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MPC for Continuous Piecewise-Affine Systems – Addendum

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Abstract

This addendum contains some extra information in connection with the worked examples of Section 6 of the paper "MPC for continuous piecewise-affine systems" (by B. De Schutter and T.J.J. van den Boom, *Systems & Control Letters*, vol. 52, no. 3–4, pp. 179–192, July 2004). In particular, we give the explicit form of the optimization problems for each of the four solution approaches used in Section 6.

All references in this addendum that are not preceded by a capital letter A refer to sections, equations, etc. of the paper [A2].

Example 6.1 (continued) For each of the solution approaches considered in Section 6 we get the following explicit form for the optimization problems:

1. **the new LP based approach of Section 5.2**:
   In this case we have to solve the six LPs that correspond to (21) subject to[1]

$$
\begin{aligned}
&-0.2 \leqslant u(k) - u(k-1) \leqslant 0.2 && \text{(A.1)}\\
&-0.2 \leqslant u(k+1) - u(k) \leqslant 0.2 && \text{(A.2)}\\
&u(k) \geqslant 0 && \text{(A.3)}\\
&u(k+1) \geqslant 0 \ ; && \text{(A.4)}
\end{aligned}
$$

2. **a nonlinear nonconvex SQP approach**:
   Here we have to solve

$$
\min_{u(k),u(k+1)} \max\big(|y(k)-r(k)|, |y(k+1)-r(k+1)|\big) + \lambda\left(u(k)+u(k+1)\right)
$$

subject to

$$
\begin{aligned}
y(k) &= \min\big(0.5\,x(k-1) + 4\,u(k) - 1, 0.2\,u(k) + 1\big)\\
y(k+1) &= \min\big(0.25\,x(k-1) + 2\,u(k) + 4\,u(k+1) - 1.5,\\
&\qquad 0.1\,u(k) + 4\,u(k+1) - 0.5, 0.2\,u(k+1) + 1\big)
\end{aligned}
$$

and (A.1)–(A.4);

[1] I.e., (18) for k and $k+1$.

3. **the mixed integer linear programming (MILP) approach of [4]**:
   This results in[2]

$$
\min_{\substack{u(k),u(k+1),\delta(k),\delta_{k+1},\\ z_1(k+1),z_2(k),z_2(k+1),t(k)}} \quad t(k)+\lambda u(k)+\lambda u(k+1)
$$

subject to

$$
\begin{aligned}
& t(k) \geqslant x(k)-r(k) \\
& t(k) \geqslant r(k)-x(k) \\
& t(k) \geqslant x(k+1)-r(k+1) \\
& t(k) \geqslant r(k+1)-x(k+1) \\
& x(k) = 0.5\,x(k-1)\delta(k)+3.8\,z_2(k)-2\,\delta(k)+0.2\,u(k)+1 \\
& x(k+1) = 0.5\,z_1(k+1)+3.8\,z_2(k+1)-2\,\delta(k+1)+0.2\,u(k+1)+1 \\
& \varepsilon-(M_f+\varepsilon)\delta(k) \leqslant 0.5\,x(k-1)+3.8\,u(k)-2 \leqslant M_f(1-\delta(k)) \\
& -M_u\delta(k) \leqslant z_2(k) \leqslant M_u\delta(k) \\
& u(k)-M_u(1-\delta(k)) \leqslant z_2(k) \leqslant u(k)+M_u(1-\delta(k)) \\
& \varepsilon-(M_f+\varepsilon)\delta(k+1) \leqslant 0.5\,x(k)+3.8\,u(k+1)-2 \leqslant M_f(1-\delta(k+1)) \\
& -M_x\delta(k+1) \leqslant z_1(k+1) \leqslant M_x\delta(k+1) \\
& x(k)-M_x(1-\delta(k+1)) \leqslant z_1(k+1) \leqslant x(k)+M_x(1-\delta(k+1)) \\
& -M_u\delta(k+1) \leqslant z_2(k+1) \leqslant M_u\delta(k+1) \\
& u(k+1)-M_u(1-\delta(k+1)) \leqslant z_2(k+1) \leqslant u(k+1)+M_u(1-\delta(k+1)) \\
& \delta(k),\delta(k+1) \in \{0,1\} \\
& \text{and (A.1)–(A.4),}
\end{aligned}
$$

   with ε a small positive number, and with M_x an upper bound[3] for $|x(k)|$ for all k, and M_u an upper bound for $|u(k)|$ for all k, and $M_f = 0.5\,M_x+3.8\,M_u+2$;

