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Model Predictive Control for Perturbed Max-Plus-Linear Systems: A Stochastic Approach

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Abstract

Model predictive control (MPC) is a popular controller design technique in the process industry. Conventional MPC uses linear or nonlinear discrete-time models. Recently, we have extended MPC to a class of discrete event systems that can be described by a model that is "linear" in the (max,+) algebra. In our previous work we have only considered MPC for the perturbations-free case and for the case with bounded noise and/or modeling errors. In this paper we extend these results on MPC for max-plus-linear systems to a stochastic setting. We show that under quite general conditions the resulting optimization problems turns out to be convex and can thus be solved very efficiently.

Keywords: discrete event systems, model predictive control, max-plus-linear systems, noise and modeling errors, stochastic setting.

1 Introduction

Discrete event systems (DES) are dynamic, asynchronous systems the state of which changes due to the occurrence of events; this in contrast to continuous variable systems, whose behavior is governed by the progression of time or the ticks of a clock and which can be modeled by a system of differential or difference equations. Typical examples of DES are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, and logistic systems. An event corresponds to the start or the end of an activity. In the case of a production system possible events are: the completion of a part on a machine, a machine breakdown, or a buffer becoming empty. There exist many modeling frameworks for DES such as queueing theory, (extended) state machines, formal languages, automata, temporal logic models, generalized semi-Markov processes, Petri nets, and computer simulation models (see [6, 13, 20, 28] and the references therein). In general, models that describe the behavior of a DES are nonlinear in conventional algebra. However, there is a class of DES that can be described by a model that is "linear" in the max-plus algebra [1, 8]. Such DES are called max-plus-linear (MPL) DES. Essentially, they can be characterized as DES in which only synchronization and no concurrency or choice occurs. So typical examples are serial production lines, production systems with a fixed routing schedule, and railway networks.

Model predictive control (MPC) [2, 4, 7, 11] is a proven technology for the control of multivariable systems in the presence of input, output and state constraints and is capable of tracking pre-scheduled reference signals. These attractive features make MPC widely accepted in the process industry. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to MPL systems [9, 10, 22]. In [24] we have presented some results on MPL-MPC in the presence of bounded noise and/or bounded modeling errors. In this paper we will extend these results to cases with noise and/or modeling errors in a stochastic setting, where the noise and/or modeling errors are not bounded a priori.

In contrast to conventional linear systems, where noise and disturbances are usually modeled by including an extra term in the system equations (i.e., the noise is considered to be additive), the influence of noise and disturbances in MPL systems is not max-plus-additive, but max-plus-multiplicative (see [1] or the worked example of Section 6). This means that the system matrices will be perturbed and as a consequence the system properties will change. Ignoring the noise can lead to a bad tracking behavior or even to an unstable closed loop. A second important feature is modeling errors. Uncertainty in the modeling or identification phase leads to errors in the system matrices. It is clear that both modeling errors, and noise/disturbances perturb the system by introducing uncertainty in the system matrices. Sometimes it is difficult to distinguish the two from one another, but usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. Results for handling uncertainty of some specific classes of DES are given in [5, 15, 19, 27] and the references therein. Note however, that there are few results in the literature on noise and modeling errors in an MPL context. Similar to the results in [24], we will show that both model mismatch and disturbances can be treated in one single framework; the characterization of the perturbation will then determine whether it describes model mismatch or disturbance. We will also show that under quite general conditions the resulting MPC optimization problem can be solved very efficiently.

This paper is organized as follows. In Section 2 we first give a concise introduction to MPL systems and MPC for MPL systems (without noise or modeling errors). Next, we present a noise and uncertainty model for MPL systems in a stochastic framework. In Section 3 the MPC method for stochastic MPL systems is presented and we derive algorithms to make predictions in this setting. In Section 4 we proof convexity of the MPL-MPC method. Section 5 discusses the computational aspects of the algorithm and in Section 6 we give a worked example.

2 Max-plus algebra and stochastic max-plus-linear systems

2.1 Max-plus algebra

In this section we give the basic definition of the max-plus algebra and we present some results on a class of (max,+) functions.

