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Control of PWA systems using a stable receding horizon method: Extended Report*

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Abstract

In this paper we derive stabilization conditions for the class of PWA systems using the linear matrix inequality (LMI) framework. We consider the class of piecewise affine feedback controllers and the class of piecewise quadratic Lyapunov functions that guarantee stability of the closed-loop system. We take into account the piecewise structure of the system and therefore the matrix inequalities that we solve are less conservative. We prove that the infinite-horizon quadratic cost is bounded if certain LMIs are satisfied. Using the upper bound of the infinite-horizon quadratic cost as a terminal cost and constructing also a convex terminal set we show that the receding horizon control stabilizes the PWA system. We derive also an algorithm for enlarging the terminal set based on a backward procedure; therefore, the prediction horizon can be chosen shorter, removing some computations off-line.

1 Introduction

Hybrid systems model the interaction between continuous and logic components. Recently, hybrid systems have attracted the interest of both academia and industry [5, 6, 10, 12, 27, 28, 31], but general analysis and control design methods for hybrid systems are not yet available. For this reason, several authors have studied special subclasses of hybrid systems for which analysis and control techniques are currently being developed: discrete-event systems [7], piecewise affine systems (PWA) [2–4, 14, 24, 27], etc.

Model Predictive Control (MPC) is the most successful control technology implemented in industry due to its ability to handle complex systems with hard input-output constraints. MPC is a control scheme in which the current input is computed by solving, at each sample step, a optimal control problem; the optimization yields an optimal input sequence and the current control action is chosen to be the first input in

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this sequence. The theory of the MPC for linear systems is quite mature, but its extension to hybrid systems is still an active area of research. Recently, research have been focused on developing stabilizing controllers for hybrid systems and in particular for PWA systems. PWA models are very popular, since they represent a powerful tool for approximating nonlinear systems with arbitrary accuracy and since a rich class of hybrid systems can be described by PWA systems. PWA systems are defined by partitioning the state space of the system in a finite number of polytopes and associating to each polytope a different affine dynamic. Several results about stability of PWA systems and MPC schemes for such systems can be found in the literature, see [2, 4, 14, 17, 18, 20, 22, 23, 27, 29] and the references therein. In [22] are derived piecewise linear (PWL) controllers and quadratic Lyapunov functions, based on LMIs approach that guarantee stability of the closed-loop PWA system. One of the first results in guaranteeing stability of MPC for PWA systems is obtained in [4], where a terminal equality constraint approach is presented. This type of constraint is very restrictive, therefore in order to guarantee feasibility of the optimal problem we need a long prediction horizon, that leads to an optimization problem very demanding from computational point of view. In [20] a terminal set and a terminal cost approach is presented for guarantee stability of the MPC scheme for PWA systems in which the origin lies in the interior of one of the polytopes. In [17] this approach is extended to PWA systems, constructing a terminal set corresponding to the piecewise linear (PWL) dynamics of the systems and the terminal cost is derived from solving some linear matrix inequalities (LMI) for the same PWL dynamics.

In this paper we continue in the same line of research. We derive LMIs condition for stabilization of PWA systems using PWA feedback controllers and also piecewise quadratic Lyapunov functions. We take into account also the structure of our system, introducing less conservatism in the LMIs. We also derive LMIs conditions that assure the controller satisfies constraints on input and output. We will show that the infinite-horizon quadratic cost is bounded if certain LMIs are satisfied. Moreover from these LMIs we derive a feedback controller that guarantees asymptotic stability of the closed-loop system with a convex region of attraction. In general this set is small, therefore we show that applying the receding horizon controller we can also guarantee stability, and we prove that this controller is better than the original feedback controller, i.e. by this method the region of attraction increases such that for an infinite horizon we obtain the maximal region of attraction. We derive a stable MPC scheme with a convex terminal set and the upper bound of the infinite-horizon quadratic cost is used as a terminal cost. If the terminal set is small, we need a long prediction horizon. Therefore we present an algorithm for enlarging this set based on *backward procedure* and then we show that a certain inner polytope approximation of this set can be used also as a terminal set. By enlarging the terminal set the prediction horizon can be chosen shorter, therefore the computational complexity decreases.

We use the following notations: a PWA system is defined as

$$\begin{cases} x(k+1) = \tilde{A}_i x(k) + \tilde{B}_i u(k) + \tilde{a}_i, \text{ if } x(k) \in \mathcal{P}_i \\ y(k) = \tilde{C}_i x(k) + \tilde{c}_i, \end{cases} \tag{1}$$

where $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ is a partition of $\mathbb{R}^n$ into a number of polyhedral cells (n is the number of states). Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be the set of indexes for the cells that contain origin in their closure. Similarly, let $\mathcal{I}_1 \subseteq \mathcal{I}$ be the set of indexes that do not contain the origin in their closure. Each polyhedral cell is given by: $\mathcal{P}_i = \{x \in \mathbb{R}^n : E_1^i x \geq e_1, E_2^i x > e_2\}$, but we assume that the closure $\text{cl}(\mathcal{P}_i) = \{x \in \mathbb{R}^n : \bar{E}_i x \geq e_i\}$ with $e_i = 0$ for $i \in \mathcal{I}_0$.

So $\tilde{E}_i = [E_1^{i,T} \; E_2^{i,T}]^T$. We also introduce

$$A_i = \begin{bmatrix} \tilde{A}_i & \tilde{a}_i \\ 0 & 1 \end{bmatrix},\; B_i = \begin{bmatrix} \tilde{B}_i \\ 0 \end{bmatrix}, \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, C_i = \begin{bmatrix} \tilde{C}_i & \tilde{c}_i \end{bmatrix},\; E_i = \begin{bmatrix} \tilde{E}_i & -e_i \end{bmatrix}.$$

with $\tilde{a}_i = 0,\ \tilde{c}_i = 0$ for $i \in \mathcal{I}_0$.

With the previous notations, the PWA system (1) can be written as a piecewise linear (PWL) system (see also [24] for continuous time case):

$$\begin{cases} \bar{x}(k+1) = A_i \bar{x}(k) + B_i u(k),\ \text{if}\ x(k) \in \mathcal{P}_i \\ y(k) = C_i \bar{x}(k), \end{cases} \tag{2}$$

where the closure of $\mathcal{P}_i$ is given by $\mathrm{cl}(\mathcal{P}_i) = \{x \in \mathbb{R}^n : E_i \bar{x}^T \geq 0\}$ for any $i \in \mathcal{I}$.

2 Boundaries on the infinite-horizon quadratic cost using static feedback controllers

2.1 Lower bounds

In this section we consider a generalization of the standard linear quadratic control for the system (1). The problem is to bring the system to the equilibrium point $x(\infty) = 0$ from an arbitrary initial state $x(0) = x_0$, satisfying constraints on input and output, limiting also the infinite-horizon quadratic cost:

$$J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} x^T(k) Q x(k) + u^T(k) R u(k) \tag{3}$$

with $Q = Q^T \geq 0$, $R = R^T > 0$ and $\mathbf{u} = (u(0), u(1), ...)$.

In this paper we consider the following type of constraints[1]:

$$U_{\mathrm{c}} = \{u \in \mathbb{R}^{n_u} : |u_j| \leq u_{j,\max},\ \text{for}\ j = 1, ..., n_u\} \tag{4}$$

$$X_{\mathrm{c}} = \{x \in \mathbb{R}^n : |y_j| \leq y_{j,\max},\ \text{for}\ j = 1, ..., n_y\}. \tag{5}$$

with $u_{j,\max}, y_{j,\max} > 0$, therefore the sets $U_{\mathrm{c}}, X_{\mathrm{c}}$ are convex, compact and contain the origin in the interior. We define now the problem that we would like to solve:

Definition 2.1 ***Problem P***
Design a feedback controller $u = F(x)$ for system (1) that:
(i) limits the infinite-horizon quadratic cost in a positively invariant set $\mathcal{E}$, i.e. $\forall x_0 \in \mathcal{E} \Rightarrow x(k) \in \mathcal{E},\ \forall k \geq 0$
(ii) makes the closed-loop asymptotically stable, with $x(\infty) = 0$
(iii) satisfies the constraints $u(k) \in U_{\mathrm{c}},\ y(k) \in X_{\mathrm{c}},\ \forall k \geq 0$. $\diamondsuit$

We denote also with $\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, then we have the following proposition:

[1]We can consider more general constraints, for instance: $u_{\mathrm{j,min}} \leq u_j \leq u_{\mathrm{j,max}}$, with $u_{\mathrm{j,min}} < 0,\ u_{\mathrm{j,max}} > 0$. Similar for output.

