent constraints and a vector $\lambda(\xi) \leq \lambda^+$ can be chosen which provides the same solution $U^*$ (cf. [15, Lemma 4.4.5] and [20, Theorem 2.6.12]).
An alternative proof follows by considering the KKT conditions (13) as a linear complementarity problem [20] $v = (GH^{-1}G')\lambda + [W + (GH^{-1}F' + S)\xi]$, $0 \leq v \perp \lambda \geq 0$, and directly applying [15, Lemma 7.6.14], showing that, for all $\xi \in \Xi$ such that the QP (10) is feasible, there exists a unique least-norm solution $\lambda(\xi)$ satisfying, for some scalar $\alpha \in \mathbb{R}$, $\|\lambda(\xi)\| \leq \alpha \|W + (GH^{-1}F' + S)\xi\| \leq \alpha(\|W\| + \|GH^{-1}F' + S)\| \cdot \max_{\xi \in \Xi} \|\xi\|)$. □

**Remark 4** A more efficient way of computing $\lambda^+$ than enumerating all possible combinations of linearly independent active constraints (as proposed in the first part of the proof of Lemma 1) consists of computing the solution to the mp-QP problem (10) by applying the algorithm of [10], which provides all and only the combinations of linearly independent active constraints which are optimal for some $\xi \in \Xi$ ($\Xi$ is partitioned into polyhedral cells, each one characterized by a different combination). □

Using arguments similar to those used to prove the relationship between LC and MLD models of Fact 1, we obtain the following result:

**Proposition 1** *Every cl-MPC system (7), (9)–(12) can be written as an MLD system, provided that bounds on the states, inputs, and references, are specified.*

**Proof:** Introduce a vector of binary variables $\delta(k) \in \{0, 1\}^q$. The idea is to represent $v_i(k) = 0$, $w_i(k) \geq 0$ with $\delta_i(k) = 1$, and $v_i(k) \geq 0$, $w_i(k) = 0$ with $\delta_i(k) = 0$. This can be achieved by introducing the constraints

$$\begin{array}{rclcrcl} w(k) & \leq & M_w \delta(k) & \quad & v(k) & \leq & M_v(e - \delta(k)) \\ w(k) & \geq & 0 & & v(k) & \geq & 0, \end{array}$$

where $M_w$ and $M_v$ are diagonal matrices containing upper bounds on $w(k)$ and $v(k)$ (provided by Lemma 1), respectively, and $e$ denotes the vector for which all entries are equal to one. By setting $z(k) = w(k)$ and replacing $v(k)$ as in (15c), it is easy to rewrite the MPC closed-loop system in the

MLD form

$$
\begin{aligned}
&\begin{bmatrix}\chi(k+1)\\ x(k+1)\\ x_u(k+1)\end{bmatrix} = \begin{bmatrix}\mathcal{A} & \mathcal{B}I_1T & \mathcal{B}(I_m+I_1V)\\ K_e\mathcal{C} & A-K_eC+BI_1T & B(I_m+I_1V)\\ 0 & I_1T & (I_m+I_1V)\end{bmatrix}\cdot\\
&\qquad\begin{bmatrix}\chi(k)\\ x(k)\\ x_u(k)\end{bmatrix} + \begin{bmatrix}\mathcal{B}I_1Z & \mathcal{H}\\ BI_1Z & K_e\mathcal{D}\\ I_1Z & 0\end{bmatrix}\begin{bmatrix}r(k)\\ d(k)\end{bmatrix} + \begin{bmatrix}\mathcal{B}I_1\Lambda\\ BI_1\Lambda\\ \Lambda\end{bmatrix} z(k);\\
&y(k) = \mathcal{C}\chi(k) + \mathcal{D}d(k);\\
&\begin{bmatrix}0 & 0 & 0\\ 0 & M-GT & N-GV\\ 0 & 0 & 0\\ 0 & -M+GT & -N+GV\end{bmatrix}\begin{bmatrix}\chi(k)\\ x(k)\\ x_u(k)\end{bmatrix} + \begin{bmatrix}0 & 0\\ L-GZ & 0\\ 0 & 0\\ -L+GZ & 0\end{bmatrix}\begin{bmatrix}r(k)\\ d(k)\end{bmatrix} +\\
&\qquad\begin{bmatrix}-M_w\\ M_v\\ 0\\ 0\end{bmatrix}\delta(k) + \begin{bmatrix}I\\ -G\Lambda\\ -I\\ G\Lambda\end{bmatrix} z(k) \le \begin{bmatrix}0\\ M_ve-W\\ 0\\ W\end{bmatrix}.
\end{aligned}
\tag{16}
$$

