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Model predictive control for perturbed continuous piecewise affine systems with bounded disturbances

I. Necoara, B. De Schutter, T.J.J. van den Boom, J. Hellendoorn

Abstract— **Continuous piecewise-affine systems are a powerful tool for describing or approximating both nonlinear and hybrid systems. In this paper we extend the Model Predictive Control (MPC) framework for continuous piecewise-affine systems that we have developed previously to deterministic uncertainty. We show that the resulting MPC optimization problem can be transformed into a sequence of linear optimization problems (LP), which can be solved very efficiently.**

I. INTRODUCTION

Recently, hybrid systems have attracted the interest of both academia and industry due to their ability to model the interaction between continuous and logic components. In particular, several authors have studied a subclass of hybrid systems, piecewise affine systems (PWA) [1], [6], [10], since they represent a powerful tool for approximating nonlinear systems with arbitrary accuracy and since a rich class of hybrid systems can be described by PWA systems. PWA systems are defined by partitioning the state space in a finite number of polyhedral regions and associating to each region a different affine dynamic. Another subclass of hybrid systems is the class of max-min-plus-scaling (MMPS) systems, the evolution equations of which can be described using the operations maximization, minimization, additions and scalar multiplication. Using the results of [4], [8] we can prove that *continuous* PWA systems are equivalent with MMPS systems. In this paper we consider MMPS systems, and thus also continuous PWA systems.

The relation between PWA and MMPS systems is useful for the investigation of structural properties of PWA systems such as observability and controllability [10], but also in designing controller schemes like model predictive control (MPC) [2], [4]. Using the work of [4] in which MPC for MMPS (and equivalently for continuous PWA) systems for the deterministic noise-free case without modeling errors is proposed, we further extend MPC for the cases with noise and modeling errors.

An important difference between MPC and some other control methods is the explicit use of a prediction model. Because the models play such an important role in MPC, we must also take into account noise and error modeling when we implement MPC. Ignoring the noise can lead to a bad tracking or even to unstable closed-loop behavior. Also uncertainty in the modeling phase leads to errors in the system equations. Therefore, both modeling errors and noise and disturbances perturb the system by introducing uncertainty in the system equations. In general it is difficult to distinguish them one from another, but usually fast changes will be considered as noise and disturbance, whereas slow changes or permanent errors are considered as modeling errors. In this paper both are treated in one single framework. We model noise and disturbances by including extra additive terms in the system equations for MMPS systems. We consider the deterministic case, therefore the uncertainty is bounded. Note that there are some results in the literature on noise and modeling errors for some classes of hybrid systems (see [7], [11], [12]) but to the authors' best knowledge this is the first time that such an approach is used for the MMPS framework.

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This paper is organized as follows. A brief review of PWA and MMPS systems is given and MPC for them as it was developed in [4] is presented in Section II. Further we show that the optimal solution of a multi-parametric linear programming problem is an MMPS function of the parameter. In Section III we discuss MPC for perturbed MMPS systems. We obtain an efficient MPC method which is based on minimizing the worst-case cost criterion. More specifically, we prove that the optimization problem at each step of MPC can be transformed into a sequence of linear programming problems (LP), for which efficient solution methods exist. We conclude with an worked example in Section IV.

II. PRELIMINARIES

A. Equivalence between Continuous PWA and MMPS systems

Definition 2.1: A vector-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be a *continuous* PWA function if there exists a finite family $\mathcal{C}_1, ..., \mathcal{C}_N$ of closed polyhedral regions that covers $\mathbb{R}^n$ and for each $i \in \{1, ..., N\}$, $j \in \{1, ..., m\}$, each component f_j of f can be expressed as $f_j(x) = \alpha_{i,j}^T\, x + \beta_{i,j}$ for any $x \in \mathcal{C}_i$, with $\alpha_{i,j} \in \mathbb{R}^n$, $\beta_{i,j} \in \mathbb{R}$.

Note that because the polyhedral regions $\mathcal{C}_i$ are closed, it results that f is indeed continuous (i.e. each component of f is continuous), because f is continuous on the boundary between any two regions.

A *continuous* PWA system in state space representation is a system of the form:

$$x(k+1) = \mathcal{P}_x(x(k), u(k)) \tag{1}$$

$$y(k) = \mathcal{P}_y(x(k), u(k)), \tag{2}$$

where $\mathcal{P}_x$ and $\mathcal{P}_y$ are continuous PWA functions, with input u, output y and state x.

Definition 2.2: A scalar-valued MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ is generated from the set of affine functions by the recursive relation:

$$f(x) = \max(f_k(x), f_l(x)) | \min(f_k(x), f_l(x)) \tag{3}$$

where $f_k, f_l : \mathbb{R}^n \to \mathbb{R}$ are again MMPS functions, and the symbol $|$ stands for *"or"*. For vector-valued MMPS functions the above statements hold component-wise.