Proposition 2.2 *If there exists symmetric matrices $\{\overline{P}_i, \overline{U}_{ij}, \overline{V}_i\}_{i,j\in I}$, such that $\overline{U}_{ij}, \overline{V}_i$ has all elements non-negative and that verify the following LMIs:*

$$\left[\begin{array}{cc} R+B_i^T\overline{P}_jB_i & B_i^T\overline{P}_jA_i \\ A_i^T\overline{P}_jB_i & A_i^T\overline{P}_jA_i-\overline{P}_i+\bar{Q}-E_i^T\overline{U}_{ij}E_i \end{array}\right] \geq 0, \quad \overline{P}_i > E_i^T\overline{V}_iE_i \quad (6)$$

for any $i,j\in I$, then any trajectory (u,x) of the system (1) with $x(\infty)=0$ and $x(0)=x_0\in\mathcal{P}_{i_0}$ verifies

$$J_\infty(x_0,\boldsymbol{u}) \geq \sup\{\bar{x}_0^T\overline{P}_{i_0}\bar{x}_0\colon (\overline{P}_i,\overline{U}_{ij},\overline{V}_i) \text{ is solution of } (6)\} \quad (7)$$

Moreover any trajectory (u,x) of the system (1) with $x(\infty)=0$ and $x(0)=x_0\in\mathcal{P}_{i_0}$, satisfying the constraints (4), (5) on the input and output verifies also the lower bound (7) but this time subject to the following LMIs: for any $i,j\in\mathcal{I}$

$$\left[\begin{array}{cc} R+B_i^T\overline{P}_jB_i & B_i^T\overline{P}_jA_i \\ A_i^T\overline{P}_jB_i & A_i^T\overline{P}_jA_i-\overline{P}_i+Q \end{array}\right] - \left[\begin{array}{ccc} E_u & 0 & e_u \\ 0 & E_y & e_y \\ 0 & E_i & 0 \end{array}\right]^T \overline{U}_{ij} \left[\begin{array}{ccc} E_u & 0 & e_u \\ 0 & E_y & e_y \\ 0 & E_i & 0 \end{array}\right] \geq 0 \quad (8)$$

Proof: We define the following piecewise quadratic function:

$$V(k) = \begin{cases} \left[\begin{array}{c} x(k) \\ 1 \end{array}\right]^T \left[\begin{array}{cc} \tilde{P}_i & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ 1 \end{array}\right] & \text{if } i\in\mathcal{I}_0 \\ \left[\begin{array}{c} x(k) \\ 1 \end{array}\right]^T \overline{P}_i \left[\begin{array}{c} x(k) \\ 1 \end{array}\right] & \text{if } i\in\mathcal{I}_1 \end{cases} = \begin{cases} x(k)^T\tilde{P}_ix(k) & \text{if } i\in\mathcal{I}_0 \\ \left[\begin{array}{c} x(k) \\ 1 \end{array}\right]^T \overline{P}_i \left[\begin{array}{c} x(k) \\ 1 \end{array}\right] & \text{if } i\in\mathcal{I}_1 \end{cases}$$

with $\overline{P}_i\in\mathbb{R}^{n+1\times n+1}$, $\tilde{P}_i\in\mathbb{R}^{n\times n}$ and $\overline{P}_i > E_i^T\overline{V}_iE_i$ for all $i\in\mathcal{I}_1$, $\tilde{P}_i > E_i^T\overline{V}_iE_i$ for all $i\in\mathcal{I}_0$ (i.e. it is not necessary $\overline{P}_i,\tilde{P}_i>0$ on the entire state space, but rather on $\mathcal{P}_i$). With these conditions we have that $V(k)>0$ for any $x(k)\neq 0$.

We introduce the notations:

$$\overline{P}_i = \left[\begin{array}{cc} \tilde{P}_i & 0 \\ 0 & 0 \end{array}\right], \text{ if } i\in\mathcal{I}_0$$

$$l(x(k),u(k)) = x^T(k)Qx(k)+u^T(k)Ru(k) = \bar{x}^T(k)\bar{Q}\bar{x}(k)+u^T(k)Ru(k)$$

$$V(k) = \bar{x}^T(k)\overline{P}(k)\bar{x}(k) \text{ with } \overline{P}(k)=\overline{P}_i \text{ if } x(k)\in\mathcal{P}_i.$$

If we assume that at sample step $k\geq 0$, the state $x(k)\in\mathcal{P}_i$ and $x(k+1)=\tilde{A}_ix(k)+\tilde{B}_iu(k)+\tilde{a}_i\in\mathcal{P}_j$ then imposing

$$V(k)-V(k+1)\leq l(x(k),u(k)) \text{ when } x(k)\in\mathcal{P}_i \quad (9)$$

and applying the S-procedure [15, 25] for $\bar{x}(k)\in\mathrm{cl}(\mathcal{P}_i)=\{E_i\bar{x}\geq 0\}$ and using Fact 1 (see Appendix), we get

$$V(k)-V(k+1)\leq l(\bar{x}(k),u(k))-\bar{x}(k)^TE_i^T\overline{U}_{ij}E_i\bar{x}(k) \quad (10)$$

for any $\bar{x}(k)\in\mathbb{R}^{n+1}, u(k)\in\mathbb{R}^{n_u}$.

From the last relation we obtain for any $\bar{x}(k)\in\mathbb{R}^{n+1}, u(k)\in\mathbb{R}^{n_u}$ that

$\bar{x}^T(k)\overline{P}_i\bar{x}(k) - (A_i\bar{x}(k) + B_iu(k))^T\overline{P}_j(A_i\bar{x}(k) + B_iu(k)) \leq \bar{x}^T(k)\bar{Q}\bar{x}(k) + u^T(k)Ru(k) - \bar{x}(k)^T E_i^T \overline{U}_{ij}E_i\bar{x}(k)$. This relation leads us to LMI (6).

For the second part, we observe that writing (9) for $k = 0, 1, 2...$, with $x(\infty) = 0$ and thus $V(\infty) = 0$ and adding these relations yields:

$(V(0) - V(1)) + (V(1) - V(2)) + ... \leq l(x(0), u(0)) + l(x(1), u(1)) + ...$ which leads us to (7).

Let us now also take into account the constraints. The constraints on the input (4) can be written as $[E_u\ e_u][u^T\ 1]^T \geq 0$ and the constraints on the output (5) as $[E_y\ e_y][x^T\ 1]^T \geq 0$. Then by applying the S-procedure and Fact 1 for $[u^T, x^T, 1]^T \in \mathbb{R}^{n_u+n+1}$ we have to replace the LMIs (6) with the LMIs (8). Then any trajectory (x, u) of system (1) with $x(\infty) = 0$ and $x(0) = x_0 \in \mathcal{P}_{i_0}$, satisfying the constraints (4), (5) verifies also the lower bound (7) but this time subject to the above LMIs (8). $\diamondsuit$

Remark 2.3 Upper and lower bounds for the infinite-horizon quadratic cost were derived for continuous time PWA systems in [24].