4. **the ELCP approach (cf. [12])**:
   Here we have to solve the following optimization problem[4]:

$$
\min_{\nu} \max\big(|y(k,\nu)-r(k)|,|y(k+1,\nu)-r(k+1)|\big)+\lambda\,\big(u(k,\nu)+u(k+1,\nu)\big)
$$

   where ν contains the parameters of the parameterized solution set of the ELCP given below (this solution set can be computed with the ELCP algorithm of [A1]), and where $y(k,\nu)$, $y(k+1,\nu)$,

[2]See [4] for the way to transform a PWA model into a mixed logical dynamical model (MLD) (i.e., a system with both boolean and real state variables, with linear state and output equations, and with additional linear inequality constraints on the state variables). The piecewise linear objective function $J(k) = \max\big(|y(k)-r(k)|,|y(k+1)-r(k+1)|\big)+\lambda\,\big(u(k)+u(k+1)\big)$ has been transformed into a linear objective function by introducing an extra variable $t(k) = \max\big(|y(k)-r(k)|,|y(k)-r(k+1)|\big)$. The 5 other extra variables ($\delta(k)$, $\delta(k+1)$, $z_1(k+1) = x(k)\delta(k+1)$, $z_2(k) = u(k)\delta(k)$, $z_2(k+1) = u(k+1)\delta(k+1)$) originate from the transformation from PWA into MLD equations. Note that since the value of $x(k-1)$ is known at sample step k the term $0.5x(k-1)\delta(k)$ is in fact a linear term. As a consequence, the equation for $x(k)$ is linear in $x(k)$, $\delta(k)$, $z_2(k)$ and $u(k)$.

[3]The upper bounds M_x and M_u could be determined based on physical insight or on operational constraints.

[4]This problem can be solved using an SQP approach.

$u(k,\nu)$, $u(k+1,\nu)$ are respectively the $y(k)$, $y(k+1)$, $u(k)$, $u(k+1)$ that correspond to the parameter vector ν. The ELCP is given by[5]

$$
\begin{aligned}
&0.5\,x(k-1)+4\,u(k)-1-y(k)\geqslant 0\\
&0.2\,u(k)+1-y(k)\geqslant 0\\
&\big(0.5\,x(k-1)+4\,u(k)-1-y(k)\big)\cdot\big(0.2\,u(k)+1-y(k)\big)=0\\
&0.25\,x(k-1)+2\,u(k)+4\,u(k+1)-1.5-y(k+1)\geqslant 0\\
&0.1\,u(k)+4\,u(k+1)-0.5-y(k+1)\geqslant 0\\
&0.2\,u(k+1)+1-y(k+1)\geqslant 0\\
&\big(0.25\,x(k-1)+2\,u(k)+4\,u(k+1)-1.5-y(k+1)\big)\cdot\\
&\qquad\big(0.1\,u(k)+4\,u(k+1)-0.5-y(k+1)\big)\cdot\\
&\qquad\qquad\big(0.2\,u(k+1)+1-y(k+1)\big)=0\\
&\text{and (A.1)–(A.4).}
\end{aligned}
$$

Remark A.1 For the system (16)–(17) we can also allow output constraints of the form[6]

$$y(k)\geqslant r(k)\qquad\text{for all } k\ . \tag{A.5}$$

Indeed, for k and $k+1$ this constraint leads to

$$
\begin{aligned}
y(k)&=\min\big(0.5\,x(k-1)+4\,u(k)-1,\,0.2\,u(k)+1\big)\geqslant r(k)\\
y(k+1)&=\min\big(0.25\,x(k-1)+2\,u(k)+4\,u(k+1)-1.5,\\
&\qquad\qquad 0.1\,u(k)+4\,u(k+1)-0.5,\,0.2\,u(k+1)+1\big)\geqslant r(k+1)
\end{aligned}
$$

or equivalently

$$0.5\,x(k-1)+4\,u(k)-1\geqslant r(k) \tag{A.6}$$
$$0.2\,u(k)+1\geqslant r(k) \tag{A.7}$$
$$0.25\,x(k-1)+2\,u(k)+4\,u(k+1)-1.5\geqslant r(k+1) \tag{A.8}$$
$$0.1\,u(k)+4\,u(k+1)-0.5\geqslant r(k+1) \tag{A.9}$$
$$0.2\,u(k+1)+1\geqslant r(k+1)\ . \tag{A.10}$$

