Delft University of Technology

Fac. of Information Technology and Systems **Control Systems Engineering**

Technical report CSE02-007

Complexity reduction in MPC for stochastic max-plus-linear systems by variability expansion*

T.J.J. van den Boom, B. De Schutter, and B. Heidergott

If you want to cite this report, please use the following reference instead:

T.J.J. van den Boom, B. De Schutter, and B. Heidergott, "Complexity reduction in MPC for stochastic max-plus-linear systems by variability expansion," *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, pp. 3567–3572, Dec. 2002.

Control Systems Engineering
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands
Current URL: https://www.dcsc.tudelft.nl

* This report can also be downloaded via https://pub.bartdeschutter.org/abs/02_007

Complexity reduction in MPC for stochastic max-plus-linear systems by variability expansion

T.J.J. van den Boom[‡], B. De Schutter[‡] and B. Heidergott[§]

[‡]Control Systems Engineering, Faculty of Information Technology and Systems
Delft University of Technology, P.O.Box 5031, 2600 GA Delft, The Netherlands

[§]Department of Mathematics and Computing Science
Eindhoven University of Technology, P.O.Box 513, 5600 MB Eindhoven, The Netherlands

email: {t.j.j.vandenboom,b.deschutter}@its.tudelft.nl, b.heidergott@tue.nl

Abstract

Model predictive control (MPC) is a popular controller design technique in the process industry. Conventional MPC uses linear or nonlinear discrete-time models. Recently, we have extended MPC to a class of discrete event systems that can be described by a model that is "linear" in the max-plus algebra. In our previous work we have considered MPC for the perturbations-free case and for the case with noise and/or modeling errors in a bounded or stochastic setting. In this paper we consider a method to reduce the computational complexity of the resulting optimization problem, based on variability expansion. We show that the computational load is reduced if we decrease the level of "randomness" in the system.

1 Introduction

Model predictive control (MPC) [7, 11] is a proven technology for the control of multivariable systems in the presence of input, output and state constraints and is capable of tracking pre-scheduled reference signals. These attractive features make MPC widely accepted in the process industry. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to discrete event systems. Typical examples of discrete event systems (DES) are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, and logistic systems. The class of DES essentially consists of man-made systems that contain a finite number of resources (such as machines, communications channels, or processors) that are shared by several users (such as product types, information packets, or jobs) all of which contribute to the achievement of some common goal (the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation) [1]. There exist many different modeling and analysis frameworks for DES such as Petri nets, finite state machines, automata, languages, process algebra, computer models, etc. [3, 9]. In this paper we consider the class of DES with synchronization but no concurrency. Such DES can be described by models that are "linear" in the max-plus algebra [1, 4], and therefore they are called max-plus-linear (MPL) DES.

In [5, 6, 15] we have extended MPC to MPL systems. In [14, 15] we have presented some results on MPL-MPC in the presence of noise and/or modeling errors, both in a bounded and stochastic setting. In contrast to conventional linear systems, where noise and disturbances are usually modeled by including an extra term in the system equations (i.e., the noise is considered to be additive), the influence of noise and disturbances in MPL systems is not max-plus-additive, but max-plus-multiplicative (see [1] or the worked example of Section 5). A second important feature is modeling errors. Uncertainty in the modeling or identification phase leads to errors in the system matrices. It is clear that both modeling errors, and noise/disturbances perturb the system by introducing uncertainty in the system matrices. Therefore, both features can be treated in one single framework, as was already shown in [14, 15]. The characterization of the perturbation will then determine whether it describes model mismatch or disturbance. In [14] we have derived an MPC controller for this framework and we have also shown that under quite general conditions the resulting MPC optimization problem is a convex optimization problem. However, for many practical situations, the problem complexity will grow fast for an increasing prediction horizon and a higher system order. In this paper we will approximate the predicted future outputs using the method of variability expansion [8]. We introduce a parameter that allows us to control the level of random-

ness in the system, and letting the parameter go from 0 to 1 increases the level of stochasticity in the system. We will show that by using this approximation the MPC optimization problem can be solved very efficiently.