We can define a set $\Omega = \{(i,\ j) : x(k) \in \mathcal{P}_i,\ x(k+1) \in \mathcal{P}_j\}$ that represents all possible transitions from one region to another and then to restrict $i, j \in \Omega$ only. The set Ω can be determined via reachability analysis (see [2]). $\diamondsuit$

2.2 Upper bounds for the infinite-horizon cost

In order to obtain an upper bound for $J_\infty(x_0)$ we should take a particular control law. Our first impulse would be to take the ordinary linear quadratic control:

$$F_i = -(R + B_i^T\overline{P}_iB_i)^{-1}B_i\overline{P}_iA_i = [\tilde{F}_i\ f_i] \tag{11}$$

with $f_i = 0$ when $i \in \mathcal{I}_0$. But, in general, the PWA control law

$$u(k) = \begin{cases} \tilde{F}_ix(k) + f_i, & \text{if } i \in \mathcal{I}_1 \\ \tilde{F}_ix(k), & \text{if } i \in \mathcal{I}_0 \end{cases}$$

cannot guarantee stability of system (2). If the closed-loop system

$$\bar{x}(k+1) = (A_i + B_iF_i)\bar{x}(k),\ x(k) \in \mathcal{P}_i \tag{12}$$

is asymptotically stable then we can choose this controller in order to obtain an upper bound for the infinite-horizon cost (we can check stability via LMI feasibility as we will see in the sequel). Indeed, similar to Proposition 2.2 we introduce the piecewise quadratic function:

$$V(x) = \begin{cases} x^T\tilde{P}_ix, & \text{if } i \in \mathcal{I}_0 \\ x^T\tilde{P}_ix + 2p_i^Tx + p_{ii}, & \text{if } i \in \mathcal{I}_1 \end{cases}$$

We want to find $P_i \in \mathbb{R}^{n+1\times n+1},\ i \in I$ with

$$P_i = \begin{bmatrix} \tilde{P}_i & 0 \\ 0 & 0 \end{bmatrix},\ \text{if } i \in \mathcal{I}_0, P_i = \begin{bmatrix} \tilde{P}_i & p_i \\ p_i^T & p_{ii} \end{bmatrix},\ \text{if } i \in \mathcal{I}_1$$

such that $x(\infty) = 0$, $P_i > E_i^TV_iE_i$ for all $i \in \mathcal{I}$ (V_i has all elements non-negative which implies that $P_i > 0$ only on a set that contains $\mathcal{P}_i$) and

$$V(k+1) - V(k) \leq -l(\bar{x}(k), F(k)\bar{x}(k)) \text{ for any } k \tag{13}$$

If $x(k) \in \mathcal{P}_i$ and $\bar{x}(k+1) = (A_i + B_iF_i)\bar{x}(k) \in \mathcal{P}_j$ then (13) becomes

$$\begin{aligned} &\bar{x}^T(k)(A_i + B_iF_i)^T P_j (A_i + B_iF_i)\bar{x}(k) - \bar{x}^T(k)P_i\bar{x}(k) \\ &\leq -\bar{x}^T(k)\bar{Q}\bar{x}(k) - \bar{x}^T(k)F_i^T RF_i\bar{x}(k) \text{ for } x(k) \in \mathcal{P}_i \end{aligned} \tag{14}$$

Because we need (14) to be valid only for $x \in \mathcal{P}_i$, we can use *S-procedure* in order to reduce conservatism when we solve (14). Using *Fact* 1 (see Appendix), one method found in the literature (see [24]) is to relax the matrix inequality (14) to: find $P_i, U_{ij},\ i,j \in \mathcal{I}$, such that U_{ij} has all entries non-negative that satisfies the following matrix inequalities

$$\begin{aligned} &\bar{x}^T(A_i + B_iF_i)^T P_j (A_i + B_iF_i)\bar{x} - \bar{x}^T P_i\bar{x} \leq -\bar{x}^T\bar{Q}\bar{x} - \\ &- \bar{x}^T F_i^T RF_i\bar{x} - \bar{x}^T E_i^T U_{ij} E_i\bar{x}, \ \text{ for any } \bar{x} \in \mathbb{R}^{n+1} \end{aligned} \tag{15}$$

This gives rise to the following LMIs in P_i, U_{ij}, V_i (all entries of U_{ij}, V_i non-negative):

$$\begin{aligned} (A_i + B_iF_i)^T P_j (A_i + B_iF_i) - P_i + \bar{Q} + F_i^T RF_i + E_i^T U_{ij} E_i \leq 0, &\ \text{ for all } i,j \in \mathcal{I} \\ P_i > E_i^T V_i E_i, \ \text{ for all } i \in \mathcal{I}. & \end{aligned} \tag{16}$$

Proposition 2.4 *If $x_0 \in \mathcal{P}_{i_0}$ then*

$$J_\infty(x_0) \leq \inf\{\bar{x}_0^T P_{i_0}\bar{x}_0 : (P_i, U_{ij}, V_i) \textit{ is solution of } (16)\} \tag{17}$$

Proof: From (13) we have

$$(V(1) - V(0)) + (V(2) - V(1)) + \ldots \leq -l(\bar{x}(0), F(0)\bar{x}(0)) - l(\bar{x}(1), F(1)\bar{x}(1)) - \ldots$$

and because $x(\infty) = 0$ then $V(\infty) = 0$, which implies $-V(0) \leq -J_\infty(x_0)$. Therefore (17) holds. $\diamondsuit$

If the linear quadratic controller (11) is not stabilizing for (12), i.e. the LMIs (16) do not have a solution, or does not satisfy the constraints on input/output, we look for another controller. In the sequel we derive this controller that solves problem **P**.

We denote with $F_i = [\tilde{F}_i \ \ f_i]$ with $f_i = 0$ for $i \in \mathcal{I}_0$ and the piecewise quadratic function: $V(k) = \bar{x}^T(k)P(k)\bar{x}(k)$ with $P(k) = P_i$ if $x(k) \in \mathcal{P}_i$. We want to find $F_i, P_i \in \mathbb{R}^{n+1 \times n+1},\ i \in \mathcal{I}$, with

$$P_i = \begin{bmatrix} \tilde{P}_i & 0 \\ 0 & 0 \end{bmatrix}, \ \text{if } i \in \mathcal{I}_0; P_i = \begin{bmatrix} \tilde{P}_i & p_i \\ p_i^T & p_{ii} \end{bmatrix}, \ \text{if } i \in \mathcal{I}_1$$

such that:

1. $x(\infty) = 0$
2. $V(k+1) - V(k) \leq -l(\bar{x}(k), F(k)\bar{x}(k)),\ k \geq 0$
3. the resulting input-output sequence should satisfy the input/output constraints.

We do not require $P_i > 0$ on the entire space but rather $P_i > 0$ on $\mathcal{P}_i$, which using S-procedure can be expressed as $P_i > E_i^T V_i E_i$, where V_i has all elements non-negative. Condition 2 is implied by the existence of a solution F_i, P_i of the matrix inequalities

$$\bar{x}^T(A_i + B_iF_i)^T P_j (A_i + B_iF_i)\bar{x} - \bar{x}^T P_i\bar{x} + \bar{x}^T\bar{Q}\bar{x} + \bar{x}^T F_i^T RF_i\bar{x} < 0,\ \forall\, x \in \mathcal{P}_i, \tag{18}$$

and applying S-procedure we obtain the following matrix inequalities:

$$(A_i + B_iF_i)^T P_j(A_i + B_iF_i) - P_i + \bar{Q} + F_i^T RF_i + E_i^T U_{ij}E_i < 0, \ \forall\, i, j \in \mathcal{I}, \quad (19)$$

$$P_i > E_i^T V_iE_i, \ \text{for all } i \in \mathcal{I} \quad (20)$$

where we require U_{ij}, V_i have all entries non-negative. In the sequel the symbol * is used to induce a symmetric structure in an LMI. The following proposition give a solution to (19)-(20):

Proposition 2.5 *The matrix inequalities* (19)–(20) *have a solution if and only if the following matrix inequalities have a solution*

$$\begin{bmatrix} B_i^T P_jB_i + \theta R - I & B_i^T P_jA_i + F_i \\ * & A_i^T P_jA_i - P_i + E_i^T U_{ij}E_i + \theta\bar{Q} - F_i^T F_i \end{bmatrix} < 0$$
$$P_i > E_i^T V_iE_i \quad (21)$$

where U_{ij}, V_i have all entries non-negative and $\theta > 0$.