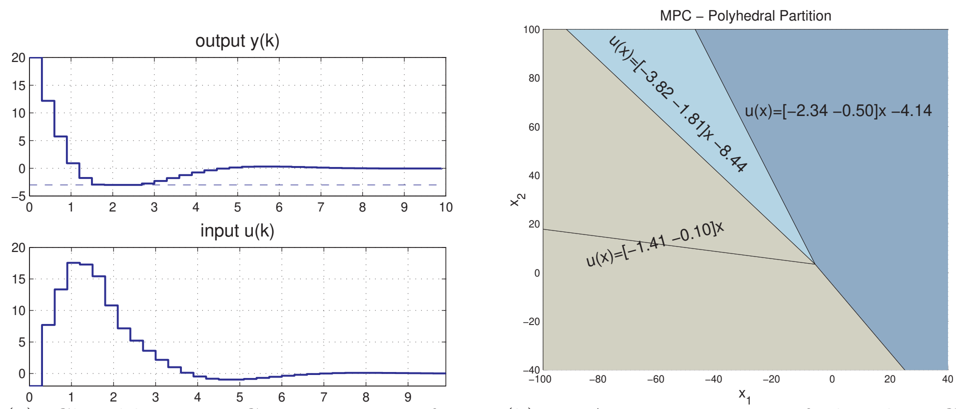
□

Note that the number $q$ of integer variables equals the number of constraints of the MPC optimization problem (10). Hence, if the MLD system were translated into PWA form as in [6], the resulting PWA system would have at most $2^q$ regions. This confirms the result of [10], where the explicit PWA form of the MPC controller (obtained by using multiparametric programming) is defined over a polyhedral partition of the state space composed by at most $2^q$ regions (note that $2^q$ equals the number of all possible combinations of active constraints). Since many of such combinations are infeasible, in general the resulting number of regions is much lower than $2^q$.

**Remark 5** For each weight matrix $R \succ 0$, the cl-MPC system (7), (9)–(12) is well-posed on the set of $x(k)$, $x_u(k)$, $r(k)$ where (10) is feasible. In fact, the Hessian matrix $H \succ 0$ in (10), and therefore $\Delta u_k^*$ is uniquely determined once $x(k)$, $r(k)$, $x_u(k)$ are assigned. Consequently, the equivalent LC form (15) and MLD model 16 is well-posed on the feasible set, despite the fact that $w(k)$ might not be uniquely defined by the KKT conditions (e.g., in case of primal degeneracy of the QP problem (10)), and the fact that for MLD systems the variable $\delta_i(k)$ is non-unique, when $v_i(k) = w_i(k) = 0$ in the proof of Proposition 1 (i.e., a constraint is active and the corresponding Lagrange multiplier in (13) is zero). □

Note that the result of Theorem 1 and Proposition 1 also holds when model (11) is replaced by any of the hybrid models described in the previous sections. Consequently, stability, feasibility/safety, constraint fulfillment and performance properties of cl-MPC where a simple linear model is used (a common choice for obtaining an easily implementable controller) in the *synthesis* of the controller, and a more accurate hybrid model approximating the plant dynamics is used for *analysis*, can be tested using tools developed for hybrid systems. The hybrid model can be for instance a PWA system obtained by linearizing a nonlinear process model at different operating points, an LC system obtained by a discretizing a mechanical model, or an MLD system obtained by using the description language HYSDEL [13]. These considerations prove immediately the following corollary.