An MMPS system is written in the following form:

$$x(k+1) = \mathcal{M}_x(x(k), u(k)) \tag{4}$$

$$y(k) = \mathcal{M}_y(x(k), u(k)), \tag{5}$$

where $\mathcal{M}_x$, $\mathcal{M}_y$ are vector-valued MMPS functions.

Proposition 2.3: [4] Any scalar-valued MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ can be written into *min-max canonical form*

$$f(x) = \min_{j=1,\ldots,\hat{l}} \max_{i \in T_j} (\alpha_{i,j}^T x + \beta_{i,j}), \tag{6}$$

or into *max-min canonical form*

$$f(x) = \max_{j=1,\ldots,l} \min_{i \in S_j} (\gamma_{i,j}^T x + \delta_{i,j}), \tag{7}$$

for some integers $\hat{l}$, l, N; $\{S_j\}_{j=1}^{l}$ and $\{T_j\}_{j=1}^{\hat{l}}$ each are a family of incomparable (with respect to $\subseteq$) subsets of $\{1, ..., N\}$ and $\alpha_{i,j}, \gamma_{i,j} \in \mathbb{R}^n$, $\beta_{i,j}, \delta_{i,j} \in \mathbb{R}$.

Proposition 2.4: [8] Any continuous PWA function can be written as MMPS function and vice-versa.

Corollary 2.5: Continuous PWA systems and MMPS systems are equivalent in the sense that for a given continuous PWA model there exists an MMPS model (and vice-versa) such that the input-output behavior of both models coincides.

Note that the above propositions imply that any continuous PWA system (1)–(2) can be written in the form (4)–(5), with each component of $\mathcal{M}_x$ and $\mathcal{M}_y$ in min-max canonical form (6) or max-min canonical form (7).

B. MPC for MMPS systems

In this section we give a short overview of the main results of [4]. Note that in that paper modeling errors or noise and disturbance in the model are not included.

In MMPS-MPC we define for each sample step k, a cost criterion

$$J(k) = J_{\text{out}}(k) + \lambda J_{\text{in}}(k),$$

over the period $[k, k + N_{\text{p}} - 1]$, where N_{p} is the prediction horizon and $\lambda > 0$ is a weighting factor. By optimizing this cost criterion we obtain an optimal input sequence $u^*(k), ..., u^*(k + N_{\text{p}} - 1)$, but we apply only the first input sample $u^*(k)$ according to a receding horizon strategy. At the next sample step the whole procedure is repeated.

Now we explain in more details how the MPC for MMPS systems can be implemented efficiently in the case when the cost criterion $J(k)$ is an MMPS function of the input. Assuming that at each step k, the state $x(k)$ can be measured or predicted, we can make an estimation of the output of the model (4)–(5):

$$\hat{y}(k+j|k) = \mathcal{M}_j(x(k), u(k), ..., u(k+j)) \tag{8}$$

at sample step $k + j$ for $j = 0, ..., N_{\text{p}} - 1$ where $\mathcal{M}_j$ is an MMPS function of $x(k), u(k), ..., u(k + j)$.

Our goal is to track a reference signal r. Define $\tilde{u}(k) = [u^T(k), ..., u^T(k + N_{\text{p}} - 1)]^T$, $\tilde{y}(k) = [\hat{y}^T(k|k), ..., \hat{y}^T(k + N_{\text{p}} - 1|k)]^T$, $\tilde{r}(k) = [r^T(k), ..., r^T(k + N_{\text{p}} - 1)]^T$.

We consider only linear constraints on the input[1]

$$P(k)\tilde{u}(k) + q(k) \le 0. \tag{9}$$

In practical situations, such constraints occur when we have to guarantee that the input signal must stay within certain bounds[2], e.g. $m(k+j) \le u(k+j) \le M(k+j)$, where $m(\cdot)$ and $M(\cdot)$ are the lower and upper bounds respectively.

As output cost functions we will take:

$$J_{\text{out},1}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_1$$
$$J_{\text{out},\infty}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_\infty, \tag{10}$$

which reflect the tracking error, and are MMPS functions of $x(k), \tilde{u}(k), \tilde{r}(k)$. As input cost function one could take:

$$J_{\text{in},1}(k) = \|\tilde{u}(k)\|_1, \quad J_{\text{in},\infty}(k) = \|\tilde{u}(k)\|_\infty, \tag{11}$$

which are also MMPS functions of $\tilde{u}(k)$. Or we can use any other output or input cost criterion that can be expressed as an MMPS function of $\tilde{u}(k)$.

We introduce a control horizon N_{c} such that

$$u(k+j) = u(k+N_{\text{c}}-1) \quad \text{for} \quad j = N_{\text{c}}, ..., N_{\text{p}}-1, \tag{12}$$

to decrease the number of degrees of freedom for $\tilde{u}(k)$ and thus we obtain a reduction in computational effort but this also makes the control signal smooth and the controller more robust. Note that (12) can also be expressed in the form (9).