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows [1, 8]:

$$
\begin{aligned}
x \oplus y &= \max(x, y) \\
x \otimes y &= x + y
\end{aligned}
$$

for numbers $x, y \in \mathbb{R}_\varepsilon$ and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n}(a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$. The matrix $\mathcal{E}$ is the max-plus-algebraic zero matrix: $[\mathcal{E}]_{ij} = \varepsilon$ for all i, j.

Let $\mathcal{S}_{\text{mpns}}$ be the set of max-plus-nonnegative-scaling functions, i.e., functions f of the form

$$f(z) = \max_i(\alpha_{i,1}z_1 + \ldots + \alpha_{i,n}z_n + \beta_i)$$

with variable $z \in \mathbb{R}_\varepsilon^n$ and constants $\alpha_{i,j} \in \mathbb{R}^+$ and $\beta_i \in \mathbb{R}$, where $\mathbb{R}^+$ is the set of the nonnegative real numbers. If we want to stress that f is a function of z we will denote this by $f \in \mathcal{S}_{\text{mpns}}(z)$.

Lemma 1 *The set $\mathcal{S}_{\text{mpns}}$ is closed under the operations $\oplus$, $\otimes$, and scalar multiplication by a nonnegative scalar.*

Proof: This is a consequence of the fact that for $x, y, z, v \in \mathbb{R}_\varepsilon$ and $\rho \in \mathbb{R}^+$ we have $\max(x, y) \oplus \max(z, v) = \max(\max(x, y), \max(z, v)) = \max(x, y, z, v)$, $\max(x, y) \otimes \max(z, v) = \max(x, y) + \max(z, v) = \max(x + z, x + v, y + z, y + v)$ and $\rho \max(x, y) = \max(\rho x, \rho y)$. $\diamond$

2.2 Max-plus-linear systems

In [1, 8] it has been shown that (time-invariant) discrete event systems (DES) in which there is synchronization but no concurrency can be described by a model of the form

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \tag{1}$$
$$y(k) = C \otimes x(k)\ . \tag{2}$$

DES that can be described by this model will be called time-invariant max-plus-linear (MPL). The index k is called the event counter. The state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instants at which the input events occur for the kth time, and the output $y(k)$ contains the time instants at which the output events occur for the kth time[1].

2.3 Stochastic max-plus-linear systems

In this paper we proceed on the results of [24], in which we have extended the deterministic model (1)–(2) to include uncertainty. Consider the following MPL system:

$$x(k) = A(k) \otimes x(k-1) \oplus B(k) \otimes u(k) \tag{3}$$
$$y(k) = C(k) \otimes x(k) \tag{4}$$

[1]More specifically, for a manufacturing system, $x(k)$ contains the time instants at which the processing units start working for the kth time, $u(k)$ the time instants at which the kth batch of raw material is fed to the system, and $y(k)$ the time instants at which the kth batch of finished product leaves the system.

where $A(k)$, $B(k)$ and $C(k)$ represent uncertain system matrices due to modeling errors or disturbances. Usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. In this paper both features will be treated in one single framework. The uncertainty caused by disturbances and errors in the estimation of physical variables, is gathered in the uncertainty vector $e(k)$. In this paper we assume that the uncertainty has stochastic properties. Hence, $e(k)$ is a stochastic variable.

We assume that the uncertainty vector $e(k)$ captures the complete time-varying aspect of the system. Furthermore, the system matrices of an MPL model usually consist of sums or maximizations of internal process times, transportation times, etc. (see, e.g., [1] or Section 6). Since the entries of $e(k)$ directly correspond to the uncertainties in these duration times, it follows from Lemma 1 that the entries of the uncertain system matrices belong to $\mathcal{S}_{\mathrm{mpns}}$:

$$A(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times n}(e(k)), \quad B(k) \in \mathcal{S}_{\mathrm{mpns}}^{n\times m}(e(k)), \quad C(k) \in \mathcal{S}_{\mathrm{mpns}}^{l\times n}(e(k)). \tag{5}$$