Since these constraints are affine in $\tilde{u}(k)=\begin{bmatrix}u(k) & u(k+1)\end{bmatrix}^T$, the new optimization approach of Section 5.2 can still be applied[7]. This also holds for constraints of the form

$$x(k)\geqslant x_{\mathrm{low}}(k)\quad\text{and}\quad y(k)\geqslant y_{\mathrm{low}}(k)\qquad\text{for all } k$$

[5] See [12] for more information on how this ELCP should be constructed.

[6] Since the output saturates at $0.2u(k)+1$, we will have to adapt the reference signal r if the constraint (A.5) is added, since otherwise the MPC problem will be infeasible for some values of k (cf. conditions (A.7) and (A.10)).

[7] If the constraint (A.5) is added, the terms $r(k)-y(k)$ and $r(k+1)-y(k+1)$ in expression (19) for the objective function become redundant. As a consequence, the terms $m_1,\ldots,m_5$ will disappear from (21), but the constraints (A.6)–(A.10) will be added. Hence, we still have 6 LPs with 13 inequalities and 2 variables.

for lower bound signals x_{low} and y_{low}, or for any *nonnegative* linear combination of these constraints[8]. This shows that our approach can also deal with constraints on the output or the state provided that after substitution they result in constraints that are convex in the input $\tilde{u}(k)$.

Example 6.2 (continued) Recall that we have selected the following MPC objective function $J(k)$:

$$J(k) = J_{\mathrm{out},\infty}(k) + \lambda J_{\mathrm{in},1}(k) \quad .$$

Using the system equations (24)–(25) and the constraints (26)–(28) we obtain

$$\begin{aligned}
J(k) &= \max\big(\,|y(k)-r(k)|,\, |y(k+1)-r(k+1)|\,\big) + \lambda\,\big(\,|u(k)| + |u(k+1)|\,\big) \\
&= \max\big(y(k)-r(k),\, y(k+1)-r(k+1)\big) + \qquad\qquad \text{(by (28))} \\
&\qquad \lambda\,\big(\max(u(k),-u(k)) + \max(u(k+1),-u(k+1))\big) \\
&= \max\big(y(k)-r(k),\, y(k+1)-r(k+1)\big) + \\
&\qquad + \lambda \max\big(u(k)+u(k+1),\, u(k)-u(k+1),\, -u(k)+u(k+1),\, -u(k)-u(k+1)\big) \\
&= \max\big(y(k)-r(k),\, y(k+1)-r(k+1)\big) + \\
&\qquad + \lambda \max\big(u(k)-u(k+1),\, -u(k)+u(k+1),\, -u(k)-u(k+1)\big) \qquad \text{(by (27))} \\
&= \max\big(y(k)-r(k)+\lambda u(k)-\lambda u(k+1),\, y(k)-r(k)-\lambda u(k)+\lambda u(k+1), \\
&\qquad y(k)-r(k)-\lambda u(k)-\lambda u(k+1),\, y(k+1)-r(k+1)+\lambda u(k)-\lambda u(k+1), \\
&\qquad y(k+1)-r(k+1)-\lambda u(k)+\lambda u(k+1), \\
&\qquad y(k+1)-r(k+1)-\lambda u(k)-\lambda u(k+1)\big) \quad .
\end{aligned}$$

By using successive substitution and by applying the properties given in Section 3 of the main paper, $y(k)$ and $y(k+1)$ can be expressed as functions of the current state $x(k-1)$ and the future inputs $u(k)$ and $u(k+1)$:

$$\begin{aligned}
y(k) = x(k) &= \min\big(x(k-1)+u(k),\, 1\big) \\
y(k+1) = x(k+1) &= \min\big(x(k)+u(k+1),\, 1\big) \\
&= \min\big(\min\big(x(k-1)+u(k),\, 1\big)+u(k+1),\, 1\big) \\
&= \min\big(\min\big(x(k-1)+u(k)+u(k+1),\, 1+u(k+1)\big),\, 1\big) \\
&= \min\big(x(k-1)+u(k)+u(k+1),\, u(k+1)+1,\, 1\big) \quad .
\end{aligned}$$