The paper is organized as follows. In Section 2 we introduce the max-plus algebra and the concept of stochastic MPL systems. In Section 3 we give a short overview of the MPC algorithm for MPL systems. Section 4 describes how the complexity of the problem can be reduced by using the method of variability expansion. Finally, Section 5 gives a worked example and a comparison in computational load.

2 Max-plus algebra and stochastic max-plus-linear systems

2.1 Max-plus algebra

In this section we give the basic definition of the max-plus algebra and we present some results on a class of max-plus functions.

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows [1, 4]:

$$x \oplus y = \max(x, y) \qquad x \otimes y = x + y$$

for numbers $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n} (a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$.

Let $\mathcal{S}_{\text{mpns}}$ be the set of max-plus-nonnegative-scaling functions, i.e., functions f of the form

$$f(z) = \max_i(\alpha_{i,1} z_1 + \ldots + \alpha_{i,n} z_n + \beta_i)$$

(in conventional algebra) with variable $z \in \mathbb{R}_\varepsilon^n$ and constants $\alpha_{i,j} \in \mathbb{R}^+$ and $\beta_i \in \mathbb{R}$, where $\mathbb{R}^+$ is the set of the nonnegative real numbers. If we want to stress that f is a function of z we will denote this by $f \in \mathcal{S}_{\text{mpns}}(z)$.

Lemma 1 *The set $\mathcal{S}_{\text{mpns}}$ is closed under the operations $\oplus$, $\otimes$, and scalar multiplication by a nonnegative scalar.*

The proof is in [15].

2.2 Stochastic max-plus-linear systems

In [14, 15] we have studied discrete event systems (DES) in which there is synchronization but no concurrency and which include uncertainty. It has been shown that these systems can be described by a model of the form

$$x(k) = A(k) \otimes x(k-1) \oplus B(k) \otimes u(k) \tag{1}$$
$$y(k) = C(k) \otimes x(k) \; . \tag{2}$$

DES that can be described by this model will be called max-plus-linear (MPL). The index k is called the event counter. The state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instants at which the input events occur for the kth time, and the output $y(k)$ contains the time instants at which the output events occur for the kth time[1]. The entries of system matrices $A(k)$, $B(k)$ and $C(k)$ are uncertain due to modeling errors or disturbances. Usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. In this paper both features will be treated within one single framework. The uncertainty caused by disturbances and errors in the estimation of physical variables, is gathered in the uncertainty vector $e(k)$. In this paper we assume that the uncertainty has stochastic properties. Hence, $e(k)$ is a stochastic variable.

We assume that the uncertainty vector $e(k)$ captures the complete time-varying aspect of the system. Furthermore, the system matrices of an MPL model usually consist of sums or maximizations of internal process times, transportation times, etc. (see, e.g., [1] or Section 5). Since the entries of $e(k)$ directly correspond to the uncertainties in these duration times, it follows from Lemma 1 that the entries of the uncertain system matrices belong to $\mathcal{S}_{\text{mpns}}$:

$$A(k) \in \mathcal{S}_{\text{mpns}}^{n \times n}(e(k)), \quad B(k) \in \mathcal{S}_{\text{mpns}}^{n \times m}(e(k)),$$
$$C(k) \in \mathcal{S}_{\text{mpns}}^{l \times n}(e(k)). \tag{3}$$

Results for handling uncertainty for some other classes of DES are given in [2, 10, 13, 17] and the references therein.

3 Model predictive control for stochastic MPL systems

In [5, 14, 15] we have extended the MPC framework to MPL models (1)–(2) as follows. Just as in conventional MPC [7, 11] we define a cost criterion J that reflects the input and output cost functions (J_{in} and J_{out}, respectively) in the event period $[k, k + N_{\text{p}} - 1]$:

$$J(k) = J_{\text{out}}(k) + \lambda J_{\text{in}}(k) \tag{4}$$

[1] More specifically, for a manufacturing system, $x(k)$ contains the time instants at which the processing units start working for the kth time, $u(k)$ the time instants at which the kth batch of raw material is fed to the system, and $y(k)$ the time instants at which the kth batch of finished product leaves the system.