3 Model predictive control for stochastic MPL systems

In [9, 10] we have extended the MPC framework to time-invariant MPL models (1)–(2) as follows. Just as in conventional MPC [7, 11] we define a cost criterion J that reflects the input and output cost functions (J_{in} and J_{out}, respectively) in the event period $[k, k+N_{\mathrm{p}}-1]$:

$$J(k) = J_{\mathrm{out}}(k) + \lambda J_{\mathrm{in}}(k) \tag{6}$$

where N_{p} is the prediction horizon and λ is a weighting parameter. Possible choices for J_{out} and J_{in} are given in [9, 10]. The aim is now to compute an optimal input sequence $u(k), \ldots, u(k+N_{\mathrm{p}}-1)$ that minimizes $J(k)$ subject to linear constraints on the inputs and outputs. Since the $u(k)$'s correspond to consecutive event occurrence times, we have the additional condition $\Delta u(k+j) = u(k+j) - u(k+j-1) \geq 0$ for $j = 0, \ldots, N_{\mathrm{p}}-1$. Furthermore, in order to reduce the number of decision variables and the corresponding computational complexity we introduce a control horizon N_{c} ($\leq N_{\mathrm{p}}$) and we impose the additional condition that the input rate[2] should be constant from the point $k+N_{\mathrm{c}}-1$ on: $\Delta u(k+j) = \Delta u(k+N_{\mathrm{c}}-1)$ for $j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$, or equivalently $\Delta^2 u(k+j) = \Delta u(k+j) - \Delta u(k+j-1) = 0$ for $j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$.

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k), \ldots, u(k+N_{\mathrm{c}}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one event step, and the optimization is restarted with new information of the measurements. Define the vectors

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_{\mathrm{p}}-1) \end{bmatrix} \qquad \tilde{y}(k) = \begin{bmatrix} \hat{y}(k) \\ \vdots \\ \hat{y}(k+N_{\mathrm{p}}-1) \end{bmatrix}$$

[2] For a manufacturing system the input rate corresponds to the rate at which raw material or external resources are fed to the system

Now the MPL-MPC problem for event step k can be defined as:

$$\min_{\tilde{u}(k)} J_{\text{out}}(k) + \lambda J_{\text{in}}(k)$$

subject to

$$x(k+j) = A \otimes x(k+j-1) \oplus B \otimes u(k+j) \tag{7}$$

$$y(k+j) = C \otimes x(k+j) \tag{8}$$

$$\Delta u(k+j) \geq 0 \tag{9}$$

$$\Delta^2 u(k+\ell) = 0 \tag{10}$$

$$A_{\text{c}}(k)\tilde{u}(k) + B_{\text{c}}(k)\tilde{y}(k) \leq c_{\text{c}}(k) \tag{11}$$

for $j = 0, \ldots, N_{\text{p}}-1,\ \ell = N_{\text{c}}, \ldots, N_{\text{p}}-1$

where (11) represents the linear constraints on the inputs and the outputs.

In this paper J_{out} and J_{in} are chosen as follows:

$$J_{\text{out}}(k) = \sum_i \mathbb{E}[\tilde{\eta}_i(k)] \tag{12}$$

$$J_{\text{in}}(k) = -\sum_j \tilde{u}_j(k) \tag{13}$$

where $\mathbb{E}[\tilde{\eta}_i(k)]$ denotes the expectation of the i-th "tardiness" error $\tilde{\eta}_i(k)$, which is given by

$$\tilde{\eta}_i(k) = \max(\ \tilde{y}_i(k) - \tilde{r}_i(k)\ ,\ 0\ )\ , \tag{14}$$

where the due date signal $r(k)$ is stacked in the vector

$$\tilde{r}(k) = \begin{bmatrix} r(k) \\ \vdots \\ r(k+N_{\text{p}}-1) \end{bmatrix}$$

and $\tilde{y}_i(k)$, $\tilde{u}_i(k)$ and $\tilde{r}_i(k)$ denote the i-th element of $\tilde{y}(k)$, $\tilde{u}(k)$ and $\tilde{r}(k)$, respectively. Other choices for J_{out} and J_{in} are given in [9, 10].