Hence,

$$\begin{aligned}
J(k) = \max\big(&\min\big(x(k-1)+u(k),\, 1\big)-r(k)+\lambda u(k)-\lambda u(k+1), \\
&\min\big(x(k-1)+u(k),\, 1\big)-r(k)-\lambda u(k)+\lambda u(k+1), \\
&\min\big(x(k-1)+u(k),\, 1\big)-r(k)-\lambda u(k)-\lambda u(k+1), \\
&\min\big(x(k-1)+u(k)+u(k+1),\, u(k+1)+1,\, 1\big)-r(k+1)+\lambda u(k)-\lambda u(k+1), \\
&\min\big(x(k-1)+u(k)+u(k+1),\, u(k+1)+1,\, 1\big)-r(k+1)-\lambda u(k)+\lambda u(k+1), \\
&\min\big(x(k-1)+u(k)+u(k+1),\, u(k+1)+1,\, 1\big)-r(k+1)-\lambda u(k)-\lambda u(k+1)\big)
\end{aligned}$$

[8]However, constraints of the form $x(k) \leqslant x_{\mathrm{upp}}(k)$ or $y(k) \leqslant y_{\mathrm{upp}}(k)$ for upper bound signals x_{upp} and y_{upp} lead to constraints that are not convex in $\tilde{u}(k)$. Hence, if such constraints are present, the new optimization approach of Section 5.2 cannot be applied.

$$
\begin{aligned}
&= \max\Big(\min\big(x(k-1)+u(k)-r(k)+\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad 1-r(k)+\lambda u(k)-\lambda u(k+1)\big),\\
&\quad \min\big(x(k-1)+u(k)-r(k)-\lambda u(k)+\lambda u(k+1),\\
&\qquad\qquad 1-r(k)-\lambda u(k)+\lambda u(k+1)\big),\\
&\quad \min\big(x(k-1)+u(k)-r(k)-\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad 1-r(k)-\lambda u(k)-\lambda u(k+1)\big),\\
&\quad \min\big(x(k-1)+u(k)+u(k+1)-r(k+1)+\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad u(k+1)+1-r(k+1)+\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad 1-r(k+1)+\lambda u(k)-\lambda u(k+1)\big),\\
&\quad \min\big(x(k-1)+u(k)+u(k+1)-r(k+1)-\lambda u(k)+\lambda u(k+1),\\
&\qquad\qquad u(k+1)+1-r(k+1)-\lambda u(k)+\lambda u(k+1),\\
&\qquad\qquad 1-r(k+1)-\lambda u(k)+\lambda u(k+1)\big),\\
&\quad \min\big(x(k-1)+u(k)+u(k+1)-r(k+1)-\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad u(k+1)+1-r(k+1)-\lambda u(k)-\lambda u(k+1),\\
&\qquad\qquad 1-r(k+1)-\lambda u(k)-\lambda u(k+1)\big)\Big)\\
&= \max\Big(\min\big(x(k-1)+(\lambda+1)u(k)-\lambda u(k+1)-r(k),\\
&\qquad\qquad \lambda u(k)-\lambda u(k+1)-r(k)+1\big),\\
&\quad \min\big(x(k-1)+(-\lambda+1)u(k)+\lambda u(k+1)-r(k),\\
&\qquad\qquad -\lambda u(k)+\lambda u(k+1)-r(k)+1\big),\\
&\quad \min\big(x(k-1)+(-\lambda+1)u(k)-\lambda u(k+1)-r(k),\\
&\qquad\qquad -\lambda u(k)-\lambda u(k+1)-r(k)+1\big),\\
&\quad \min\big(x(k-1)+(\lambda+1)u(k)+(-\lambda+1)u(k+1)-r(k+1),\\
&\qquad\qquad \lambda u(k)+(-\lambda+1)u(k+1)-r(k+1)+1,\\
&\qquad\qquad \lambda u(k)-\lambda u(k+1)-r(k+1)+1\big),\\
&\quad \min\big(x(k-1)+(-\lambda+1)u(k)+(\lambda+1)u(k+1)-r(k+1),\\
&\qquad\qquad -\lambda u(k)+(\lambda+1)u(k+1)-r(k+1)+1,\\
&\qquad\qquad -\lambda u(k)+\lambda u(k+1)-r(k+1)+1\big),\\
&\quad \min\big(x(k-1)+(-\lambda+1)u(k)+(-\lambda+1)u(k+1)-r(k+1),\\
&\qquad\qquad -\lambda u(k)+(-\lambda+1)u(k+1)-r(k+1)+1,\\
&\qquad\qquad -\lambda u(k)-\lambda u(k+1)-r(k+1)+1\big)\Big)\ . \qquad \text{(A.11)}
\end{aligned}
$$

Note that this is an MMPS expression in max-min canonical form, which can be written compactly as (29).