Proof: It is easy to see that (19) *can be written as*

$$\begin{bmatrix} F_i \\ I \end{bmatrix}^T \begin{bmatrix} B_i^T P_jB_i + R & B_i^T P_jA_i \\ * & A_i^T P_jA_i - P_i + E_i^T U_{ij}E_i + \bar{Q} \end{bmatrix} \begin{bmatrix} F_i \\ I \end{bmatrix} < 0$$

Now $M^\perp$ denotes the orthogonal complement of M ($M^\perp$ exists only if M has linear dependent rows). We have then $M^T M^\perp = 0$ and $M^\perp M^{\perp T} > 0$. In our case we have the relation $\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp = \begin{bmatrix} F_i \\ I \end{bmatrix}^T$. Therefore, the previous formula can be written

$$\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp Q_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp T} < 0 \quad (22)$$

where $Q_{ij} = \begin{bmatrix} B_i^T P_jB_i + R & B_i^T P_jA_i \\ * & A_i^T P_jA_i - P_i + E_i^T U_{ij}E_i + \bar{Q} \end{bmatrix}$. Using now the Finsler's lemma (see Appendix) we obtain that (22) *is equivalent with*

$$Q_{ij} < \sigma_{i,j} \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \quad (23)$$

with $\sigma_{ij} \in \mathbb{R}$. Of course (23) *has a solution if and only if*

$$Q_{ij} < \sigma \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \quad (24)$$

with $\sigma > 0$ has a solution (Take $\sigma > \max_{i,j}\{0, \sigma_{ij}\}$ for the implication "(23) ⇒ (24)*". The other implication is obvious). Now if we divide* (24) *with $\sigma > 0$ and denote with $P_i \to 1/\sigma P_i$, $U_{ij} \to 1/\sigma U_{ij}$, $V_i \to 1/\sigma V_i$ and $\theta \to 1/\sigma$ we obtain* (21). ♢

The matrix inequalities (21) are not LMIs due to the term $F_i^T F_i$. Therefore we have to use standard algorithms for solving bilinear matrix inequalities (BMI). The algorithms for solving BMIs cover both local and global approaches. Local approaches are less computationally intensive , and they consist in searching a feasible solution: if it exists then we have solved the problem, otherwise one cannot tell whether there is a feasible solution or not. Global algorithms are able to find a solution if the problem is feasible. The branch-and-bound algorithm of Tuan [30] can be used to solve globally our problem, although in this case the computational time is increasing in comparison with local approach.

Now we discuss some possible relaxations for (19)–(20). First relaxation is to replace (20) with $P_i > 0$. In this case we can apply the Schur complement to (19). Note that (19) is equivalent with

$$(A_i+B_iF_i)^T S_j^{-1}(A_i + B_iF_i)-P_i+\bar{Q}+F_i^T RF_i + E_i^T U_{ij}E_i<0,\ \forall\, i,j \in \mathcal{I} \quad (25)$$

$$0 < P_j \leq S_j^{-1},\ \forall\, j \in \mathcal{I} \quad (26)$$

In this way we take into account also the case $S_j = P_j^{-1}$. We give now a sketch of the proof: it is clear that if (19) has a solution then there exists an $\epsilon > 0$ such that $(A_i + B_iF_i)^T P_j(A_i+B_iF_i)-P_i+\bar{Q}+F_i^T RF_i+E_i^T U_{ij}E_i < -\epsilon(A_i+B_iF_i)^T(A_i+B_iF_i)$. Then, we can take $S_j^{-1} = P_j + \epsilon I$ and thus we obtain (25)–(26). The other implication is obvious.

Now, using the Schur complement, (25)–(26) is equivalent with

$$\begin{bmatrix} P_i - \bar{Q} - E_i^T U_{ij}E_i & * & * \\ A_i + B_iF_i & S_j & 0 \\ F_i & 0 & R^{-1} \end{bmatrix} > 0 \quad (27)$$

$$0 < P_j \leq S_j^{-1} \quad (28)$$

In Appendix (Fact 5) we give an algorithm for finding a solution for (27)–(28) based on an idea from [13].

We define $X_0 = \cup_{i\in\mathcal{I}_0}\mathcal{P}_i$ and we consider an inner approximation with an ellipsoid of this set:

$$\mathcal{E}(H) = \{x \in \mathbb{R}^n : x^T Hx \leq 1\} \subseteq X_0.$$

The computation of a maximal volume ellipsoid included in a polytope can be done using convex optimization (see [33]).

Proposition 2.6 *(i) If the following LMIs in $P_i = P_i^T > 0, S_i = S_i^T > 0, F_i, U_{ij} = U_{ij}^T$, with all entries of U_{ij} non-negative*

$$\begin{bmatrix} P_i - \bar{Q} - E_i^T U_{ij}E_i & * & * \\ A_i + B_iF_i & S_j & 0 \\ F_i & 0 & R^{-1} \end{bmatrix} > 0 \quad (29)$$

and the following bilinear matrix inequalities (BMIs)

$$S_iP_i + P_iS_i \leq 2I \quad (30)$$

have a solution $P_i, S_i, F_i, U_{ij},\ i,j \in \mathcal{I}$ then they are also solution of (27)–(28).

(ii) The following set $\mathcal{E}(\rho) = \{x \in \mathbb{R}^n : x^T \tilde{P}_j x \leq \rho,\ j \in \mathcal{I}_0\},\ \rho > 0$ is a positive invariant set for the closed-loop system, convex, compact, containing the origin in the interior if the following LMIs are satisfied:

$$\begin{bmatrix} \tau H - \tilde{P}_j & 0 \\ 0 & -\tau + \rho \end{bmatrix} \leq 0 \quad (31)$$

with $\tau > 0$ and $j \in \mathcal{I}_0$.

(iii) If we require the input $u(k+l)$ to satisfy (4) for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then an additional LMI must be satisfied:

$$\begin{bmatrix} \Lambda - E_i^T W_i E_i & F_i \\ * & P_i \end{bmatrix} \geq 0 \text{ with } \Lambda_{jj} \leq u_{j,\max}^2/\rho \tag{32}$$

where the matrices W_i have all entries non-negative for any $i \in \mathcal{I}_0$.

(iv) If we require the output $y(k+1+l)$ to satisfy (5) for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then the following additional LMIs must be satisfied:

$$\begin{bmatrix} \Gamma - E_i^T \tilde{W}_i E_i & C_i(A_i + B_i F_i) \\ * & P_i \end{bmatrix} \geq 0, \ \ \Gamma_{jj} \leq y_{j,\max}^2/\rho \tag{33}$$

where the matrices $\tilde{W}_i$ have all entries non-negative for any $i \in \mathcal{I}_0$. Taking $\gamma = 1/\rho$ all formulas (29)-(33) are LMIs except (30).

Proof: (i) We have 4 possibilities: $i, j \in \mathcal{I}_1$; $i, j \in \mathcal{I}_0; i \in \mathcal{I}_0, j \in \mathcal{I}_1; i \in \mathcal{I}_1, j \in \mathcal{I}_0$. For brevity we discuss here only the first case (the other three cases can be proved similarly, see also Appendix). The BMIs (30) imply that

$$0 < S_i \leq P_i^{-1} \text{ if and only if } 0 < P_i \leq S_i^{-1}$$

(see also [26]). Applying the Schur complement to (29) and using the last inequality we get:

$$\begin{aligned} 0 <& P_i - \bar{Q} - E_i^T U_{ij} E_i - (A_i + B_i F_i)^T S_j^{-1} (*) - F_i^T R F_i \\ &\leq P_i - \bar{Q} - E_i^T U_{ij} E_i - (A_i + B_i F_i)^T P_j (*) - F_i^T R F_i \end{aligned}$$

i.e. formula (19) is valid with the requirement U_{ij} has all entries non-negative.

(ii) It is well-known [32] that inclusion of ellipsoids

$$\{x : x^T \tilde{P}_j x \leq \rho\} \subseteq \mathcal{E}(H) \subseteq X_0$$

can be expressed as an LMI (31). Then if $x(k) \in \mathcal{E}(\rho) \cap \mathcal{P}_{i_0}$, for some $i_o \in \mathcal{I}_0$, then $x^T(k) \tilde{P}_{i_0} x(k) \leq \rho$ and $x(k+1) = (\tilde{A}_{i_0} + \tilde{B}_{i_0} \tilde{F}_{i_0}) x(k)$.

Therefore, for any $j \in \mathcal{I}_0$ according to (i) we have:

$$x^T(k+1) \tilde{P}_j x(k+1) = x^T(k) (\tilde{A}_{i_0} + \tilde{B}_{i_0} \tilde{F}_{i_0})^T \tilde{P}_j (\tilde{A}_{i_0} + \tilde{B}_{i_0} \tilde{F}_{i_0}) x(k) \leq x^T(k) \tilde{P}_{i_0} x(k) \leq \rho$$

according to condition 2. By induction we can see that if $x(k) \in \mathcal{E}(\rho)$ then $x(k+l) \in \mathcal{E}(\rho)$ for all $l \geq 0$. $\mathcal{E}(\rho)$ is convex[2] and compact because it is the intersection of ellipsoids (the ellipsoids are convex and compact sets because $\tilde{P}_i > 0$) and it contains the origin in the interior because each ellipsoid contains the origin: $\tilde{P}_i > 0$.