The next step is to make predictions. We collect the uncertainty over the event interval $[k, k+N_{\text{p}}-1]$ in one vector

$$\tilde{e}(k) = \begin{bmatrix} e(k) \\ \vdots \\ e(k+N_{\text{p}}-1) \end{bmatrix} \in \mathbb{R}^{n_{\tilde{e}}}\ .$$

We assume $\tilde{e}(k)$ to be a random variable with probability density function $p(\tilde{e})$. Now it is easy to verify that the prediction model, i.e., the prediction of the future outputs for the system (3)–(4), is given by

$$\tilde{y}(k) = \tilde{C}(\tilde{e}(k)) \otimes x(k-1) \oplus \tilde{D}(\tilde{e}(k)) \otimes \tilde{u}(k)\ , \tag{15}$$

in which $\tilde{C}(\tilde{e}(k))$ and $\tilde{D}(\tilde{e}(k))$ are given by

$$\tilde{C}(\tilde{e}(k)) = \begin{bmatrix} \tilde{C}_1(\tilde{e}(k)) \\ \vdots \\ \tilde{C}_{N_{\text{p}}}(\tilde{e}(k)) \end{bmatrix}$$

$$\tilde{D}(\tilde{e}(k)) = \begin{bmatrix} \tilde{D}_{11}(\tilde{e}(k)) & \cdots & \tilde{D}_{1N_\mathrm{p}}(\tilde{e}(k)) \\ \vdots & \ddots & \vdots \\ \tilde{D}_{N_\mathrm{p}1}(\tilde{e}(k)) & \cdots & \tilde{D}_{N_\mathrm{p}N_\mathrm{p}}(\tilde{e}(k)) \end{bmatrix} \tag{16}$$

where

$$\tilde{C}_m(\tilde{e}(k)) = C(k+m-1) \otimes A(k+m-1) \otimes \ldots \otimes A(k) \tag{17}$$

and

$$\tilde{D}_{mn}(\tilde{e}(k)) = \begin{cases} C(k+m-1) \otimes A(k+m-1) \otimes \ldots \otimes A(k+n) \otimes B(k+n-1) & \text{if } m > n \\ C(k+m-1) \otimes B(k+m-1) & \text{if } m = n \\ \mathcal{E} & \text{if } m < n\,. \end{cases} \tag{18}$$

We combine the material of previous subsections, and obtain

$$J_\mathrm{out}(k) = \sum_i \mathbb{E}\left[\max\Big(\,\Big\{[\tilde{C}(k)]_i \otimes x(k) \;\oplus\; [\tilde{D}(k)]_i \otimes \tilde{u}(k)\Big\} - \tilde{r}_i(k)\;,\;0\,\Big)\right] \tag{19}$$

$$J_\mathrm{in}(k) = -\sum_j \tilde{u}_j(k) \tag{20}$$

where $[\tilde{C}(k)]_i$ and $[\tilde{D}(k)]_i$ denote the i-th row of $\tilde{C}(k)$ and $\tilde{D}(k)$, respectively. Finally the following problem is obtained:

$$\min_{\tilde{u}(k)} \; J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k) \tag{21}$$

subject to

$$A_\mathrm{c}(k)\tilde{u}(k) + B_\mathrm{c}(k)\mathbb{E}[\tilde{y}(k)] \le c_\mathrm{c}(k) \tag{22}$$

$$\Delta u(k+j) \ge 0 \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1 \tag{23}$$

$$\Delta^2 u(k+j) = 0 \qquad \text{for } j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1 \tag{24}$$

This problem will be called the stochastic MPL-MPC problem for event step k.

Recall that the MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k), \ldots, u(k+N_\mathrm{c}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one sample, and the optimization is restarted with new information of the measurements.

4 Convexity of stochastic MPL-MPC

In order to compute the optimal MPC input signal, we need the expectation of the signals $\tilde{\eta}(k)$ and $\tilde{y}(k)$. In this section we present a method to compute $\mathbb{E}[\tilde{\eta}_i(k)]$ and $\mathbb{E}[\tilde{y}(k)]$ and we show that these expectations are convex. As a consequence, the MPL-MPC problem is convex.