Since after substitution of $\tilde{y}(k)$ using (8), the cost function $J(k)$ is an MMPS function of $\tilde{u}(k)$ which can be written in min-max canonical form, it follows that at each sample step k we have to solve an optimization problem of the following form

$$\min_{\tilde{u}(k)} \min_{j=1,\ldots,\hat{l}} \max_{i \in T_j} (\alpha_{i,j}^T \tilde{u}(k) + \beta_{i,j}(k)) \tag{13}$$
$$\text{subject to } P(k)\tilde{u}(k) + q(k) \le 0,$$

so for any $j = 1, ..., \hat{l}$ we obtain a linear programming problem:

$$\min_{\tilde{u}(k), t(k)} t(k) \tag{14}$$
$$\text{subject to} \begin{cases} P(k)\tilde{u}(k) + q(k) \le 0 \\ t(k) \ge \alpha_{i,j}^T \tilde{u}(k) + \beta_{i,j}(k)), \ i \in T_j. \end{cases}$$

[1]We can take into account also constraints on states, but in this case the number of optimization problems that must be solved increases.

[2]Actually we can also allow state or output constraints provided that after substitution they lead to linear or convex constraints in input, because the algorithm used below is based on cutting plane method for convex optimization, which can deal also with convex constraints.

The linear programming problems are easy to solve using the simplex method or an interior point algorithm [9]. Let $[t^*(k)\ \tilde{u}^{*T}_{(j)}(k)]^T$ be the optimal solution of (14). To obtain the solution of (13), we solve (14) for $j = 1, ..., \hat{l}$ and afterward we select the $\tilde{u}^*_{(j)}(k)$ for which $\max_{i \in T_j}(\alpha^T_{i,j}(k)\tilde{u}^*_{(j)}(k) + \beta_{i,j}(k))$ is the smallest.

C. Multi-parametric linear programming

The following proposition characterizes the solution to a multi-parametric linear programming problem (mp-lpp) defined in the following way:

$$\max\ c^T x \tag{15}$$

$$\text{subject to} \quad Sx \leq q + U\theta, \tag{16}$$

where $x \in \mathbb{R}^n$ is the optimization variable, $\theta \in \Theta = \{\theta \in \mathbb{R}^s : W\theta \leq \omega\} \subseteq \mathbb{R}^s$ is the vector of parameters, $S \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, and $U \in \mathbb{R}^{m \times s}$.

For simplicity we assume that for any $\theta \in \Theta$ (where Θ is a closed polyhedron), the problem (15)–(16) has a finite optimal solution. Let $V^*(\theta)$ denote the maximum value of the objective function in problem (15)–(16) and $x^*(\theta)$ the optimizer[3] related to $V^*(\theta)$ for any $\theta \in \Theta$.

Proposition 2.6: With the above notations, the function $V^* : \Theta \to \mathbb{R}$ is a concave MMPS function (i.e. only a *min* expression). Furthermore, there exists an MMPS function $X^* : \Theta \to \mathbb{R}^n$ such that $X^*(\theta) \in x^*(\theta)$ for all $\theta \in \Theta$.

Proof: In [5] it is proved that $V^* : \Theta \to \mathbb{R}$ is a concave PWA function and that there exists a continuous PWA function $X^* : \Theta \to \mathbb{R}^n$ such that $X^*(\theta) \in x^*(\theta)$ for all $\theta \in \Theta$. It is well known that a concave (convex) PWA function is also continuous [3]. Therefore, V^* is a continuous PWA function, and also an MMPS function (according to Proposition 1.4). Using the same arguments we can also prove that X^* is an MMPS function. $\diamondsuit$

The reader is referred to [3] for a geometric algorithm for computing the solution to an mp-lpp.

[3] In general, $x^*(\theta)$ is set-valued.

III. MPC FOR PERTURBED CONTINUOUS PWA OR MMPS SYSTEMS

A. Perturbed continuous PWA or MMPS systems

In this section we extend the *continuous PWA* (or equivalently the *MMPS*) deterministic model (1)–(2) or (4)–(5), without noise, to take also the uncertainty into account.

The MPC method is based on a model of the system; the prediction of the future behavior is made using the respective model. Therefore we must also take into account the uncertainty when we implement MPC. If we ignore the noise we can get a bad tracking or even an unstable closed-loop behavior. Uncertainty in the modeling of the plant leads to errors in the system equations. Therefore, both modeling errors and noise and disturbances perturb the system by introducing uncertainty in the system equations. In the sequel both are treated in the same framework.