where $N_{\rm p}$ is the prediction horizon and λ is a weighting parameter and $J_{\rm out}$ and $J_{\rm in}$ are chosen as follows:

$$J_{\rm out}(k) \;=\; \sum_i \sum_{j=0}^{N_{\rm p}-1} I\!E[\eta_i(k+j)] \qquad (5)$$

$$J_{\rm in}(k) \;=\; -\sum_\ell \sum_{j=0}^{N_{\rm p}-1} u_\ell(k+j) \qquad (6)$$

where $I\!E[\eta_i(k)]$ denotes the expected value of the ith "tardiness" $\eta_i(k)$, which is given by

$$\eta_i(k) = \max(\; y_i(k) - r_i(k)\;,\; 0\;)\;\;, \qquad (7)$$

where $r(k)$ is the due date signal. Other choices for $J_{\rm out}$ and $J_{\rm in}$ are given in [5, 6].

The aim is now to compute an optimal input sequence $u(k),\ldots,u(k+N_{\rm p}-1)$ that minimizes $J(k)$ subject to linear constraints on the inputs and outputs. Since the $u(k)$'s correspond to consecutive event occurrence times, we have the additional condition $\Delta u(k+j) = u(k+j) - u(k+j-1) \geq 0$ for $j = 0,\ldots,N_{\rm p}-1$. Furthermore, in order to reduce the number of decision variables and the corresponding computational complexity we introduce a control horizon $N_{\rm c}$ ($\leq N_{\rm p}$) and we impose the additional condition that the input rate[2] should be constant from the point $k+N_{\rm c}-1$ on: $\Delta u(k+j) = \Delta u(k+N_{\rm c}-1)$ for $j = N_{\rm c},\ldots,N_{\rm p}-1$, or equivalently $\Delta^2 u(k+j) = \Delta u(k+j) - \Delta u(k+j-1) = 0$ for $j = N_{\rm c},\ldots,N_{\rm p}-1$.

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k),\ldots,u(k+N_{\rm c}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one event step, and the optimization is restarted with new information of the measurements.

Define the vectors

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_{\rm p}-1) \end{bmatrix}, \quad \tilde{r}(k) = \begin{bmatrix} r(k) \\ \vdots \\ r(k+N_{\rm p}-1) \end{bmatrix}$$

$$\tilde{y}(k) = \begin{bmatrix} \hat{y}(k) \\ \vdots \\ \hat{y}(k+N_{\rm p}-1) \end{bmatrix}, \quad \tilde{e}(k) = \begin{bmatrix} e(k) \\ \vdots \\ e(k+N_{\rm p}-1) \end{bmatrix}$$

Now the MPL-MPC problem for event step k can be defined as:

$$\min_{\tilde{u}(k)} J_{\rm out}(k) + \lambda J_{\rm in}(k) \qquad (8)$$

subject to

$$\Delta u(k+j) \geq 0 \qquad (9)$$

$$\Delta^2 u(k+\ell) = 0 \qquad (10)$$

$$A_{\rm c}(k)\tilde{u}(k) + B_{\rm c}(k) I\!E[\tilde{y}(k)] \leq c_{\rm c}(k) \qquad (11)$$

for $j = 0,\ldots,N_{\rm p}-1,\; \ell = N_{\rm c},\ldots,N_{\rm p}-1$.

In order to compute the optimal MPC input signal, we need the expected value of the signals $\eta_i(k+j)$ and $y_i(k+j)$. We will now consider the computation of $I\!E[\eta_i(k+j)]$ and $I\!E[y_i(k+j)]$. In [14] we have shown that $\eta_i(k+j)$ and $y_i(k+j)$ are max-plus-nonnegative-scaling functions in the variable $w(k) = \begin{bmatrix} -\tilde{r}^T(k) & x^T(k-1) & \tilde{u}^T(k) \end{bmatrix}^T$.