Recall that we have considered the computation of the closed-loop MPC input signal over a simulation period $[1,15]$ with $\lambda = 0.1$, $x(0) = 1$, $u(0) = -0.1$, and for the reference signal

$$
\begin{aligned}
\{r(k)\}_{k=1}^{15} = {} & 1,\ 1,\ 0.7,\ 0.5,\ -0.45,\ -0.9,\ -1.2,\ -1.5,\ -1.4,\ -2.4,\\
& -2.5,\ -2.6,\ -2.6,\ -2.75,\ -2.75\ .
\end{aligned}
$$

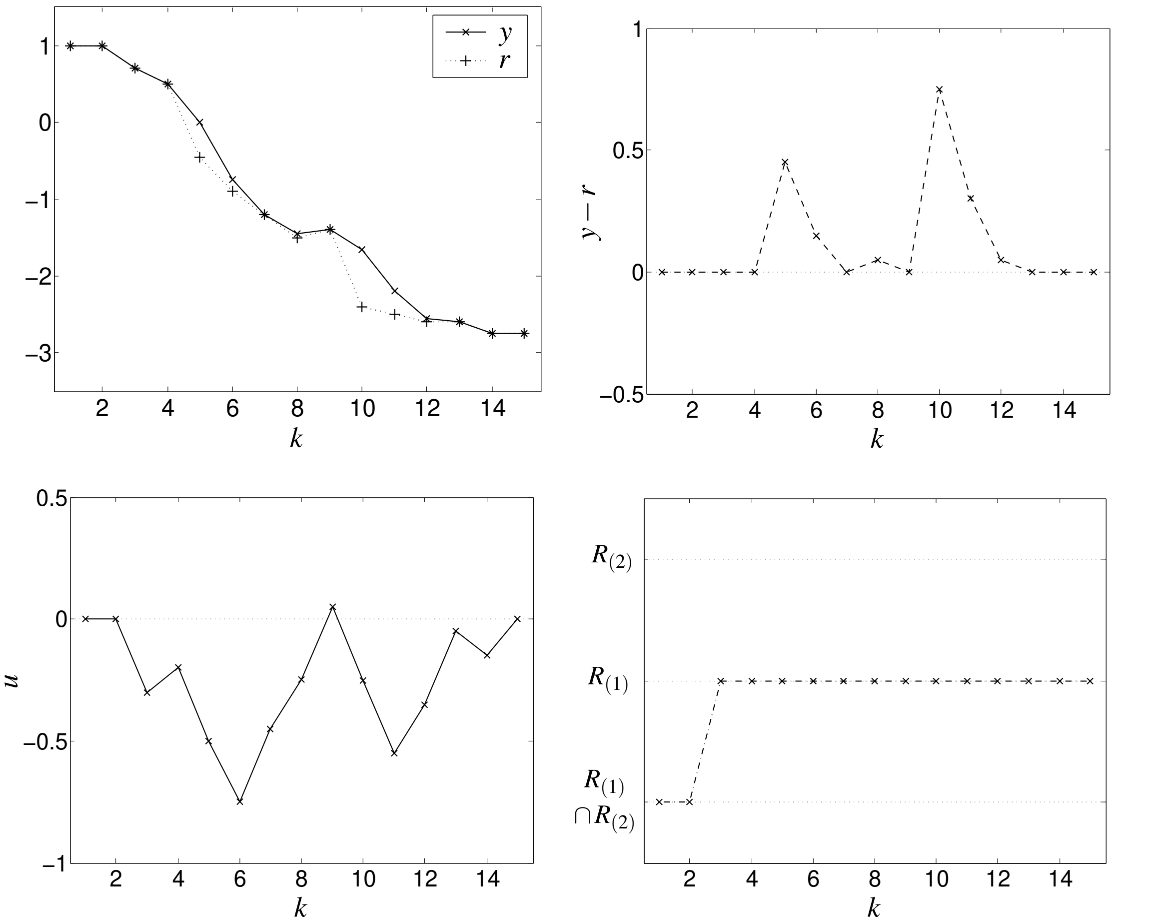

Figure A.1: The closed-loop MPC output signal y, the reference signal r, the difference signal $y-r$, the input signal u, and the region $R_{(i)}$ in which the system is at sample step k for Example 6.2.