(iii) The constraint on the input (4) is equivalent with $u_j^2(k) \leq u_{j,\max}^2$. We have $\mathcal{E}(\rho) \subseteq \{x : x^T \tilde{P}_i x \leq \rho\}$ and if $x(k) \in \mathcal{E}(\rho) \cap \mathcal{P}_i$ then

$$\begin{aligned} u_j^2(k) \leq & \max_{x(k) \in \mathcal{E}(\rho)} (\tilde{F}_i x(k))_j^2 \leq \max_{x^T \tilde{P}_i x \leq \rho} (\tilde{F}_i x)_j^2 \leq \max_{x^T \frac{\tilde{P}_i}{\rho} x \leq 1} (\tilde{F}_i x)_j^2 \leq \| \sqrt{\rho} (\tilde{F}_i \tilde{P}_i^{-1/2})_j \|_2^2 \\ &= \rho (\tilde{F}_i \tilde{P}_i^{-1} \tilde{F}_i^T)_{jj} = \rho (F_i P_i^{-1} F_i^T)_{jj} \leq\leq \rho \Lambda_{jj} \leq u_{j,\max}^2 \text{ on } \mathcal{P}_i. \end{aligned}$$

[2]We observe that $\mathcal{E}(\rho)$ is in particular an invariant set for the free switching system with the modes $i \in \mathcal{I}_0$.

Taking W_i with all entries non-negative, and applying the S-procedure, the last inequality translates in:

$$\Lambda - F_i P_i^{-1} F_i^T - E_i^T W_i E_i \geq 0$$

and $\Lambda_{jj} \leq \frac{u_{j,\max}^2}{\rho}$ which is the LMI (32) if we denote with $\gamma = \frac{1}{\rho}$.

(iv) The LMI (33) is derived in the same way. $\diamondsuit$

Remark 2.7 We have considered an ellipsoid approximation $\mathcal{E}(H)$ of X_0. If X_0 is a polytope (or we can have a inner polytope approximation of X_0) given by $\{x : c_l^T x \leq 1,\ l = 1, ..., s\}$, we can replace LMIs (31) with the following inequalities: $c_l^T \rho P_j^{-1} c_l \leq 1$ for all $l = 1, ..., s$. Using the Schur complement the last inequality can be recast as an LMI in P_j and $1/\rho$, which fits to our settings from Proposition 2.6. We used here the fact that an ellipsoid $\{x : x^T P x \leq \rho\}$ is contained in a half space $\{x : c^T x \leq 1\}$ if and only if $c^T \rho P^{-1} c \leq 1$. $\diamondsuit$

Remark 2.8 In Proposition 2.6 we can search for P_i, F_i, for which the volume of $\mathcal{E}(\rho)$ is maximal, by considering the minimization of the convex cost function:

$\min - \sum_{i \in \mathcal{I}_0} \log \det(P_i)$. There is a method to transform the matrix inequality (28) into LMIs using the fact that $S + S^{-1} \geq 2I$ for any $S > 0$. Then we can replace (28) with the more conservative LMI $P_j + S_j \leq 2I$. $\diamondsuit$

We propose now a second relaxation. If we do not apply the S-procedure for (18), i.e. we replace the condition "$x \in \mathcal{P}_i$", with $x \in \mathbb{R}^n$, then (18) becomes:

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + Q + F_i^T R F_i \leq 0 \tag{34}$$

for all $i, j \in \mathcal{I}_0$ and $P_i > 0$. The matrix inequalities (34) were solved in [16] making a so-called linearizing change of variables by introducing: $S = P^{-1}, F_i = Y_i S$. This type of linearization for (34) is also used in [17], where a stabilizing receding horizon control for PWL systems is developed. If we use this linearization we have to impose a certain structure on matrices P_i. In order to avoid this conservatism, we propose here another linearization of (34), namely $P_i = S_i^{-1}, F_i = Y_i G^{-1}$. Using this change of variables we see that the determination of the control law does not depend explicitly on the Lyapunov matrices P_i. The extra degree of freedom introduced by the matrix G which is not considered symmetric, is incorporated in the control variable, removing the special structure of P_i to G. A similar linearizing method was used in [9] in the context of stabilizing linear parameter varying (LPV) systems.

Proposition 2.9 *(i) If the following LMIs in G, Y_i, S_i*

$$\begin{bmatrix} G + G^T - S_i & * & * & * \\ A_i G + B_i Y_i & S_j & * & * \\ \bar{Q}^{1/2} G & 0 & I & * \\ R^{1/2} Y_i & 0 & 0 & I \end{bmatrix} > 0 \tag{35}$$

for all $i, j \in \mathcal{I}$ have a solution then $F_i = Y_i G^{-1}, P_i = S_i^{-1}$ are solutions of (18).

(ii) The following set $\mathcal{E}(\rho) = \{x \in \mathbb{R}^n : x^T \tilde{P}_j x \leq \rho,\ j \in \mathcal{I}_0\},\ \rho > 0$ is a positive invariant set for the closed-loop system, convex and compact, containing the origin in the interior if the following LMIs are satisfied:.

$$\begin{bmatrix} \tau H^{-1} - \tilde{S}_j & 0 \\ 0 & -\tau + 1/\rho \end{bmatrix} \geq 0 \tag{36}$$

with $\tau > 0$ and $j \in \mathcal{I}_0$.

(iii) If we require the input $u(k+l)$ to satisfy (4) for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then an additional LMI must be satisfied:

$$\begin{bmatrix} \Lambda & Y_i \\ * & G + G^T - S_i \end{bmatrix} \geq 0 \text{ with } \Lambda_{jj} \leq u_{j,\max}^2/\rho \tag{37}$$

for any $i \in \mathcal{I}_0$.

(iv) If we require the output $y(k+1+l)$ to satisfy (5) for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then the following additional LMI must be satisfied:

$$\begin{bmatrix} \Gamma & \tilde{C}_i(\tilde{A}_i G + \tilde{B}_i Y_i) \\ * & G + G^T - S_i \end{bmatrix} \geq 0 \text{ with } \Gamma_{jj} \leq y_{j,\max}^2/\rho. \tag{38}$$

for any $i \in \mathcal{I}_0$. Taking $\gamma = 1/\rho$ all previous formulas become LMIs.

Proof: (i) We have to discuss also separate 4 cases, but for brevity we give the proof only for first case (see Appendix for more details).

From (35) using the Schur complement, we observe first that G is a nonsingular matrix because

$$G + G^T > S_i$$

and also

$$0 < S_i \Rightarrow (S_i - G)^T S_i^{-1}(S_i - G) \geq 0$$

therefore we get the following relation:

$$G + G^T - S_i \leq G^T S_i^{-1} G$$

and

$$\begin{aligned} 0 <& G + G^T - S_i - (A_i G + B_i Y_i)^T S_j^{-1}(*) - G^T \bar{Q} G - Y_i^T R Y_i \\ &\leq G^T S_i^{-1} G - (A_i G + B_i Y_i)^T S_j^{-1}(*) - G^T \bar{Q} G - Y_i^T R Y_i \\ &= G^T (S_i^{-1} - (A_i + B_i Y_i G^{-1})^T S_j^{-1}(*) - \bar{Q} - G^{-T} Y_i^T R Y_i G^{-1}) G \end{aligned}$$

Taking $F_i = Y_i G^{-1}, P_i = S_i^{-1}$ we obtain from the last relation (18):

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + \bar{Q} + F_i^T R F_i \leq 0, \text{ for any } i, j \in \mathcal{I}$$

(ii) The LMIs (36) express the fact that

$$\{x : x^T \tilde{S}_j^{-1} x \leq \rho\} \subseteq \mathcal{E}(H) \subseteq X_0$$

with $\tilde{S}_j^{-1} = \tilde{P}_j$ (see [32]) and the rest of the proof is similar to the proof from Proposition 2.6 (ii).