**Corollary 1** *The cl-MPC system formed by an LC (MLD, PWA) system in feedback with a MPC controller of the form (7), (9), (12) is an LC (MLD, PWA) system.*




(a) Closed-loop MPC trajectories for plant model=prediction model (17), state-feedback

(b) PWA representation of the cl-MPC system: partition of the $x$-space and explicit MPC control law

Figure 2: Example 1

# 4 Application of the Results

Based on the above results, a long list of available hybrid tools developed for PWA, LC, MLD, MMPS and ELC systems can now be applied to analyze cl-MPC systems: controller synthesis based on MPC for MLD [5], MMPS [8] and ELC systems [21], PWA control / observer techniques for nonlinear plants [7], state estimation and fault detection [22], verification and safety analysis [23] for MLD systems, stability results for PWA systems [12, 24], controllability and observability for PWA and MLD systems [6], well-posedness of LC systems [2,3,15,16], simulation and discretization of continuous-time LC systems [14, 15], hybrid modeling languages [13]. The use of some of the above tools will be immediately shown in the following example.

**Example 1** Consider the second order open-loop unstable system

$$y = \frac{s+1}{s^2 - 0.3s + 1} u,$$

which is sampled with a sample time of $T = 0.3$ s to obtain the state-space representation

$$\left\{ \begin{array}{rcl} x(k+1) & = & \left[\begin{smallmatrix} 1.0467 & -0.30923 \\ 0.30923 & 0.95397 \end{smallmatrix}\right] x(k) + \left[\begin{smallmatrix} 0.095004 \\ 0.00060418 \end{smallmatrix}\right] u(k) \\ y(k) & = & \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right] x(k). \end{array} \right. \tag{17}$$

The task is to regulate the system to the origin while fulfilling the output constraint

$$y(k) \geq -3, \ \forall k \geq 0. \tag{18}$$

To this aim, we design an MPC controller based on the optimization problem

$$\begin{array}{ll} \displaystyle\min_{u_k, u_{k+1}} & x'_{k+3|k} P x_{k+3|k} + \displaystyle\sum_{t=0}^{2} \left[ y'_{k+t|k} y_{k+t|k} + u^2_{k+t} \right] \\ \text{subj. to} & y_{t+k} \geq -3, \ k = 1, 2 \\ & x_{k|k} = x(k), \end{array} \tag{19}$$

where $P$ solves the Riccati equation $P = (A + BK_{LQ})' P (A + BK_{LQ}) + K'_{LQ} R K_{LQ} + Q$, $K_{LQ} = -(R + B'PB)^{-1} B'PA$ is the LQ gain, $R = 1$, $Q = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Fig. 2(a) depicts the closed-loop trajectories obtained when (17) is in feedback with the MPC controller based on (19) (nominal cl-MPC system), for the initial condition $x(0) = [0 \ 20]'$.

We want to obtain the equivalent PWA, LC, and MLD form of the cl-MPC system that results from connecting the MPC controller based on (19) with the plant model

$$\Sigma : \begin{cases} \chi(k+1) &= \begin{bmatrix} 1.0567 & -0.32923 \\ 0.31923 & 0.93397 \end{bmatrix} \chi(k) \\ & \quad + \begin{bmatrix} 0.31923 \\ 0.036032 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(k) \\ y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \chi(k), \end{cases} \tag{20}$$

where $d(k) \in \mathbb{R}$ is an additive, norm-bounded, input disturbance. We assume that the state vector is measurable, so that no state observer is used ($\chi(t) = x(t)$). Note that this setting differs somewhat from the description given before in the sense that we now consider a regulation problem instead of a tracking problem (hence, there is no reference $r$ and an endpoint penalty $x'_{k+3|k} P x_{k+3|k}$ is added to the quadratic cost criterion) and no constraints on the increments $\Delta u(k)$. Moreover, the observer is a static one instead of (12). In this way we obtain a QP (10) depending on $\xi(k) = x(k)$ (and not on $u(k-1)$, $r(k) \equiv 0$). The controller is a function of $x(k)$ and the closed-loop system will turn out to have a state dimension of 2.