As in conventional linear systems, we model the noise and disturbances by including a noise term in the system equations for continuous PWA systems. Hence, we consider the *perturbed continuous PWA* model:

$$x(k+1) = \mathcal{P}_x(x(k), u(k), e(k)) \tag{17}$$

$$y(k) = \mathcal{P}_y(x(k), u(k), e(k)), \tag{18}$$

where $\mathcal{P}_x$ and $\mathcal{P}_y$ are continuous vector-valued PWA functions and the uncertainty caused by disturbances and errors in the estimation of the real system is gathered in the uncertainty vector $e(k)$. We assume that this uncertainty is included in a bounded polyhedral set $\mathcal{E} = \{e \in \mathbb{R}^s : Se \leq q\}$ and if consecutive noise samples $e(k)$,...,$e(k+j)$ are related, we assume that this relation is linear (e.g. a system of linear equalities or inequalities).

Using the equivalence between continuous PWA and MMPS systems, the perturbed continuous PWA model (17)–(18) can be also written as a MMPS system:

$$x(k+1) = \mathcal{M}_x(x(k), u(k), e(k)) \tag{19}$$

$$y(k) = \mathcal{M}_y(x(k), u(k), e(k)), \tag{20}$$

where $\mathcal{M}_x$, $\mathcal{M}_y$ are vector-valued MMPS functions.

We assume that at each step k of MPC, the state $x(k)$ is available (can be measured or estimated) and we gather the uncertainty over the interval $[k, k + N_\mathrm{p} - 1]$ in the vector $\tilde{e}(k) = [e^T(k), ..., e^T(k + N_\mathrm{p} - 1)]^T \in \tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}}$, according to our assumption, is a bounded polyhedral set. Then it is easy to see that the prediction $\hat{y}(k + j|k)$ of the future output for the system (19)–(20) can be written in MMPS form, for $j = 0, ..., N_\mathrm{p} - 1$.

Using as cost criterion a combination of (10) and (11):

$$J(k) = J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k)$$

and keeping in mind that all these cost criteria are MMPS expressions[4], we get a min-max canonical form of $J(k)$:

$$J(\tilde{e}(k), \tilde{u}(k), x(k)) = \min_{j=1,...,\hat{l}} \max_{i \in T_j}(\alpha_{i,j}x(k) + \beta_{i,j}\tilde{u}(k) + \gamma_{i,j}\tilde{e}(k) + \delta_{i,j}), \tag{21}$$

or a max-min canonical representation:

$$J(\tilde{e}(k), \tilde{u}(k), x(k)) = \max_{j=1,...,l} \min_{i \in S_j}(\bar{\alpha}_{i,j}x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \bar{\gamma}_{i,j}\tilde{e}(k) + \bar{\delta}_{i,j}). \tag{22}$$

B. Worst-case MMPS-MPC

In this section we study MPC for perturbed MMPS systems when $e(k)$ is a bounded uncertainty. We want to minimize an MMPS cost criterion $J(k) = J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k)$ subject to some constraints. As we said, we consider only linear constraints on input, i.e. constraints of the form (9).

The *worst-case MMPS-MPC problem* at step k is defined:

$$\min_{\tilde{u}(k)} \max_{\tilde{e}(k) \in \tilde{\mathcal{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)) \tag{23}$$

$$\text{subject to } P(k)\tilde{u}(k) + q(k) \leq 0, \tag{24}$$

[4] Recall that $|x| = \max(x, -x)$ for $x \in \mathbb{R}$.

where $J(\cdot)$ is given by (21) or (22).
For a given $\tilde{u}(k), x(k)$ we define the *inner* worst-case MMPS-MPC problem

$$\max_{\tilde{e}(k)\in\tilde{\mathcal{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)). \tag{25}$$

We denote[5]

$$\tilde{e}^*(\tilde{u}(k), x(k)) = \arg\max_{\tilde{e}(k)\in\tilde{\mathcal{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)), \tag{26}$$

$$J^*(\tilde{u}(k), x(k)) = J(\tilde{e}^*(\tilde{u}(k), x(k)), \tilde{u}(k), x(k)). \tag{27}$$

Proposition 3.1: For a given $\tilde{u}(k)$ and $x(k)$, $\tilde{e}^*(\tilde{u}(k), x(k))$ given by (26) can be computed using a sequence of linear programming problems.
Proof: Because the uncertainty $e(k)$ is in a bounded polyhedral set $\mathcal{E}$, $\tilde{e}(k)$ will also be in a bounded polyhedral set: $\tilde{\mathcal{E}} = \{\tilde{e}(k) : \tilde{S}\tilde{e}(k) \leq \tilde{q}\}$.