Lemma 2 *Define the vector $z(k)$ as*

$$z(k) = \begin{bmatrix} -\tilde{r}(k) \\ x(k-1) \\ \tilde{u}(k) \\ \tilde{e}(k) \end{bmatrix}$$

Then, the future tardiness error $\tilde{\eta}(k)$ and the future output signal $\tilde{y}(k)$ belong to $\mathcal{S}_{\mathrm{mpns}}(z(k))$.

Proof: Equations (16)-(18) in combination with (5) and Lemma 1 show that the entries of $\tilde{C}(\tilde{e}(k))$ and $\tilde{D}(\tilde{e}(k))$ belong to $\mathcal{S}_{\mathrm{mpns}}(\tilde{e}(k))$. Then, using (14), (15) and again Lemma 1 we find that both $\tilde{\eta}(k)$ and $\tilde{y}(k)$ belong to $\mathcal{S}_{\mathrm{mpns}}(z(k))$. $\diamond$

Let $v(k) \in \mathcal{S}_{\mathrm{mpns}}(z(k))$, where $z(k)$ is as defined in Lemma 2. In the sequel of this section we will derive how to compute the expectation $\mathbb{E}[v(k)]$, and show that $\mathbb{E}[v(k)]$ has some nice convexity properties. Define $w(k) = \begin{bmatrix} -\tilde{r}^T(k) & x^T(k-1) & \tilde{u}^T(k) \end{bmatrix}^T$ to be the non-stochastic part of $z(k)$. Then, because of Lemma 2 and the definition of max-plus-nonnegative-scaling functions, there exist scalars α_j and non-negative vectors β_j and γ_j, such that

$$v(k) \quad = \max_{j=1,\ldots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)\Big)$$

Define the sets $\Phi_j(w(k))$, $j = 1, \ldots, n_v$ such that for all $\tilde{e}(k) \in \Phi_j(w(k))$ there holds:

$$v(k) \quad = \quad \alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)$$

and

$$\bigcup_{j=1}^{n_v} \Phi_j(w(k)) = \mathbb{R}^{n_{\tilde{e}}}$$

Denote, for a given $w(k)$, the expectation of $v(k)$ by $\hat{v}(w(k)) = \mathbb{E}[v(k)]$, and the probability density function of $\tilde{e}$ by p. Then

$$\begin{aligned}
\hat{v}(w(k)) &= \mathbb{E}[v(k)] && (25)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v(k)\, p(\tilde{e})\, d\tilde{e} && (26)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_{j=1,\ldots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} && (27)\\
&= \sum_{j=1}^{n_v} \int \cdots \int_{\tilde{e}\in\Phi_j(w(k))} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} && (28)
\end{aligned}$$

where $d\tilde{e} = d\tilde{e}_1\, d\tilde{e}_2 \ldots d\tilde{e}_{n_{\tilde{e}}}$.

The following proposition shows that $\hat{v}(w(k))$ is convex in the vector $w(k)$.

Proposition 3 *The function $\hat{v}(w(k))$ as defined in (28) is convex in $w(k)$ and a subgradient $g_v(w(k))$ is given by*

$$g_v(w(k)) = \sum_{\ell=1}^{n_v} \beta_\ell^T \int \cdots \int_{\tilde{e}\in\Phi_\ell(w(k))} p(\tilde{e})\, d\tilde{e} \qquad (29)$$

Proof: Consider a vector $w_o(k)$ with the same structure as $w(k)$. Recall that (cf. (28))

$$\hat{v}(w_o(k))) \quad = \quad \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e}\in\Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w_o(k) + \gamma_\ell^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e}$$

Then, using the fact that $\bigcup \Phi_\ell(w_o(k)) = \mathbb{R}^{n_{\tilde{e}}}$, there holds for any $w(k)$:

$$\begin{aligned}
\hat{v}(w(k)) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_{j=1,\ldots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e} \qquad \text{(by (27))} \\
&= \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \max_{j=1,\ldots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e} \\
&\geq \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w(k) + \gamma_\ell^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e}
\end{aligned}$$