In order to obtain the PWA equivalent of the closed-loop MPC, we use the mp-QP Algorithm [10] to compute the explicit MPC control law, which provides the polyhedral partition (in the $x$-space) with $2^2 = 4$ regions as depicted in Fig. 2(b). It can a posteriori be verified that the optimal control value $u^*(k)$, being equal to the first computed control value out of the QP similar to (9), coincides in two regions, so that the PWA system has in fact 3 regions.

An LC equivalent of the cl-MPC system is obtained according to (15). In order to compute the MLD equivalent, we use Lemma 1 on $\Xi = \{x : \|x\|_\infty \leq 100\}$ to obtain $M_w = \text{diag}[92.0965,\ 15.3846]$, and then we compute $\max_{x \in \Xi} |(M - GT)x|_i + \max_{x \in \Xi} |G\Lambda\lambda(x)|_i \geq \max_{x \in \Xi} |v_i(x)|$, $i = 1, 2$, which provides $M_v = \text{diag}[62.14330, 51.1102]$.

In order to test if the autonomous ($d(k) \equiv 0$) closed-loop MPC system is asymptotically stable, we compute a common quadratic Lyapunov function according to the LMI-based algorithm of [24], obtaining $P_{\text{Lyap}}(x) = x' \begin{bmatrix} 20.7804 & 10.8968 \\ 10.8968 & 37.3396 \end{bmatrix} x$, which proves quadratic stability of the hybrid system.

Finally, when the plant (20) is in closed-loop with the MPC controller, for the set of initial states $x(0)$ such that $\|x(0) - [0\ 20]'\|_\infty \leq 2$, and when the disturbance $d(k)$ arbitrarily varies between -1 and 1, we want to compute the worst violation of the constraint (18). To this end, we use the MLD equivalent model and mixed-integer programming to run a verification algorithm based on the ideas of [23]. The worst violation is obtained at $t = 4$ from the initial state $x(0) = [-0.7289\ 22]'$, by applying the disturbance sequence shown in Fig. 3 (we arbitrarily set $d(k) = 0$ for $k > 4$). Fig. 3 depicts also the corresponding output trajectory. $\square$

# 5 Conclusions

In this paper we showed that closed-loop MPC systems can be treated and analyzed as hybrid systems, in particular as linear complementarity (LC) systems, mixed logical dynamical (MLD) systems, piecewise affine (PWA) systems, and indirectly, by exploiting the equivalences of [9], also as extended linear complementarity (ELC) systems and max-min-plus-scaling (MMPS) systems. The result is of paramount importance for applying the tools developed for such subclasses of hybrid systems to study any closed-loop combinations of a linear MPC controller, a linear observer, and a linear plant. This can be easily extended to any feedback interconnection of a linear MPC controller and a *hybrid* plant, such as hybrid approximations of complex nonlinear dynamic models of the process under control. Here we only demonstrated through a very simple example the application of the presented results for stability and safety analysis. We believe, however, that the development of

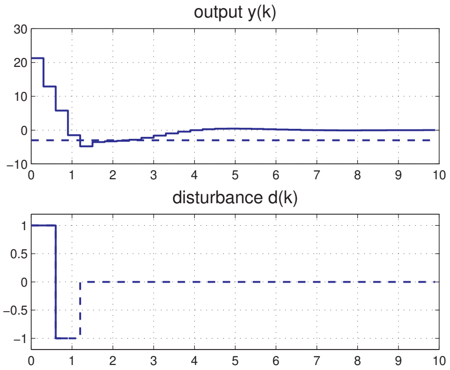



Figure 3: Worst-case closed-loop MPC trajectories (prediction model (17), plant model (20), $\|x(0) - [0\ 20]'\|_\infty \leq 2$, $|d(k)| \leq 1$)

a theory and tools for hybrid systems is still in its early stages, and therefore that the full potential of the results is still to be exploited.

# References


[1] P.J. Antsaklis, "A brief introduction to the theory and applications of hybrid systems", *Proc. IEEE, Special Issue on Hybrid Systems: Theory and Applications*, vol. 88, no. 7, pp. 879–886, July 2000.

[2] A.J. van der Schaft and J.M. Schumacher, "Complementarity modelling of hybrid systems", *IEEE Trans. Autom. Control*, vol. 43, pp. 483–490, 1998.