The following proposition is given in [14]:

Proposition 2 *Consider a signal $v(k)$ which is a max-plus-nonnegative-scaling function in $w(k)$ and $\tilde{e}(k)$, so*

$$v(k) \;=\; \max_{j=1,\ldots,n_v} \Big(\alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)\Big)$$

where α_j are scalars and β_j and γ_j are non-negative vectors and $\tilde{e}(k) \in \mathbb{R}^{n_{\tilde{e}}}$ is a stochastic variable with probability density function p. Define the sets $\Phi_j(w(k))$, $j = 1,\ldots,n_v$ such that for all $\tilde{e}(k) \in \Phi_j(w(k))$ there holds:

$$v(k) \;=\; \alpha_j + \beta_j^T w(k) + \gamma_j^T \tilde{e}(k)$$

and $\bigcup_{j=1}^{n_v} \Phi_j(w(k)) = \mathbb{R}^{n_{\tilde{e}}}$. The expected value of $v(k)$ is then given by

$$I\!E[v(k)] \;=\sum_{j=1}^{n_v} \int\!\ldots\!\int_{\tilde{e}\,\in\,\Phi_j(w)} \Big(\alpha_j+\beta_j^T w(k)+\gamma_j^T \tilde{e}\Big)\, p(\tilde{e})\, d\tilde{e} \qquad (12)$$

where $d\tilde{e} = d\tilde{e}_1\, d\tilde{e}_2 \ldots d\tilde{e}_{n_{\tilde{e}}}$.
The function $I\!E[v(k)]$ is convex in $w(k)$ and a subgradient $g_v(w(k))$ is given by

$$g_v(w(k)) = \sum_{\ell=1}^{n_v} \beta_\ell^T \int_{\tilde{e}\in\Phi_\ell(w(k))} \cdots \int p(\tilde{e})\, d\tilde{e}\;. \qquad (13)$$

Now consider the MPC problem (8) – (11). First note that because of Proposition 2, $I\!E[\eta_i(k+j)]$ and $I\!E[y(k+j)]$ are convex in $w(k)$. This means that $J_{\rm out}(k)$ and $J(k)$ are convex in $\tilde{u}(k)$.

Property 3 *If the linear constraints are monotonically nondecreasing as a function of $I\!E[\tilde{y}(k)]$ (in other words, if $[B_{\rm c}]_{ij} \geq 0$ for all i,j), the constraint (11) becomes convex in $\tilde{u}(k)$.*

[2] For a manufacturing system the input rate corresponds to the rate at which raw material or external resources are fed to the system

So, if the linear constraints are monotonically nondecreasing, the MPL-MPC problem turns out to be a convex problem in $\tilde{u}(k)$, and both a subgradient of the constraints and a subgradient of the cost criterion can easily be derived using Proposition 2. Note that convex optimization problems can be solved using reliable and efficient optimization algorithms, based on interior point methods [12, 16].

In general, the computation of the predictions requires a numerical integration. However, in the case of piecewise polynomial probability density functions, this numerical integration can be prevented and the integrals can be computed analytically (see [14]).

4 Reducing complexity

The algorithm described in the previous section, has a complexity that is growing fast with an increasing number of stochastic variables $n_{\tilde{e}}$ due to the numerical integration. In this section we will approximate the expected value of $v(\tilde{e}(k))$ using the method of variability expansion [8]. To this end, we assume that the entries of $\tilde{e}(k)$ are i.i.d. and introduce an artificial parameter θ. We now replace with probability $1-\theta$ the ith entry of random vector $\tilde{e}(k)$ by its mean, denoted by $[\tilde{e}_0]_i$. The result is denoted by $\tilde{e}_\theta(k)$. The parameter θ allows controlling the level of randomness in the system, and letting θ go from 0 to 1 increases the level of stochasticity in the system.

The main result of [8] is the following. For any $\theta \in [0,1]$, $I\!E[v(\tilde{e}_\theta(k)))]$ can be developed into a Taylor series in θ whose domain of convergence is $[0,1]$, where we take one-sided limits at the boundary points. In particular, denote the limit of $d^m/d\theta^m\, I\!E[v(\tilde{e}_\theta(k))]$ for $\theta \downarrow 0$ by $d^m/d\theta^m\, I\!E[v(\tilde{e}_0(k))]$, then $I\!E[v(\tilde{e}_\theta(k))]$, the "true" expected value of $v(\tilde{e}(k))$, is given by

$$I\!E[v(\tilde{e}(k)))] = \sum_{m=0}^{M} \frac{1}{m!} \frac{d^m}{d\theta^m} I\!E[v(\tilde{e}_0(k))] + R_M(k)$$

where, for $M < n_{\tilde{e}}$,

$$R_M \leq \frac{1}{(M+1)!} \left| \frac{d^{M+1}}{d\theta^{M+1}} I\!E[v(\tilde{e}_0(k))] \right|$$

and $R_M = 0$ otherwise.