Note that the sets $\Phi_\ell(w_o(k))$ are computed for $w_o(k)$, where as $\hat{v}(w(k))$ is computed for $w(k)$. Now we derive:

$$\begin{aligned}
&\sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w(k) + \gamma_\ell^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e} \\
&\quad = \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w_o(k) + \gamma_\ell^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e} \\
&\qquad + \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \Big(\beta_\ell^T (w(k) - w_o(k)) \Big)\, p(\tilde{e})\, d\tilde{e} \\
&\quad = \sum_{\ell=1}^{n_v} \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} \Big(\alpha_\ell + \beta_\ell^T w_o(k) + \gamma_\ell^T \tilde{e} \Big)\, p(\tilde{e})\, d\tilde{e} \\
&\qquad + \left(\sum_{\ell=1}^{n_v} \beta_\ell^T \int \cdots \int_{\tilde{e} \in \Phi_\ell(w_o(k))} p(\tilde{e})\, d\tilde{e} \right) \Big(w(k) - w_o(k) \Big) \\
&\quad = \hat{v}(w_o(k)) + g_v(w_o(k)) \Big(w(k) - w_o(k) \Big)
\end{aligned}$$

and we conclude:

$$\hat{v}(w(k)) \geq \hat{v}(w_o(k)) + g_v(w_o(k)) \Big(w(k) - w_o(k) \Big) \tag{30}$$

From [21] it follows that equation (30) proves that $\hat{v}$ is convex in $w(k)$ and that g_v, defined by (29), is a subgradient of $\hat{v}$. $\diamond$

Now consider the MPC problem (21) – (24). First note that because of Lemma 3, $\mathbb{E}[\tilde{\eta}_i(k)]$ and $\mathbb{E}[\tilde{y}(k)]$ are convex in $w(k)$. This means that $J_{\text{out}}(k)$ and $J(k)$ are convex in $\tilde{u}(k)$.

Property 4 *If the linear constraints are monotonically nondecreasing as a function of $\mathbb{E}[\tilde{y}(k)]$ (in other words, if $[B_c]_{ij} \geq 0$ for all i, j), the constraint (22) becomes convex in $\tilde{u}(k)$.*

So, if the linear constraints are monotonically nondecreasing, the MPL-MPC problem turns out to be a convex problem in $\tilde{u}(k)$, and both a subgradient of the constraints and a subgradient of the cost criterion can easily be derived using Lemma 3. Note that convex optimization problems can be solved using reliable and efficient optimization algorithms, based on interior point methods [18, 26].

5 Piecewise affine and piecewise polynomial probability density functions

So far, we did not make any assumption on the characterization of probability function $p(\tilde{e})$. For the computation of the cost criterion and the constraints we need the values of $\mathbb{E}[\tilde{y}(k)]$ and $\mathbb{E}[\tilde{\eta}(k)]$. If we choose for example a Gaussian distribution, they can be calculated from (28) using numerical integration. Numerical integration is usually time-consuming and cumbersome, but can be avoided by choosing piecewise affine or piecewise polynomial probability density functions (possibly as an approximation of the real probability density function).

Let $p(\tilde{e})$ be piecewise affine functions, so consider sets P_ℓ , $\ell = 1, \ldots, n_p$, such that for $\tilde{e} \in P_\ell$ the probability density function is given by:

$$p(\tilde{e}) = \mu_\ell + \zeta_\ell^T \tilde{e}$$

Consider a signal $v(k) \in \mathcal{S}_{\mathrm{mpns}}(z(k))$ and let $w(k)$ be its non-stochastic part. Let $E_{j\ell}(w(k)) = \Phi_j(w(k)) \cap P_\ell$ for $j = 1, \ldots, n_v$, $\ell = 1, \ldots, n_p$, then $\hat{v}(w(k))$ is given by

$$\hat{v}(w(k)) = \sum_{\ell=1}^{n_p} \sum_{j=1}^{n_v} \int \cdots \int_{\tilde{e} \in E_{j\ell}(w(k))} \left(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e} \right) (\mu_\ell + \zeta_\ell^T \tilde{e})\, d\tilde{e}$$

This is an integral of a quadratic function in $\tilde{e}$ and can be solved analytically for all regions $E_{j\ell}$. In general, for piecewise polynomial probability density functions, the integral will be a polynomial function in $\tilde{e}$, and can be solved analytically for all regions $E_{j\ell}$ [3, 14].