(iii) The constraint on the input (4) is equivalent with $u_j^2(k) \leq u_{j,max}^2$. We have $\mathcal{E}(\rho) \subseteq \{x : x^T \tilde{P}_i x \leq \rho\}$ and if $x(k) \in \mathcal{E}(\rho) \cap \mathcal{P}_i$ then

$$\begin{aligned} &u_j^2(k) \leq \max_{x(k) \in \mathcal{E}(\rho)} (Y_i G^{-1} x(k))_j^2 \leq \max_{\bar{x}^T P_i \bar{x} \leq \rho} (Y_i G^{-1} x)_j^2 \\ &\leq \max_{\bar{x}^T \frac{P_i}{\rho} \bar{x} \leq 1} (Y_i G^{-1} x)_j^2 \leq \|\sqrt{\rho}(Y_i G^{-1} S_i^{1/2})_j\|_2^2 = \rho (Y_i G^{-1} S_i G^{-T} Y_i^T)_{jj} \\ &\leq \rho \Lambda_{jj} \leq u_{j,\max}^2. \end{aligned}$$

Therefore (37) holds. We used here $P_i = S_i^{-1}$.

(iv) This proof is similar with (c). ♢

Now we assume that the LMIs from Proposition 2.2 are feasible, and that we have available $\bar{P}_i,\ i \in \mathcal{I}$ and also applying one of the approaches proposed before we obtained $F_i, P_i,\ i \in \mathcal{I}$. In this case we have the following two consequences:

Corollary 2.10 *The origin of system (1) is globally asymptotically stable with the PWA feedback controller $u(k) = \tilde{F}_i x(k) + f_i,\ x(k) \in \mathcal{P}_i$ and the infinite-horizon quadratic cost is bounded by:*

$$\sup \bar{x}_0^T \bar{P}_{i_0} \bar{x}_0 \leq J_\infty(x_0) \leq \inf \bar{x}_0^T P_i \bar{x}_0 \tag{39}$$

for any $x_0 \in \mathcal{P}_{i_0}$.

We define $\gamma = \frac{1}{\rho}$ and replace then in Proposition 2.6 and Proposition 2.9. The PWA feedback controller $u(k) = F_i \bar{x}(k), x(k) \in \mathcal{P}_i$ makes the origin locally asymptotically stable, the input and output satisfying the constraints (4), (5) with a region of attraction

$$\mathcal{E} = \cup_{i \in \mathcal{I}_0} (\{x : x^T P_i x \leq 1/\gamma\} \cap \mathcal{P}_i)$$

*i.e. the feedback controller $u(k) = F_i \bar{x}(k), x(k) \in \mathcal{P}_i$ solves locally the problem (**P**), and moreover*

$$J_\infty(x_0) \leq 1/\gamma$$

for any $x_0 \in \mathcal{E}$.

Proof: From the construction of P_i we know that $x(\infty) = 0$. Actually we have that the function

$$V(x) = \bar{x}^T P_i \bar{x},\ x \in \mathcal{P}_i$$

is a piecewise quadratic Lyapunov function for the closed-loop system:

$$x(k+1) = \tilde{A}_i x(k) + \tilde{B}_i [\tilde{F}_i\ f_i][x(k)^T\ \ 1]^T + \tilde{a}_i,\ x(k) \in \mathcal{P}_i$$

Contrary to the continuous time case the Lyapunov function can be discontinuous across cell boundaries for discrete case.

*For the second part we know that $\mathcal{E}(1/\gamma)$ is an invariant set, but a larger invariant set is $\mathcal{E}$, that is a union of convex sets. From LMIs (37)-(32) the controller satisfies the input constraints. The output constraints are satisfied due to LMIs (38)-(33). Asymptotic stability is proved using the same Lyapunov function. In this way we can solve problem **P** locally.* $\diamondsuit$

3 Model predictive control law

3.1 MPC using an ellipsoidal terminal set

In the previous section we have found a PWA feedback controller $u(x) = F\bar{x}$ that solves problem **P** with a positive invariant set $\mathcal{E} = \mathcal{E}(\rho)$. In general this set is small in comparison with $\mathcal{E}_{\max}$ defined as the largest domain of attraction achievable by a control law solving problem **P**. In this section we show the benefits of MPC applied to solve problem **P**.

If at sample time k we want to minimize the infinite-horizon quadratic cost (11), this is very difficult because our system is nonlinear, and therefore we will have an infinite dimensional optimization problem. Therefore we apply a *quasi-infinite* method (see [16, 19]) to cope with this drawback. We consider a prediction horizon N, we assume that at sample time k the state $x(k)$ is available (i.e. can be measured or estimated) and we split the infinite-horizon cost into two parts:

$$J_\infty(x(k),\mathbf{u}) = \sum_{j=k}^{k+N-1} x^T(k)Qx(j) + u^T(j)Ru(j) + \sum_{j=k+N}^{\infty} x^T(j)Qx(j) + u^T(j)Ru(j)$$
$$= J_N(x(k)) + J_\infty(x(k+N)).$$

From Section 2 we have available $\overline{P}_i, K_i, P_i,\ i \in \mathcal{I}$ and moreover we have obtained an upper and lower bound for $J_\infty(x(k+N))$:

$$\inf_{P_i,K_i,U_{ij}} \bar{x}(k+N)^T P_{i_0}\bar{x}(k+N) \geq J_\infty(x(k+N)) \geq \sup_{\overline{P}_i,\overline{U}_{ij}} \bar{x}(k+N)^T \overline{P}_{i_0}\bar{x}(k+N)$$

for any $x(k+N) \in \mathcal{P}_{i_0}$.

The quasi-infinite method replaces the second infinite term with its upper bound [19]. Then at each sample step k we have to solve the following optimization problem which will be called **QI**(N):

$$J^*(k) = \min_{\mathbf{u}_k} \sum_{j=k}^{k+N-1} x^T(j)Qx(j) + u^T(j)Ru(j) + \bar{x}(k+N)^T P(k+N)\bar{x}(k+N)$$

subject to

$$\begin{cases} \mathbf{u}_k = (u(k), ..., u(k+N-1) \in U_c^N) \\ \text{equation } (1) \\ (y(k+1), ..., y(k+N)) \in X_c^N \\ \text{hard constraint: } x(k+N) \in \mathcal{E}(\rho). \end{cases}$$

where $P(k+N) = P_i$ if $x(k+N) \in \mathcal{P}_i$.

In the above formulation we can detect the standard *ingredients* for a stable MPC scheme: a terminal cost and constraint set (see [21]). According to the authors of [21], ideally, the terminal cost should be the infinite-horizon value cost (in this way is constructed a stable MPC for linear systems), but due to the nonlinearity of the system, this cannot be computed explicitly (as in linear case), therefore we replace it with its upper bound that we have derived in Section 2. MPC schemes using the upper bound of the infinite-horizon cost as a terminal cost has been employed also in [16, 19] in the context of MPC for LPV systems with polytopic uncertainty.

We apply a receding horizon principle, therefore after we solve the optimization problem at step k with optimal solution $\mathbf{u}_k^*$ we apply only the first sample $u(k) = \mathbf{u}_k^*(0) = F^{\mathrm{RH,N}}(x(k))$ and at next step $k+1$ we update the state and repeat the whole procedure.

In order to prove that receding horizon controller solves problem **P**, we introduce first some definitions.

Definition 3.1 *Let $\mathcal{F}(N, x_0)$ be the set of all feasible inputs corresponding to **QI**(N) and let $\mathcal{E}^{\mathrm{RH}}(N)$ be the set of initial states x_0 such that $\mathcal{F}(N, x_0)$ is non-empty.*

Definition 3.2 *Consider the closed loop system given by receding horizon control:*

$$\Sigma^{\mathrm{RH}} \begin{cases} x(k+1) = \tilde{A}_i x(k) + \tilde{B}_i F^{\mathrm{RH},N}(x(k)) + \tilde{a}_i, \\ y(k) = \tilde{C}_i x(k) + \tilde{c}_i, \ \textit{if}\ x(k) \in \mathcal{P}_i \end{cases}$$

Proposition 3.3 *We assume that we obtained $F_i, P_i, \mathcal{E}(\rho)$ applying one of the methods from Section 2. Then we have:*

(i) $\mathcal{E}^{\mathrm{RH}}(N)$ is a positive invariant set for Σ^{RH} and

$$\mathcal{E}(\rho) \subseteq \mathcal{E}^{\mathrm{RH}}(N), \ \forall\, N > 0 \tag{40}$$

*(ii) the quasi-infinite program **QI**(N) asymptotically stabilizes the system (2) with $u(k) = \boldsymbol{u}_k^*(0) = F^{\mathrm{RH},N}(x(k))$. Therefore this quasi-infinite receding horizon control solves problem **P**.*

(iii) $\mathcal{E}^{\mathrm{RH}}(N) \subset \mathcal{E}^{\mathrm{RH}}(N+1)$ and
*$\lim_{N\to\infty} \mathcal{E}^{\mathrm{RH}}(N) = \cup_{N=1}^{\infty} \mathcal{E}^{\mathrm{RH}}(N) = \mathcal{E}_{\max}$, where $\mathcal{E}_{\max}$ denotes the largest domain of attraction achievable by a control law solving problem **P**. Moreover, if there exists an N^* such that*

$$\mathcal{E}^{\mathrm{RH}}(N^*) = \mathcal{E}^{\mathrm{RH}}(N^*+1)$$

then $\mathcal{E}_{\max} = \mathcal{E}^{\mathrm{RH}}(N^*)$.