Expressions for the mth order derivative $d^m/d\theta^m\, I\!E[v(\tilde{e}_0(k))$ are given in [8]. In this paper we only give expressions for the first three derivatives ($m = 1,2,3$). Set for $0 \leq m \leq n_{\tilde{e}}$ and $i_1 < i_2 < \ldots < i_m$:

$$V(i_1, i_2, \ldots, i_m) = I\!E[v(\tilde{e}_\theta(k, i_1, i_2, \ldots, i_m))]$$

where $[\tilde{e}_\theta(k, i_1, i_2, \ldots, i_m)]_j = [\tilde{e}_0(k)]_j$ for $j \notin \{i_1, i_2, \ldots, i_m\}$ and $V(0) = v(\tilde{e}_0(k))$. This means that $V(i_1, i_2, \ldots, i_m)$ is the estimation of v in the case where only the elements $\tilde{e}_j$ for $j \in \{i_1, i_2, \ldots, i_m\}$ are stochastic, and the elements $\tilde{e}_j$ for $j \notin \{i_1, i_2, \ldots, i_m\}$ are fixed to their mean. Now the derivatives ($m = 1,2,3$) are given by

$$\begin{aligned}
\frac{d}{d\theta} I\!E[v(\tilde{e}_0(k))] &= \sum_{i=1}^{n_{\tilde{e}}} (V(i) - V(0)) \\
\frac{d^2}{d\theta^2} I\!E[v(\tilde{e}_0(k))] &= 2 \sum_{i_1=1}^{n_{\tilde{e}}-1} \sum_{i_2=i_1+1}^{n_{\tilde{e}}} \Big(V(i_1, i_2) + V(0) \\
&\quad - V(i_1) - V(i_2) \Big) \\
\frac{d^3}{d\theta^3} I\!E[v(\tilde{e}_0(k))] &= 6 \sum_{i_1=1}^{n_{\tilde{e}}-2} \sum_{i_2=i_1+1}^{n_{\tilde{e}}-1} \sum_{i_3=i_2+1}^{n_{\tilde{e}}} \Big(V(i_1, i_2, i_3) \\
&\quad + V(i_1) + V(i_2) + V(i_3) \\
&\quad - V(i_1, i_2) - V(i_1, i_3) \\
&\quad - V(i_2, i_3) - V(0) \Big)
\end{aligned}$$

By ignoring the error-term R_4, the estimation of $v(\tilde{e}(k))$ can be approximated by a third-order Taylor expansion:

$$\begin{aligned}
I\!E[v(\tilde{e}(k))] &\approx v(\tilde{e}_0(k)) + \frac{d}{d\theta} I\!E[v(\tilde{e}_0(k)) \\
&\quad + \frac{1}{2} \frac{d^2}{d\theta^2} I\!E[v(\tilde{e}_0(k)) + \frac{1}{6} \frac{d^3}{d\theta^3} I\!E[v(\tilde{e}_0(k)) \\
&\approx c_0 v(\tilde{e}_0(k)) + c_1 \sum_{i=1}^{n_{\tilde{e}}} V(i) \\
&\quad + c_2 \sum_{i_1=1}^{n_{\tilde{e}}-1} \sum_{i_2=i_1+1}^{n_{\tilde{e}}} V(i_1, i_2) \\
&\quad + c_3 \sum_{i_1=1}^{n_{\tilde{e}}-2} \sum_{i_2=i_1+1}^{n_{\tilde{e}}-1} \sum_{i_3=i_2+1}^{n_{\tilde{e}}} V(i_1, i_2, i_3) \quad (14)
\end{aligned}$$