If piecewise affine or polynomial probability density functions are used as an approximation of "true" non-polynomial probability functions, the quality of the approximation can be improved by increasing the number of sets n_p.

6 Example

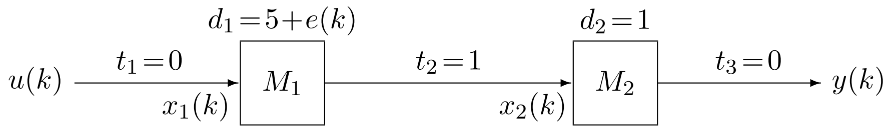

Figure 1: A production system.

Consider the production system of Figure 1. This system consists of two machines M_1 and M_2 and operates in batches. The raw material is fed to machine M_1 where preprocessing is done. Afterwards the intermediate product is fed to machine M_2 and finally leaves the system. We assume that each machine starts working as soon as possible on each batch, i.e., as soon as the raw material or the required intermediate product is available, and as soon as the machine is idle (i.e., the previous batch of products has been processed and has left the machine). Define:

$$\begin{array}{rcl} u(k) & : & \text{time instant at which the system is fed for the } k\text{th time} \\ y(k) & : & \text{time instant at which the } k\text{th product leaves the system} \\ x_i(k) & : & \text{time instant at which machine } i \text{ starts for the } k\text{th time} \\ t_j(k) & : & \text{transportation time for the } k\text{th batch.} \\ d_i(k) & : & \text{processing time on machine } i \text{ for the } k\text{th batch.} \end{array}$$

The system equations are given by

$$\begin{aligned} x_1(k) &= \max(x_1(k-1) + d_1(k-1), u(k) + t_1(k)) \\ x_2(k) &= \max(x_1(k) + d_1(k) + t_2(k), x_2(k-1) + d_2(k-1)) \\ &= \max(x_1(k-1) + d_1(k-1) + d_1(k) + t_2(k), \\ &\qquad u(k) + d_1(k) + t_1(k) + t_2(k), x_2(k-1) + d_2(k-1)) \\ y(k) &= x_2(k) + d_2(k) + t_3(k) \end{aligned}$$

In matrix notation this becomes

$$\begin{aligned} x(k) &= A(k) \otimes x(k-1) \oplus B(k) \otimes u(k) \\ y(k) &= C(k) \otimes x(k) \ . \end{aligned}$$

where the system matrices A, B and C are given by

$$A = \begin{bmatrix} d_1(k-1) & \varepsilon \\ d_1(k-1) + d_1(k) + t_2(k) & d_2(k-1) \end{bmatrix} \quad B = \begin{bmatrix} t_1(k) \\ d_1(k) + t_1(k) + t_2(k) \end{bmatrix}$$

$$C = \begin{bmatrix} \varepsilon & d_2(k) + t_3(k) \end{bmatrix}$$

Let us now solve the stochastic MPC problem for this perturbed MPL system. Assume that the transportation times are constant: $t_1(k) = 0$, $t_2(k) = 1$, $t_3(k) = 0$, that the production time of the second machine is constant: $d_2(k) = 1$, and that the processing time of the second machine is corrupted by noise:

$$d_1(k) = 5 + e(k)$$

where $e(k)$ is a random signal with probability density function

$$p(e) = \begin{cases} 0 & \text{for} \quad e < -1 \\ 1/2 & \text{for} \quad -1 \leq e \leq 1 \\ 0 & \text{for} \quad e > 1 \end{cases} \tag{31}$$

Assume that the due date signal is given by

$$r(k) \quad = \quad 10 + 5 \cdot k \ , \tag{32}$$

the initial state is equal to $x(0) = [\ 0 \ \ 10 \ \ 10\ ]^T$, and the cost criterion (21) is optimized for $N_\mathrm{p} = 3$ and $N_\mathrm{c} = 2$. With the above choice of the cost criterion, we can rewrite the stochastic MPC problem into a convex optimization problem. The optimal input sequence is computed for $k = 1, \ldots, 100$, and at for each k, the first element $u(k)$ of the sequence $\tilde{u}(k)$ is applied to the perturbed system (due to the receding horizon strategy). In the experiment, the true