*Proof: (i) Let $x_0 \in \mathcal{E}^{\mathrm{RH}}(N) \cap \mathcal{P}_i$, then the optimization problem **QI**(N) has an optimal solution*

$$\boldsymbol{u}_0^* = (u(0)^*, ..., u(N-1)^*) \in U_{\mathrm{c}}^N, \ (y(1)^*, ..., y(N)^*) \in X_{\mathrm{c}}^N.$$

At the next step we have
$(u(1)^*, ..., u(N-1)^*, F_j x(N)^*) \in \mathcal{F}(N, \tilde{A}_i x_0 + \tilde{B}_i F^{\mathrm{RH},N}(x_0) + \tilde{a}_i)$ if $x(N)^* \in \mathcal{E}(\rho) \cap \mathcal{P}_j$ with $j \in \mathcal{I}_0$, because according to LMIs (37)-(38) or (32)-(33): $\tilde{F}_j x(N)^* \in U_{\mathrm{c}}$ and $y(N+1) = \tilde{C}_j(\tilde{A}_j + \tilde{B}_j \tilde{F}_j) x(N)^* \in X_{\mathrm{c}}$.

In conclusion $x_1 = \tilde{A}_i x_0 + \tilde{B}_i F^{\mathrm{RH},N}(x_0) + \tilde{a}_i \in \mathcal{E}^{\mathrm{RH}}(N)$, therefore (applying induction) we can prove that $\mathcal{E}^{\mathrm{RH}}(N)$ is a positive invariant set for Σ^{RH}.

*Moreover, for any $x_0 \in \mathcal{E}(\rho)$ there exists a feasible input sequence for **QI**(N), namely $(F(0)x_0, ..., F(N-1)x(N-1))$, where $F(\cdot) \in \{F_i, \ i \in I\} \Rightarrow x_0 \in \mathcal{E}^{\mathrm{RH}}(N)$, so that $\mathcal{E}(\rho) \subseteq \mathcal{E}^{\mathrm{RH}}(N), \ \forall\, N > 0$.*

*(ii) At sample step $k = 0$ the optimization problem **QI**(N) has an optimal solution*

$$\boldsymbol{u}_0^* = (u(0)^*, ..., u(N-1)^* \in U_c^N), \ (y(1)^*, ..., y(N)^*) \in X_c^N, \ x(N)^* \in \mathcal{E}(\rho^*).$$

Then at step $k = 1$ we have a feasible solution:

$$\boldsymbol{u}_1 = (u(1)^*, ..., u(N-1)^*, \tilde{F}_j x(N)^* \text{ if } x(N)^* \in \mathcal{E}(\rho^*) \cap \mathcal{P}_j)$$

Indeed by applying $\boldsymbol{u}_1$ we have $\tilde{F}_j x(N)^* \in U_{\mathrm{c}}$, $x(N+1) = (\tilde{A}_j + \tilde{B}_j \tilde{F}_j) x(N)^* \in \mathcal{E}(\rho)$ therefore $(y(2)^*, ..., y(N)^*, \tilde{C}_j(\tilde{A}_j + \tilde{B}_j \tilde{F}_j) x(N)^*)] \in X_{\mathrm{c}}^N$.

Moreover,

$$J^*(1) \leq J(\boldsymbol{u}_1) = J^*(0) - l(x(0)^*, u(0)^*) - x(N)^{*T} P_i x(N)^* + l(x(N)^*, \tilde{F}_i x(N)^*) + \\ x(N)^{*T} (\tilde{A}_i + \tilde{B}_i \tilde{F}_i)^T P_j (\tilde{A}_i + \tilde{B}_i \tilde{F}_i) x(N)^* \leq J^*(0) - l(x(0)^*, u(0)^*)$$

where we used in the last inequality the formula (19). Therefore

$$J^*(1) - J^*(0) \leq -l(x(0)^*, u(0)^*) \leq -\|x(0)^*\|_Q^2$$

By induction we can prove that:

$$J^*(k+1) - J^*(k) \leq -\|x(k)^*\|_Q^2$$

*i.e. the optimal quasi-infinite cost $J^*(k)$ is a Lyapunov function for the closed-loop system, and due to the previous inequality we have asymptotic stability. Therefore, in this way we can solve the problem **P** with the feedback controller $u(k) = F^{\mathrm{RH,N}}(x(k))$ and the positive invariant set $\mathcal{E}^{\mathrm{RH}}(N)$.*

(iii) Let $x_0 \in \mathcal{E}^{\mathrm{RH}}(N)$ then there exists $(u(0), ..., u(N-1)) \in \mathcal{F}(N, x_0)$, therefore $(u(0), ..., u(N-1), F(N)x^*(N) \in \mathcal{F}(N+1, x_0))$, so that $x_0 \in \mathcal{E}^{\mathrm{RH}}(N+1)$ i.e.

$$\mathcal{E}^{\mathrm{RH}}(N) \subseteq \mathcal{E}^{\mathrm{RH}}(N+1)$$

*As $N \to \infty$ the problem **QI**(N) becomes an infinite-horizon model predictive control problem implying that $\lim_{N\to\infty} \mathcal{E}^{\mathrm{RH}}(N) = \mathcal{E}_{\max}$.*

Moreover, from the equality

$$\mathcal{E}^{\mathrm{RH}}(N^*) = \mathcal{E}^{\mathrm{RH}}(N^*+1)$$

it follows that there does not exist a state $x_0 \notin \mathcal{E}^{\mathrm{RH}}(N^*)$ such that with a feasible input u the state $x_1 \in \mathcal{E}^{\mathrm{RH}}(N^*)$. Therefore $\mathcal{E}_{\max} = \mathcal{E}^{\mathrm{RH}}(N)$. $\diamondsuit$

Remark 3.4

- Point (i) is essential in order to ensure that it is worth replacing the auxiliary controller F with the receding horizon controller $F^{\mathrm{RH},N}$.

- Point (iii) shows that at the cost of an increasing of the computational effort associated with the optimization problem **QI**(N), the domain of attraction can be enlarged toward the maximum achievable one. Therefore N is a tuning parameter that realizes a trade-off between complexity/performance.

- When $N = 1$ we have to solve at each step k a convex optimization problem. If $N > 1$, **QI**(N) is a nonlinear optimization problem: the objective function is convex subject to linear and convex inequality constraints and nonlinear equality constraints.

3.2 Enlargement of the ellipsoidal terminal set

We have derived a stable MPC scheme that solved problem **P**. Stability is guaranteed by using a terminal set and a terminal cost. We derived a piecewise quadratic terminal cost together with a convex terminal set . The optimization problem that we have to solve on-line at each sample step k is nonlinear, non-convex, the computational time increasing with the prediction horizon N. If the terminal set is small, then we need a long prediction horizon in order to have feasibility for **QI**(N). Therefore, the optimization problem will be computationally intensive. A larger terminal set is $\mathcal{E} = \cup_{i\in\mathcal{I}_0}(\{x : x^T P_i x \leq \rho\} \cap \mathcal{P}_i)$, but this is not a convex set (it is a union of convex sets). In the sequel we develop a method to enlarge the terminal set based on *backward procedure* that can be done *off-line*, and thus we can efficiently implement on-line the stable MPC scheme derived before using a shorter prediction horizon. We consider the PWL dynamics of the system:

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k), \text{ if } x(k) \in \mathcal{P}_i \\ y(k) = C_i x(k), \end{cases} \tag{41}$$

where $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ is a partition of $\mathbb{R}^n$ into a number of polyhedral cells, with the closure of $\mathcal{P}_i$ given by $\mathrm{cl}(\mathcal{P}_i) = \{x \in \mathbb{R}^n : E_i x^T \geq 0\}$. Moreover we assume the constraints

(4)-(5) on input and output. Similar with Proposition 2.9 we derive an initial terminal region and cost based on LMIs.