where the coefficients are given by $c_0 = \frac{1}{6}(-n_{\tilde{e}}^3 + 6\, n_{\tilde{e}}^2 - 11\, n_{\tilde{e}} + 6)$, $c_1 = \frac{1}{2}(n_{\tilde{e}}^2 - 5\, n_{\tilde{e}} + 6)$, $c_2 = 3 - n_{\tilde{e}}$, $c_3 = 1$. Expression (14) can also be used for the approximation with $M = 0$, $M = 1$ and $M = 2$. For $M = 0$ we find the coefficients $c_0 = 1$, $c_1 = c_2 = c_3 = 0$, for $M = 1$ we find the coefficients $c_0 = 1 - n_{\tilde{e}}$, $c_1 = 1$, $c_2 = c_3 = 0$, and for $M = 2$ we find the coefficients $c_0 = \frac{1}{2}(n_{\tilde{e}}^2 - 3\, n_{\tilde{e}} + 2)$, $c_1 = 12 - n_{\tilde{e}}$, $c_2 = 1$, $c_3 = 0$.

The subgradient $\nabla_{\tilde{u}} I\!E[v(\tilde{e}(k))]$ can be computed using the same weighted summation. For example, the approximate subgradient for $M \leq 3$ becomes:

$$\begin{aligned}
\nabla_{\tilde{u}} I\!E[v(\tilde{e}(k))] &\approx c_0 \nabla_{\tilde{u}} v(\tilde{e}_0(k)) + c_1 \sum_{i=1}^{n_{\tilde{e}}} \nabla_{\tilde{u}} V(i) \\
&\quad + c_2 \sum_{i_1=1}^{n_{\tilde{e}}-1} \sum_{i_2=i_1+1}^{n_{\tilde{e}}} \nabla_{\tilde{u}} V(i_1, i_2) \\
&\quad + c_3 \sum_{i_1=1}^{n_{\tilde{e}}-2} \sum_{i_2=i_1+1}^{n_{\tilde{e}}-1} \sum_{i_3=i_2+1}^{n_{\tilde{e}}} \nabla_{\tilde{u}} V(i_1, i_2, i_3)
\end{aligned}$$

with the corresponding coefficients c_0, c_1, c_2 and c_3. The values of V and $\nabla_{\tilde{u}} V$ can be computed using Proposition 2. Because of the dramatic reduction in number of stochastic variables, these values are computed much faster than a full estimation of $v(\tilde{e}(k))$ and $\nabla_{\tilde{u}} I\!E[v(\tilde{e}(k))]$.

Note that because of the approximations, full convexity might be lost. However, if the approximations are close to the original functions, we still have a well-conditioned optimization problem.

5 Example

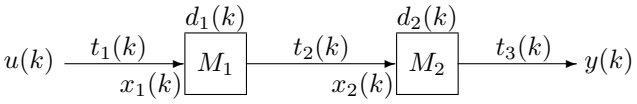

Figure 1: A production system.

Consider the production system in Figure 1. This system consists of two machines M_1 and M_2 and operates in batches. The raw material is fed to machine M_1 where preprocessing is done. Afterwards the intermediate product is fed to machine M_2 and finally leaves the system. We assume that each machine starts working as soon as possible on each batch, i.e., as soon as the raw material or the required intermediate product is available, and as soon as the machine is idle (i.e., the previous batch of products has been processed and has left the machine). Define:

- $u(k)$: time instant at which the system is fed for the kth time
- $y(k)$: time instant at which the kth product leaves the system
- $x_i(k)$: time instant at which machine i starts for the kth time
- $t_j(k)$: transportation time on link j for the kth batch.
- $d_i(k)$: processing time on machine i for the kth batch.