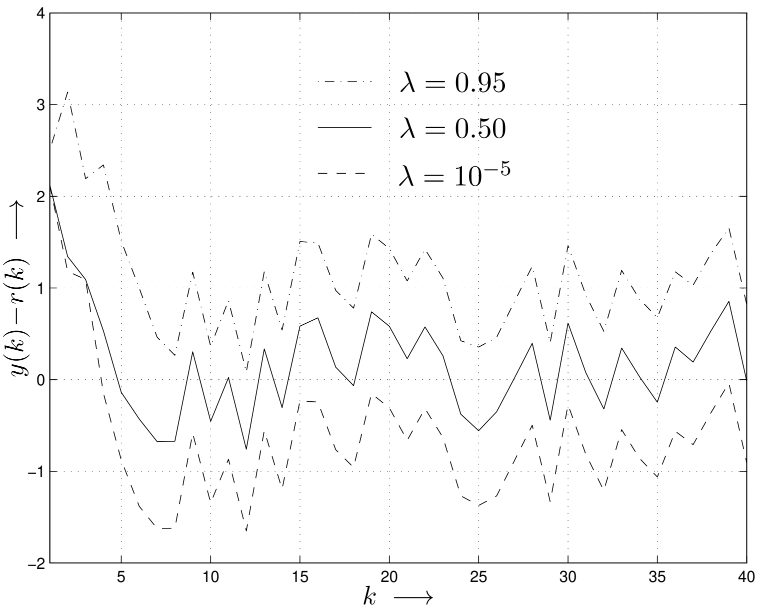

Figure 2: The difference $y(k) - r(k)$ between the output date signal y and the due date signal r for different values of λ).

system is simulated for a random sequence $e(k)$, $k = 1, \ldots, 100$, satisfying probability density function (31).

Figure 2 gives the difference between the output date signal y and the due date signal r. To see the influence of the parameter λ, the design is done for different values (λ makes a trade-off between minimization of the due-date violation and the just-in-time feeding). For $\lambda = 10^{-5}$, the input sequence is hardly weighted, and for $k > 3$ all due dates are satisfied. For $\lambda = 0.5$, the values of the input dates are taken into account, and the scheme leads to a frequent violation of the due dates. For $\lambda = 0.95$ the input cost criterion J_{in} will be dominant in the optimization, which results in a maximization of the control input (last-moment feed). Consequently, the due date is continuously violated ($y(k) - r(k) > 0$ for all k).

7 Discussion

In this paper we have extended the work on analysis and control of uncertain discrete-event systems. Instead of a deterministic approach ([15, 19, 24, 27]), we have considered the uncertainties in a stochastic setting. Further we have introduced an MPC framework for the control of these stochastic max-plus-linear discrete event systems. We have shown that, if the constraints are a nondecreasing function of the output, the resulting optimization problem is a convex optimization problem, and thus can be solved very efficiently. In general, the computation of the predictions requires a numerical integration. However, in the case of piecewise polynomial probability density functions, this numerical integration can be prevented and the integrals can be computed analytically.

Topics for future research are: determination of rules of thumb for appropriate values for the tuning parameters (control horizon N_{c}, prediction horizon N_{p}, and performance weighting parameter λ) in the stochastic case, and complexity reduction and approximation to further improve the efficiency of our approach. In the following paragraphs we will discuss both topics in more detail.

In [22] we derived rules of thumb for determining appropriate values for the MPC tuning parameters in the case of deterministic max-plus-linear discrete event systems. Although our first implementations do not show a different behavior with respect to the advised initial parameter settings of [22], we will need further study on the question if and how much these settings will change in the stochastic case. This research will be done by considering the optimal tuning parameters for a large number of generic simulation examples.

Problem complexity may lead to an excessive computation time or memory overflow. In practical applications the computation time and memory capacity are always limited, and therefore a high problem complexity will not be acceptable. A possible solution to this problem may be the concept of variability expansion [12], in which approximate calculation is done of the stochastic integrals. A first implementation of this approximation was done in [25], in which it was shown that the complexity of the MPC optimization problem may be reduced drastically. Further research on this topic is necessary.

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