We determine for any fixed $[\tilde{u}^T(k)\ x(k)]$ the optimal $\tilde{e}^*(\tilde{u}(k), x(k))$, using the *max-min canonical form* (22) of $J(\cdot)$, by solving the following optimization problem:

$$\begin{aligned}\max_{\tilde{e}(k)} \max_{j=1,\ldots,l} \min_{i\in S_j} (\bar{\alpha}_{i,j} x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \\ + \bar{\gamma}_{i,j}\tilde{e}(k) + \bar{\delta}_{i,j}) \\ \text{subject to } \tilde{S}\tilde{e}(k) \leq \tilde{q},\end{aligned} \tag{28}$$

which is equivalent with:

$$\begin{aligned}\max_{j=1,\ldots,l} \max_{\tilde{e}(k)} \min_{i\in S_j} (\bar{\alpha}_{i,j} x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \\ + \bar{\gamma}_{i,j}\tilde{e}(k) + \bar{\delta}_{i,j}) \\ \text{subject to } \tilde{S}\tilde{e}(k) \leq \tilde{q}.\end{aligned} \tag{29}$$

Now for any $j = 1, \ldots, l$ we have to solve the following optimization problem:

$$\begin{aligned}\max_{\tilde{e}(k)} \min_{i\in S_j} (\bar{\alpha}_{i,j} x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \\ + \bar{\gamma}_{i,j}\tilde{e}(k) + \bar{\delta}_{i,j}) \\ \text{subject to } \tilde{S}\tilde{e}(k) \leq \tilde{q},\end{aligned}$$

which is equivalent with the following linear programming problem:

$$\max_{\tilde{e}(k), t_{(j)}(k)} t_{(j)}(k) \tag{30}$$

subject to

$$\begin{cases} t_{(j)}(k) \leq \bar{\alpha}_{i,j} x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \\ \qquad + \bar{\gamma}_{i,j}\tilde{e}(k) + \bar{\delta}_{i,j} \quad \text{for each } i \in S_j \\ \tilde{S}\tilde{e}(k) \leq \tilde{q}. \end{cases} \tag{31}$$

To obtain the solution of (28) we solve (30)–(31) for each $j = 1, \ldots, l$, with the optimal solution $[t^*_{(j)}(\tilde{u}(k), x(k))\ \tilde{e}^{*T}_{(j)}(\tilde{u}(k), x(k))]^T$ and then we select as $\tilde{e}^*(\tilde{u}(k), x(k))$, the optimal solution $\tilde{e}^*_{(j)}(\tilde{u}(k), x(k))$ for which $\min_{i\in S_j}(\bar{\alpha}_{i,j}x(k) + \bar{\beta}_{i,j}\tilde{u}(k) + \bar{\gamma}_{i,j}\tilde{e}^*_{(j)}(\tilde{u}(k), x(k)) + \bar{\delta}_{i,j})$ is the largest. $\diamondsuit$

[5]Note that in general $\tilde{e}^*(\tilde{u}(k), x(k))$ may be set-valued, but as we will use Proposition 2.6, this does not matter.

Remark 3.2 We assume that the feasible set of the states is a closed polyhedral set X and we denote with $U = \{\tilde{u}(k) : P(k)\tilde{u}(k) + q(k) \leq 0\}$ which is also a closed polyhedron.

Remark 3.3 Note that in practice the input $\tilde{u}(k)$ will always be bounded. Furthermore, we only consider the behavior of the system over finite horizons. As a consequence, the state $x(k)$ will also be bounded for any k. This implies that for any $[\tilde{u}^T(k)\ x^T(k)] \in U \times X$ (closed polyhedron), the multi-parametric linear programming problem (25) has a finite optimal solution.

Proposition 3.4: With the notations (26)–(27), $J^* : U \times X \to \mathbb{R}$ is an MMPS function and $\tilde{e}^* : U \times X \to \mathbb{R}^s$ is a PWA function.
Proof: For each $j = 1, ..., l$ we denote with

$$[t^*_{(j)}(\tilde{u}(k), x(k))\ \ \tilde{e}^{*T}_{(j)}(\tilde{u}(k), x(k))]^T$$

the optimal solution of the mp-lpp (30)–(31), with the parameter $\theta = [\tilde{u}^T(k)\ x^T(k)]^T \in \Theta$ with $\Theta = U \times X$ a closed polyhedral set (see Remark 3.2). From Proposition 2.6 we know that $t^*_{(j)}(\cdot, \cdot)$ is an MMPS function of the argument $(\tilde{u}(k), x(k))$. But

$$J^*(\tilde{u}(k), x(k)) = \max_{j=1,\ldots,l} (t^*_{(j)}(\tilde{u}(k), x(k))).$$

We obtain thus directly an MMPS expression of $J^*(\cdot, \cdot)$. Furthermore $\tilde{e}^*(\tilde{u}(k), x(k)) = \tilde{e}^*_{(j)}(\tilde{u}(k), x(k))$ if $t^*_{(j)}(\tilde{u}(k), x(k)) \geq t^*_{(i)}(\tilde{u}(k), x(k))$ for any $i \in \{1, ..., l\} \setminus \{j\}$. But each $\tilde{e}^*_{(j)}(\cdot, \cdot)$ is an MMPS function, and therefore a continuous PWA function. This implies that $\tilde{e}^*(\cdot, \cdot)$ is a PWA function, but not necessarily continuous. $\diamondsuit$

The *outer* worst-case MMPS-MPC problem is now defined as:

$$\min_{\tilde{u}(k)} J^*(\tilde{u}(k), x(k)) \tag{32}$$

$$\text{subject to } P(k)\tilde{u}(k) + q(k) \leq 0. \tag{33}$$

where we assume that at sample step k, the state $x(k)$ is given.