This results in the following closed-loop MPC input sequence:

$$\begin{aligned}\{u_{\mathrm{mpc}}(k)\}_{k=1}^{15} = {} & 0,\, 0,\, -0.3,\, -0.2,\, -0.5,\, -0.75,\, -0.45,\, -0.25,\, 0.05,\, -0.25,\\ & -0.55,\, -0.35,\, -0.05,\, -0.15,\, 0\ \ .\end{aligned}$$

In Figure A.1 we have plotted the closed-loop MPC input signal u, the output signal y, the reference signal r, and the difference signal $y-r$.

Remark A.2 Note that the constraint (28) leads to

$$\begin{aligned} y(k) &= \min\big(x(k-1)+u(k),\, 1\big) \geqslant r(k)\\ y(k+1) &= \min\big(x(k-1)+u(k)+u(k+1),\, u(k+1)+1,\, 1\big) \geqslant r(k+1)\end{aligned}$$

or equivalently

$$x(k-1)+u(k) \geqslant r(k) \tag{A.12}$$

$$1 \geqslant r(k) \tag{A.13}$$

$$x(k-1)+u(k)+u(k+1) \geqslant r(k+1) \tag{A.14}$$

$$u(k+1)+1 \geqslant r(k+1) \tag{A.15}$$

$$1 \geqslant r(k+1) \quad . \tag{A.16}$$

We have solved the MPC optimization problems using 4 different approaches:

1. **the new LP based approach of Section 5.2**:
   In this case we have to solve the four LPs that correspond to (31) subject to[9]

$$-0.3 \leqslant u(k)-u(k-1) \leqslant 0.3 \tag{A.17}$$

$$-0.3 \leqslant u(k+1)-u(k) \leqslant 0.3 \tag{A.18}$$

$$u(k)+u(k-1) \leqslant 0 \tag{A.19}$$

$$u(k+1)+u(k) \leqslant 0 \tag{A.20}$$

$$x(k-1)+u(k) \geqslant r(k) \tag{A.21}$$

$$x(k-1)+u(k)+u(k+1) \geqslant r(k+1) \tag{A.22}$$

$$u(k+1)+1 \geqslant r(k+1) \quad ; \tag{A.23}$$

2. **a nonlinear nonconvex SQP approach**:
   Here we have to solve

$$\begin{aligned}
&\min_{u(k),u(k+1)} \max\big(y(k)-r(k), y(k+1)-r(k+1)\big)+\lambda\left(|u(k)|+|u(k+1)|\right)\\
&\text{subject to}\\
&\quad y(k)=\min\big(x(k-1)+u(k),\,1\big)\\
&\quad y(k+1)=\min\big(x(k-1)+u(k)+u(k+1),\,u(k+1)+1,\,1\big)\\
&\quad y(k) \geqslant r(k)\\
&\quad y(k+1) \geqslant r(k+1)\\
&\quad \text{and (A.17)–(A.20);}
\end{aligned}$$

3. **the mixed integer linear programming approach of [4]**:
   This results in[10]

$$\begin{aligned}
&\min_{\substack{u(k),u(k+1),\delta(k),\delta_{k+1},\\ z_1(k+1),z_2(k),z_2(k+1),t_1(k),t_2(k),t_3(k)}} t_1(k)+\lambda t_2(k)+\lambda t_3(k)\\
&\text{subject to}\\
&\quad t_1(k) \geqslant x(k)-r(k)\\
&\quad t_1(k) \geqslant x(k+1)-r(k+1)\\
&\quad t_2(k) \geqslant u(k)
\end{aligned}$$

---

[9] See (26), (27), and (A.12), (A.14)–(A.15).