The system equations are given by

$$\begin{aligned} x_1(k) &= \max(x_1(k-1)+d_1(k-1), u(k)+t_1(k)) \\ x_2(k) &= \max(x_1(k)+d_1(k)+t_2(k), \\ &\qquad x_2(k-1)+d_2(k-1)) \\ &= \max(x_1(k-1)+d_1(k-1)+d_1(k)+t_2(k), \\ &\qquad u(k)+d_1(k)+t_1(k)+t_2(k), \\ &\qquad x_2(k-1)+d_2(k-1)) \\ y(k) &= x_2(k)+d_2(k)+t_3(k) \end{aligned}$$

In matrix notation we obtain (1)-(2) where the system matrices A, B and C are given by

$$A = \begin{bmatrix} d_1(k-1) & \varepsilon \\ d_1(k-1)+d_1(k)+t_2(k) & d_2(k-1) \end{bmatrix}$$

$$B = \begin{bmatrix} t_1(k) \\ d_1(k)+t_1(k)+t_2(k) \end{bmatrix}$$

$$C = \begin{bmatrix} \varepsilon & d_2(k)+t_3(k) \end{bmatrix}$$

Let us now solve the stochastic MPC problem for this perturbed MPL system. Assume that two of the transportation times are constant: $t_1(k) = 0$, $t_3(k) = 0$, and that transportation time $t_2(k)$ and the production times $d_1(k)$ and $d_2(k)$ are corrupted by noise:

$$\begin{aligned} d_1(k) &= 5 + 0.1\, e_1(k) \\ d_2(k) &= 1 + 0.1\, e_2(k) \\ t_2(k) &= 1 + 0.1\, e_3(k) \end{aligned}$$

where $e(k) = \begin{bmatrix} e_1(k) & e_2(k) & e_3(k) \end{bmatrix}^T$ is a random signal with probability density function

$$p(e) = \begin{cases} 1/8 & \text{for} \quad \max\limits_{i=1,2,3}(|e_i|) \le 1 \\ 0 & \text{for} \quad \max\limits_{i=1,2,3}(|e_i|) > 1 \end{cases} \tag{15}$$

Assume that the initial state is equal to $x(0) = [0\ 6]^T$, the due date signal is given by $r(k) = 4 + 6 \cdot k$ and the cost criterion (8) is optimized for $N_{\mathrm{p}} = 3$, $N_{\mathrm{c}} = 2$ and $\lambda = 0.1$. With the choice of the cost criterion (5)-(6), we can rewrite the stochastic MPC problem into a convex optimization problem. For the computation of the cost criterion we use a Taylor approximation with $M = 0, 1, 2, 3$. The optimal input sequence is computed for $k = 1, \ldots, 40$, and for each k, the first element $u(k)$ of the sequence $\tilde{u}(k)$ is applied to the perturbed system (due to the receding horizon strategy). In the experiment, the true system is simulated for a random sequence $e(k)$, $k = 1, \ldots, 40$, satisfying probability density function (15).

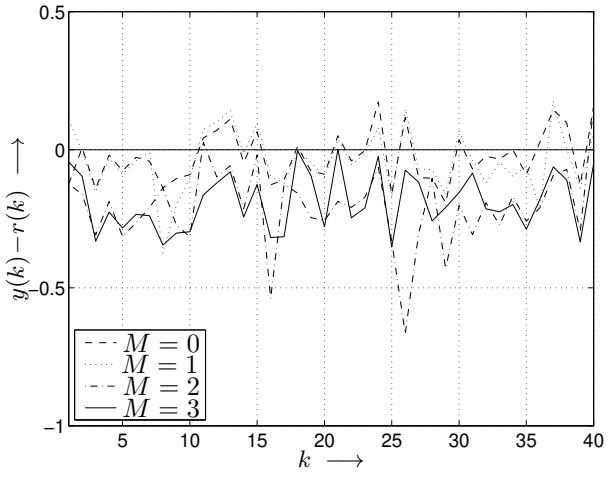

Figure 2: The difference $y(k) - r(k)$ between the output date signal y and the due date signal r for the M-th order approximation, $M \in \{0, 1, 2, 3\}$.

Figure 2 gives the difference between the output date signal y and the due date signal r for approximations

with $M \in \{0, 1, 2, 3\}$. The zero-th order approximation is in fact equal to the case where no disturbance is taken into account. We see that the scheme leads to a frequent violation of the due dates (i.e. the difference signal $y(k) - r(k)$ is frequently positive). The same holds for the first order approximation. The second order approximation already gives a better result and in the third-order case nearly all due dates are satisfied. In Table 1 the (scaled) CPU-times are given for the computation of the cost-criterion and its subgradient for $M \in \{0, 1, 2, 3\}$.