Proposition 3.5: Given $x(k)$, the outer worst-case MMPS-MPC problem can be solved using a sequence of LPs.
Proof: From Proposition 3.4 we know that $J^* : U \times X \to \mathbb{R}$ is an MMPS function. Therefore it can be written in the following min-max canonical form

$$J^*(\tilde{u}(k), x(k)) = \min_{j=1,\ldots,\hat{l}} \max_{i\in T_j} (\alpha_{i,j} x(k) + \beta_{i,j}\tilde{u}(k) + \delta_{i,j}).$$

Then, the outer worst-case MMPS-MPC problem (32)–(33) can be written as

$$\begin{aligned}\min_{\tilde{u}(k)} \min_{j=1,..,\hat{l}} \max_{i\in T_j} (\alpha_{i,j} x(k) + \beta_{i,j}\tilde{u}(k) + \delta_{i,j}) \\ \text{subject to } P(k)\tilde{u}(k) + q(k) \leq 0.\end{aligned}$$

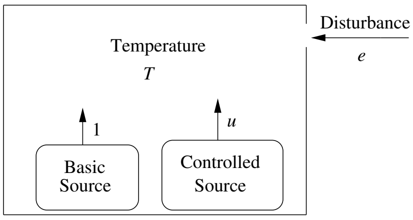

Fig. 1. Temperature in a room.

For any $j = 1, ..., \hat{l}$ we must solve the following LP:

$$\min_{\tilde{u}(k),t_{(j)}} t_{(j)}$$
$$\text{subject to} \qquad (34)$$
$$\begin{cases} t_{(j)} \geq \alpha_{i,j}x(k) + \beta_{i,j}\tilde{u}(k) + \delta_{i,j}, \text{ for each } i \in T_j \\ P(k)\tilde{u}(k) + q(k) \leq 0. \end{cases}$$

To obtain the solution of (32)–(33), we solve (34), obtaining the optimal solution $[t^*_{(j)}(x(k)) \;\; \tilde{u}^{*T}_{(j)}(x(k))]^T$, for each $j = 1, ..., \hat{l}$ and then we select the optimal $\tilde{u}^*(x(k))$ as the optimal solution $\tilde{u}^*_{(j)}(x(k))$ for which $\max_{i\in T_j}(\alpha_{i,j}x(k) + \beta_{i,j}\tilde{u}^*_{(j)}(x(k)) + \delta_{i,j})$ is the smallest. $\diamondsuit$

Corollary 3.6: According to this algorithm, the worst-case MMPS-MPC problem can be solved using a sequence of linear programming problems. Moreover the associate controller is a PWA function of the argument, the state $x(k)$.

Proof: In order to solve the problem (23)–(24), first we look for the worst-case uncertainty $\tilde{e}(k)$ as a function of $\tilde{u}(k), x(k)$ (Proposition 3.1) while in the second step we want to find the optimal input $\tilde{u}(k)$ corresponding to the worst-case uncertainty (Proposition 3.5). First step is computed off-line. Second step can be solved using a sequence of linear programming problems according to Proposition 3.5.

For the second part of the corollary, we consider the multi-parametric linear programming problem (34), with the parameter $x(k) \in X$ with X a polyhedral set. Then the optimal solution $[t^*_{(j)}(\cdot) \; u^{*T}_{(j)}(\cdot)]^T$ is an MMPS function of the argument $x(k)$ (according to Proposition 2.6). Therefore $\tilde{u}^*_{(j)}(\cdot)$ is a PWA function. But

$$\tilde{u}^*(k) = \tilde{u}^*_{(j)}(x(k)) \quad \text{if} \quad t^*_{(j)}(x(k)) \leq t^*_{(i)}(x(k)),$$

for $i \in \{1, ..., \hat{l}\} \setminus \{j\}$. So, the worst-case MMPS-MPC controller $u^*(k)$ is a PWA function of the argument $x(k)$. $\diamondsuit$

Note that the reduction to canonical form is computationally intensive, but can be done off-line (for both the inner and outer worst-case MMPS-MPC problem). Furthermore, the complexity of the reduction process can also be reduced by already eliminating redundant terms during the intermediate steps of the transformations. Also note that this elimination of the redundant terms can be done off-line.