[3] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland, "Linear complementarity systems", *SIAM J. Appl. Math.*, vol. 60, no. 4, pp. 1234–1269, 2000.

[4] B. De Schutter and B. De Moor, "The extended linear complementarity problem and the modeling and analysis of hybrid systems", in *Hybrid Systems V*, P. Antsaklis, W. Kohn, M. Lemmon, A. Nerode, and S. Sastry, Eds. 1999, vol. 1567 of *Lecture Notes in Computer Science*, pp. 70–85, Springer.

[5] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints", *Automatica*, vol. 35, no. 3, pp. 407–427, Mar. 1999.

[6] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems", *IEEE Trans. Autom. Control*, vol. 45, no. 10, pp. 1864–1876, 2000.

[7] E.D. Sontag, "Nonlinear regulation: the piecewise linear approach", *IEEE Trans. Autom. Control*, vol. 26, no. 2, pp. 346–357, 1981.

[8] B. De Schutter and T. van den Boom, "On model predictive control for max-min-plus-scaling discrete event systems", *Automatica*, vol. 37, no. 7, pp. 1049–1056, 2001.

[9] W.P.M.H. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models", *Automatica*, vol. 37, no. 7, July 2001.

[10] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems", *Automatica*, vol. 38, no. 1, pp. 3–20, 2002.

[11] A. Bemporad and M. Morari, "Robust model predictive control: A survey", in *Robustness in Identification and Control*, A. Garulli, A. Tesi, and A. Vicino, Eds., number 245 in Lecture Notes in Control and Information Sciences, pp. 207–226. Springer-Verlag, 1999.

[12] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems", *IEEE Trans. Autom. Control*, vol. 43, no. 4, pp. 555–559, Apr. 1998.

[13] F.D. Torrisi, A. Bemporad, and D. Mignone, "HYSDEL - A language for describing hybrid systems", Tech. Rep. AUT00-03, ETH Zurich, 2000, `http://control.ethz.ch/~hybrid/hysdel`.

[14] M.K. Çamlıbel, W.P.M.H. Heemels, and J.M. Schumacher, "Consistency of a time-stepping method for a class of piecewise linear networks", To appear in *IEEE Trans. Circuits Syst. – I*.

[15] W.P.M.H. Heemels, *Linear complementarity systems: a study in hybrid dynamics*, PhD thesis, Dept. of Electrical Engineering, Eindhoven University of Technology, The Netherlands, 1999.

[16] A.J. van der Schaft and J.M. Schumacher, *An Introduction to Hybrid Dynamical Systems*, vol. 251 of *Lecture Notes in Control and Information Sciences*, Springer, London, 2000.

[17] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert, "Constrained model predictive control: Stability and optimality", *Automatica*, vol. 36, no. 6, pp. 789–814, June 2000.

[18] R. Fletcher, *Practical Methods of Optimization*, Wiley, New York, 2nd edition, 1987.

[19] B.C. Eaves and C.E. Lemke, "Equivalence of LCP and PLS", *Math. Oper. Res.*, vol. 6, no. 4, pp. 475–484, 1981.

[20] R.W. Cottle, J.-S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[21] B. De Schutter, "Optimal control of a class of linear hybrid systems with saturation", *SIAM J. Control Optim.*, vol. 39, no. 3, pp. 835–851, 2000.

[22] A. Bemporad, D. Mignone, and M. Morari, "Moving horizon estimation for hybrid systems and fault detection", in *Proceedings of the American Control Conference*, 1999, pp. 2471–2475.

[23] A. Bemporad, F.D. Torrisi, and M. Morari, "Optimization-based verification and stability characterization of piecewise affine and hybrid systems", in *Hybrid Systems: Computation and Control*, B. Krogh and N. Lynch, Eds. 2000, vol. 1790 of *Lecture Notes in Computer Science*, pp. 45–58, Springer Verlag.

[24] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An LMI approach", in *Proc. 39th IEEE Conf. on Decision and Control*, Dec. 2000, pp. 504–509.