[10] See [4] for the way to transform a PWA model into an MLD model. The piecewise linear objective function $J(k) = \max\big(y(k)-r(k), y(k+1)-r(k+1)\big)+\lambda\left(|u(k)|+|u(k+1)|\right)$ has been transformed into a linear objective function by introducing 3 extra variables $t_1(k) = \max\big(y(k)-r(k), y(k)-r(k+1)\big)$, $t_2(k) = |u(k)| = \max\big(u(k), -u(k)\big)$, and $t_3(k) = |u(k+1)| = \max\big(u(k+1), -u(k+1)\big)$; the six other extra variables ($\delta(k)$, $\delta(k+1)$, $z_1(k)$, $z_1(k+1)$, $z_2(k)$, $z_2(k+1)$) originate from the transformation from PWA into MLD equations.

$$
\begin{aligned}
& t_2(k) \geqslant -u(k) \\
& t_3(k) \geqslant u(k+1) \\
& t_3(k) \geqslant -u(k+1) \\
& x(k) = x(k-1)\delta(k) + z_2(k) - \delta(k) + 1 \\
& x(k+1) = z_1(k+1) + z_2(k+1) - \delta(k+1) + 1 \\
& \varepsilon - (M_x + M_u + 1 + \varepsilon)\delta(k) \leqslant x(k-1) + u(k) - 1 \leqslant (M_x + M_u + 1)(1 - \delta(k)) \\
& -M_u\delta(k) \leqslant z_2(k) \leqslant M_u\delta(k) \\
& u(k) - M_u(1-\delta(k)) \leqslant z_2(k) \leqslant u(k) + M_u(1-\delta(k)) \\
& \varepsilon - (M_x + M_u + 1 + \varepsilon)\delta(k+1) \leqslant x(k) + u(k+1) - 1 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leqslant (M_x + M_u + 1)(1 - \delta(k+1)) \\
& -M_x\delta(k+1) \leqslant z_1(k+1) \leqslant M_x\delta(k+1) \\
& x(k) - M_x(1-\delta(k+1)) \leqslant z_1(k+1) \leqslant x(k) + M_x(1-\delta(k+1)) \\
& -M_u\delta(k+1) \leqslant z_2(k+1) \leqslant M_u\delta(k+1) \\
& u(k+1) - M_u(1-\delta(k+1)) \leqslant z_2(k+1) \leqslant u(k+1) + M_u(1-\delta(k+1)) \\
& x(k) \geqslant r(k) \\
& x(k+1) \geqslant r(k+1) \\
& \delta(k), \delta(k+1) \in \{0,1\} \\
& \text{and (A.17)–(A.20),}
\end{aligned}
$$

with ε a small positive number, and with M_x an upper bound for $|x(k)|$ for all k, and M_u an upper bound for $|u(k)|$ for all k;

4. **the ELCP approach (cf. [12])**:
Here we have the following optimization problem:

$$
\min_{\nu} \max\big(y(k,\nu) - r(k), y(k+1,\nu) - r(k+1)\big) + \lambda\left(|u(k,\nu)| + |u(k+1,\nu)|\right)
$$

where ν contains the parameters of the parameterized solution set of the ELCP given below as it can be computed with the ELCP algorithm of [A1] and $y(k,\nu)$, $y(k+1,\nu)$, $u(k,\nu)$, $u(k+1,\nu)$ respectively the $y(k)$, $y(k+1)$, $u(k)$, $u(k+1)$ that correspond to the parameter vector ν. The ELCP is given by

$$
\begin{aligned}
& x(k-1) + u(k) - y(k) \geqslant 0 \\
& 1 - y(k) \geqslant 0 \\
& \big(x(k-1) + u(k) - y(k)\big) \cdot \big(1 - y(k)\big) = 0 \\
& x(k-1) + u(k) + u(k+1) - y(k+1) \geqslant 0 \\
& u(k+1) + 1 - y(k+1) \geqslant 0 \\
& 1 - y(k+1) \geqslant 0 \\
& \big(x(k-1) + u(k) + u(k+1) - y(k+1)\big) \cdot \big(u(k+1) + 1 - y(k+1)\big) \cdot \\
& \qquad \big(1 - y(k+1)\big) = 0
\end{aligned}
$$

$y(k) \geqslant r(k)$
$y(k+1) \geqslant r(k+1)$
and (A.17)–(A.20).

Additional references

[A1] B. De Schutter and B. De Moor, "The extended linear complementarity problem," *Mathematical Programming*, vol. 71, no. 3, pp. 289–325, Dec. 1995.

[A2] B. De Schutter and T.J.J. van den Boom, "MPC for continuous piecewise-affine systems," *Systems & Control Letters*, vol. 52, no. 3–4, July 2004.