IV. EXAMPLE

Now we present an example for which we apply the above method. Consider a room with a basic heat source and an additional controlled heat source (see Figure 1). Let u be the contribution to the increase in room temperature per time unit caused by the controlled heat source (so $u \geq 0$). For the basic heat source, this value is assumed to be constant and equal to 1. The temperature in the room is assumed to be uniform and obeys the first-order differential equation

$$\dot{T}(t) = \alpha(T(t))T(t) + u(t) + 1 + e_1(t) \;,$$

the modeling error being gathered in scalar variable e_1. We assume that the temperature coefficient has the following piecewise constant form:

$$\alpha(T) = \begin{cases} 1 & \text{if} \quad T < 0 \\ -1 & \text{if} \quad T \geq 0. \end{cases}$$

We assume that the temperature is measured, but the measurement is noisy: $y(t) = T(t) + e_2(t)$.

Using the Euler discretization scheme, with a sample time of 1 time unit and denoting the state $x(k) = T(k \cdot 1)$, we get the following continuous discrete-time PWA system:

$$x(k+1) = \begin{cases} 2x(k) + u(k) + e_1(k) + 1 & \text{if } x(k) < 0 \\ u(k) + e_1(k) + 1 & \text{if } x(k) \geq 0 \end{cases} \qquad (35)$$

$$y(k) = x(k) + e_2(k) \qquad (36)$$

Let $-2 \leq e_1(k), e_2(k) \leq 2$, $e_1(k) + e_2(k) \leq 1$, i.e. the deterministic uncertainty is given by the bounded polyhedron

$$\mathcal{E} = \{[e_1 \; e_2]^T : -2 \leq e_1(k), e_2(k) \leq 2 \\ e_1(k) + e_2(k) \leq 1\}.$$

The equivalent MMPS representation of (35)–(36) is:

$$x(k+1) = \min\{2x(k) + u(k) + e_1(k) + 1, \\ u(k) + e_1(k) + 1\} \qquad (37)$$
$$y(k) = x(k) + e_2(k). \qquad (38)$$

Because at time step k the input $u(k)$ has no influence on $y(k)$, we take $N_{\rm p} = 3, N_{\rm c} = 2$, $\tilde{y}(k) = [\hat{y}(k+1|k) \; \hat{y}(k+2|k)]^T$, $\tilde{r}(k) = [r(k+1) \; r(k+2)]^T$, $\tilde{u}(k) = [u(k) \; u(k+1)]^T$. Let the uncertainty vector $e(k)$ be $e(k) = [e_1(k) \; e_2(k+1)]^T$. Therefore, $\tilde{e}(k) = [e^T(k) \; e^T(k+1)]^T$.

We consider the following constraints on the input[6]:

$$-4 \leq u(k+1) - u(k) \leq 4 \quad \text{and} \quad u(k) \geq 0 \text{ for all } k.$$

As cost criterion we take

$$J(k) = J_{\text{out},\infty}(k) + \lambda J_{\text{in},1}(k) = \\ \|\tilde{y}(k) - \tilde{r}(k)\|_\infty + \lambda\|\tilde{u}(k)\|_1 \qquad (39)$$

The first term of $J(k)$ expresses the fact that we penalize the maximum difference between the reference and the output signal, while the second term penalizes the absolute value of the control effort.

[6]Because we have only heating: $u(k) \geq 0$ and we assume that the rate of heating is bounded.

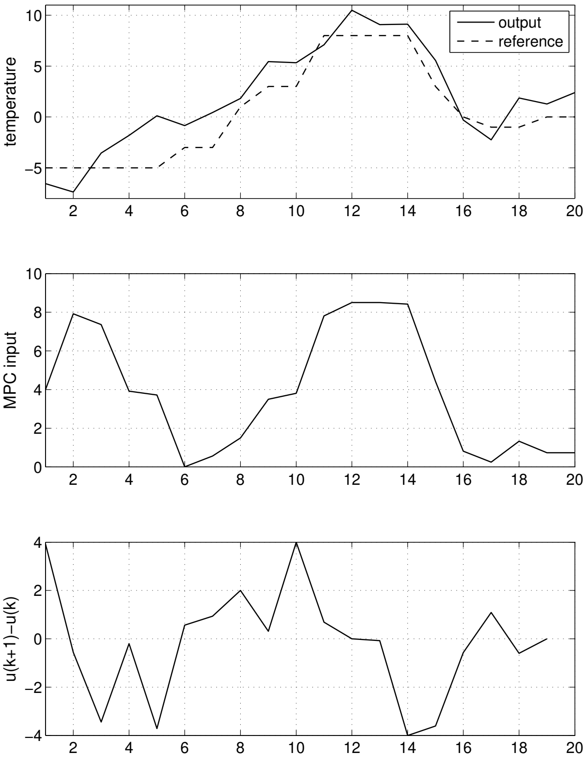

Fig. 2. Illustration of the worst-case MPC for a perturbed MMPS system.