	M=0	M=1	M=2	M=3
cpu-time	1	3.07	203	1104

Table 1: (scaled) CPU-times for different levels in approximation

From Table 1 we see that computation-time grows dramatically with increasing M. Depending on the application and the computation-interval, available between two events, we can choose the level of approximation. In general, the above trade-off will give us the best possible approximation of the optimal solution, given the constraints in computation time.

6 Discussion

We have discussed complexity reduction in MPC for max-plus-linear discrete event systems with stochastic uncertainties. From the MPC framework, a convex optimization problem results, if the constraints are a nondecreasing function of the output. With an increasing number of stochastic variables, the computational complexity of the optimization problem increases dramatically due to the numerical integrations required to evaluate the objective function. To tackle this increase of complexity, we use the method of variability expansion. The key idea of this method is to introduce a parameter θ, which controls the level of stochasticity in the system. Based on a Taylor expansion of this parameter θ, good approximations for the expectations of cost-criterion and constraints can be computed, which leads to a significant reduction of the computational complexity of our approach.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity*. New York: John Wiley & Sons, 1992.

[2] J. Cardoso, R. Valette, and D. Dubois, "Possibilistic Petri nets," *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, vol. 29, no. 5, pp. 573–582, 1999.

[3] C.G. Cassandras and S. Lafortune, "Introduction to Discrete Event Systems", Kluwer Academic Publishers, Boston, 1999.

[4] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems*. Berlin, Germany: Springer-Verlag, 1979.

[5] B. De Schutter and T.J.J. van den Boom, "Model predictive control for max-plus-linear discrete event systems," *Automatica*, vol. 37, no. 7, pp. 1049–1056, July 2001.

[6] B. De Schutter and T.J.J. van den Boom, "Model predictive control for max-plus-linear systems," in *Proceedings of the 2000 American Control Conference*, Chicago, Illinois, pp. 4046–4050, June 2000.

[7] C.E. García, D.M. Prett, and M. Morari, "Model predictive control: Theory and practice — A survey," *Automatica*, vol. 25, no. 3, pp. 335–348, May 1989.

[8] B. Heidergott, "Variablity Expansion for Performance Characteristic of (Max,Plus)-Linear Systems," Workshop on Discrete Event Systems (WODES), Zaragoza, Spain, 2-4 October, 2002.

[9] Y.C. Ho, "Discrete Event Dynamic Systems: Analyzing Complexity and Performance in the Modern World", IEEE Press, Piscataway, New Jersey, 1992.

[10] F. Lin, "Robust and adaptive supervisory control of discrete event systems," *IEEE Transactions on Automatic Control*, vol. 38, no. 12, pp. 1848–1852, Dec. 1993.

[11] J.M. Maciejowski. *Predictive control with constraints*. Prentice Hall, Pearson Education Limited, Harlow, UK, 2002.

[12] Y. Nesterov and A. Nemirovskii, "Interior-Point Polynomial Algorithms in Convex Programming", SIAM, Philadelphia, Pennsylvania, 1994.

[13] S.J. Park and J.T. Lim, "Fault-tolerant robust supervisor for discrete event systems with model uncertainty and its application to a workcell," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 2, pp. 386–391, April 1999.

[14] T.J.J. van den Boom and B. De Schutter. "MPC for perturbed max-plus-linear systems: A stochastic approach", In *Proceedings of the 2001 Conference on Decision and Control, Orlando, FL, USA*, pp. 4535–4540, 2001.

[15] T.J.J. van den Boom and B. De Schutter. "MPC for perturbed max-plus-linear systems", *Systems and Control Letters*, vol. 45, no. 1, pp. 21–33, January 2002.

[16] S.J. Wright, "Primal-Dual Interior Point Methods", SIAM, Philadephia, Pennsylvania, 1997.

[17] S. Young and V.K. Garg, "Model uncertainty in discrete event systems," *SIAM Journal on Control and Optimization*, vol. 33, no. 1, pp. 208–226, Jan. 1995.