Because $u(k) \geq 0$, we have $\|u(k)\|_1 = u(k)$ and therefore we get the following formula for $J(k)$:

$$\begin{aligned} J(k) = \max\{&y(k+1) - r(k+1) + \lambda u(k) + \lambda u(k+1), \\ &r(k+1) - y(k+1) + \lambda u(k) + \lambda u(k+1), \\ &y(k+2) - r(k+2) + \lambda u(k) + \lambda u(k+1), \\ &r(k+2) - y(k+2) + \lambda u(k) + \lambda u(k+1)\}. \end{aligned}$$

Now we have

$$\begin{aligned} y(k+1) = \min\{&2x(k) + u(k) + e_1(k) + e_2(k+1) + 1, \\ &u(k) + e_1(k) + e_2(k+1) + 1\} \end{aligned}$$

and

$$\begin{aligned} y(k+2) = \min\{&4x(k) + 2u(k) + u(k+1) + 2e_1(k) + \\ &e_1(k+1) + e_2(k+2) + 3, 2u(k) + u(k+1) + 2e_1(k) \\ &+ e_1(k+1) + e_2(k+2) + 3, \\ &u(k+1) + e_1(k+1) + e_2(k+2) + 1\}. \end{aligned}$$

Therefore, we can write now $J(k)$ in max-min canonical form:

$$J(k) = \max\{\min\{t_1, t_2\}, t_3, t_4, \min\{t_5, t_6, t_7\}, t_8, t_9, t_{10}\}$$

where t_j are appropriately defined affine functions of $x(k), u(k), u(k+1), e(k), e(k+1), r(k+1), r(k+2)$.

We compute now the closed-loop MPC controller over a simulation period $[1, 20]$, with $\lambda = 0.1$, initial state $x(0) = -6$, $u(-1) = 0$ and the reference signal $\{r(k)\}_{k=1}^{20}$=-5 -5 -5 -5 -5 -3 -3 1 3 3 8 8 8 8 8 3 0 -1 -1 0 0 using the method given in Section III.

In Figure 2, the top plot represents the output and the reference signal. We see the MPC controller performs the tracking quite well. In the second plot we show the optimal input: we can see that always $u(k) \geq 0$. Finally in the last one we plot $u^*(k+1) - u^*(k)$. We can see that also the constraint $|u^*(k+1) - u^*(k)| \leq 4$ is fulfilled, and that at some moments this constraint is indeed active.

V. Conclusions and Future Research

In this paper we have extended the MPC framework for MMPS (or equivalently for continuous PWA) systems to include also bounded modeling errors, noise and/or disturbances. We have considered the uncertainty as an extra additive term on the system equations. This allowed us to design a worst-case MMPS-MPC controller for such systems based on min-max formulation. We have shown that the resulting optimization problem can be computed efficiently using a two-level optimization approach. In first step we have to solve off-line a mp-lpp and then to write the min-max canonical expression of $J^*(x, \tilde{u})$. On-line, we solve only a sequence of LPs.

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References

[1] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine systems and hybride systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, October 2000.

[2] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, March 1999.

[3] F. Borelli, A. Bemporad, and M. Morari. A geometric algorithm for multi-parametric linear programming. *Journal of Optimization Theory and Applications*, In press, 2000.

[4] B. De Schutter and T.J.J. van den Boom. MPC for continuous piecewise-affine systems. Technical Report CSE02-004, Control Systems Engineering, Fac. of Information Technology and Systems, Delft University of Technology, Delft, The Netherlands, March 2002. Accepted for publication in Systems & Control Letters.

[5] T. Gal, editor. *Postoptimal analyses, Parametric Programming, and Related Topics*. de Gruyter, Berlin, 1995.

[6] M. Johansson. *Piecewise Linear Control Systems*, volume 284 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, Germany, 2003.

[7] E. C. Kerrigan and D. Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proceedings of the 2002 IEEE Conference on Decision and Control*, Las Vegas, Nevada, December 2002.

[8] S. Ovchinnikov. Max-min representation of piecewise linear function. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 43(1):297–302, 2002.

[9] P.M. Pardalos and M.G.C. Resende, editors. *Handbook of Applied Optimization*. Oxford University Press, Oxford, UK, 2002.

[10] E. D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Transactions on Automatic Control*, 26(2):346–357, April 1981.

[11] T.J.J. van den Boom and B. De Schutter. Model predictive control for perturbed max-plus-linear systems. *Systems & Control Letters*, 45(1):21–33, January 2002.

[12] S. Young and V.K. Garg. Model uncertainty in discrete event systems. *SIAM J. Control Optim*, 33(1):208–226, January 1995.