Step 1 Solve the following convex optimization problem:

$$\min_{G,Y_i,S_i} -\sum_{i\in\mathcal{I}} \log\det S_i$$

subject to

$$\begin{bmatrix} G+G^T-S_i & * & * & * \\ A_iG+B_iY_i & S_j & * & * \\ Q^{1/2}G & 0 & I_d & * \\ R^{1/2}Y_i & 0 & 0 & I_d \end{bmatrix} > 0 \tag{42}$$

$$\begin{bmatrix} \Lambda & Y_i \\ * & G+G^T-S_i \end{bmatrix} \geq 0, \begin{bmatrix} \Gamma & C_i(A_iG+B_iY_i) \\ * & G+G^T-S_i \end{bmatrix} \geq 0, \tag{43}$$

with $\Lambda_{jj} \leq u^2_{j,\max}, \Gamma_{jj} \leq y^2_{j,\max}$ and define

$$F_{i,1} = Y_iG^{-1}, P_{i,1} = S_i^{-1}, \mathcal{E}_1 = \{x \in \mathbb{R}^n : x^T P_{i,1} x \leq 1,\ i \in \mathcal{I}\}.$$

According to Proposition 2.9 for any $x \in \mathcal{E}_1$, the controller $u = F_{i,1}x$ satisfies the constraints on input and output and maintains the trajectory of the closed-loop system inside $\mathcal{E}_1$ but converging asymptotically to origin.

Step 2 Using the previous terminal set $\mathcal{E}_{\text{prev}} = \{x \in \mathbb{R}^n : x^T P_{i,\text{prev}} x \leq 1,\ i \in \mathcal{I}\}$, we construct a new larger terminal set $\mathcal{E}_{\text{new}}$ based on a controller $F_{i,\text{new}}$, that steers the system from $\mathcal{E}_{\text{new}}$ but not within $\mathcal{E}_{\text{prev}}$ to the last terminal set $\mathcal{E}_{\text{prev}}$.

$$\min_{G,Y_i,S_i} -\sum_{i\in\mathcal{I}} \log\det S_i$$

subject to

$$\begin{bmatrix} G+G^T-S_i & * \\ A_iG+B_iY_i & P^{-1}_{j,\text{prev}} \end{bmatrix} > 0,\ \ S_i \geq \tau_i P^{-1}_{i,\text{prev}},\ \tau_i \geq 1 \tag{44}$$

and LMIs (43) for any $i, j \in \mathcal{I}$.

Proof: We denote with $P_{i,\text{new}} = S_i^{-1}, F_{i,\text{new}} = Y_iG^{-1}$. The second LMI in (44) is equivalent with:

$$\mathcal{E}_{\text{prev}} \subseteq \mathcal{E}_{\text{new}} = \{x \in \mathbb{R}^n : x^T P_{i,\text{new}} x \leq 1,\ i \in \mathcal{I}\}$$

The first LMI in (44) after applying the Schur complement, expresses the fact that:

$$P_{i,\text{new}} = S_i^{-1} \geq (A_i + B_iF_{i,\text{new}})P_{j,\text{prev}}(*)^T$$

i.e. if $x_0 \in (\mathcal{E}_{\text{new}} \cap \mathcal{P}_i) - \mathcal{E}_{\text{prev}}$ and applying the feedback controller $u_0 = F_{i,\text{new}}x_0$ then $x_1 = (A_i + B_iF_{i,\text{new}})x_0 \in \mathcal{E}_{\text{prev}}$. The LMIs (43) guarantee that the controller $u = F_{i,\text{new}}x$ satisfies the input and output constraints. Step 2 is an iterative procedure, i.e. we repeat it as long as we want, let us say L times (e.g. we stop when there is no more increase in the volume of the set $\mathcal{E}_{\text{new}}$).

Therefore we have available a sequence of controllers $u = F_{i,l}x,$ if $x \in (\mathcal{E}_l - \mathcal{E}_{l-1}) \cap \mathcal{P}_i,\ i \in \mathcal{I},\ l \in \{1, \cdots, L\}$ where by definition $\mathcal{E}_0$ is the empty set.

Remark 3.5 The steps 1 and 2 can be seen as an off-line multi-parametric LMI version of the MPC using multi-parametric quadratic programming (see also [11]).

In order to reduce the conservatism in Step 1 we can solve the optimization problem subject to LMI+BMI from Proposition 2.6 (therefore Step 1 will be a non-convex optimization problem). Similarly, we can proceed in Step 2, namely we can solve the optimization problem :

$$\min_{F_i,P_i,U_{ij}} \sum_{i\in\mathcal{I}} \log\det P_i$$

subject to

$$\begin{bmatrix} P_i - E_i^T U_{ij} E_i & * \\ A_i + B_i F_i & P_{j,\mathrm{prev}}^{-1} \end{bmatrix} > 0 \tag{45}$$

U_{ij} having all entries non-negative and

$$P_i \le \tau_i P_{i,\mathrm{prev}},\ 0 \le \tau_i \le 1 \tag{46}$$

and LMIs (43) for any $i, j \in \mathcal{I}$. But in this case the objective function is concave, subject to convex constraints (therefore again we have a non-convex optimization problem).

Step 3 At this stage we want to find a piecewise quadratic terminal cost $P(x) = x^T P_i x$ if $x \in \mathcal{P}_i$ such that stability is guaranteed when we apply the MPC scheme **QI**(N) with the terminal set $\mathcal{E}_L$. The sequence $\{P_i\}_{i\in\mathcal{I}}$ is determined solving the following LMIs:

$$(A_i + B_i F_{i,l})^T P_j (A_i + B_i F_{i,l}) - P_i + Q + F_{i,l}^T R F_{i,l} + E_i^T U_{i,j} E_i \le 0, \tag{47}$$

for any $i, j \in \mathcal{I}$, $l \in \{1, \cdots, L\}$ (see the proof for (ii) of Proposition 3.3 where the condition $J^*(k+1) - J^*(k) \le -l(x(k), u(k))$ is implied by the LMIs (47)).

Corollary 3.6 *(i) The controller*

$$u = F_{i,l}x,\ \textit{if}\ x \in (\mathcal{E}_\mathrm{l} - \mathcal{E}_{\mathrm{l}-1}) \cap \mathcal{P}_i$$

solves problem ***P*** *associated to PWL system (41).*

(ii) $\mathcal{E}_\mathrm{L}$ *is a positive invariant set for the closed-loop system*

$$\begin{cases} x(k+1) = (A_i + B_i F_{i,l})x(k),\ \textit{if}\ x(k) \in (\mathcal{E}_\mathrm{l} - \mathcal{E}_{\mathrm{l}-1}) \cap \mathcal{P}_i \\ y(k) = C_i x(k), \end{cases} \tag{48}$$

(iii) Using $\mathcal{E}_\mathrm{L}$ *as a terminal set and the terminal cost* $P(x) = x^T P_i x$ *if* $x \in \mathcal{P}_i$*, with* P_i *given by (47), Proposition 3.3 still holds.*

Proof: It is obvious that this controller stabilizes the PWL system (41), because for any $x_0 \in \mathcal{E}_\mathrm{L}$ *in at most L steps* $x(L) \in \mathcal{E}_1$ *and then according to Corollary 2.10* $x(L)$ *will converge asymptotically towards zero. Moreover this controller fulfills the input and output constraints.*

For the last part, we observe that if $x_0 \in \mathcal{E}_\mathrm{l} \subseteq \mathcal{E}_\mathrm{L}$ *then applying this feedback controller we have* $(A_i + B_i F_{i,l})x_0 \in \mathcal{E}_{\mathrm{l}-1} \subseteq \mathcal{E}_\mathrm{L}$*, therefore* $\mathcal{E}_\mathrm{L}$ *is a positive invariant set for the closed-loop system, and the relation (47) guarantee stability for the MPC scheme* ***QI****(N).* $\diamondsuit$

Remark 3.7 It is well-known the following fact:

Lemma 3.8 *[4] Let $\mathcal{E} = \cup_{i\in\mathcal{I}_0}\mathcal{E}_i$ be a union of polyhedral sets, such that $\mathcal{E}_i \cap \mathcal{E}_j = \oslash$ (empty set) for any $i \neq j \in \mathcal{I}_0$, then the condition $x \in \mathcal{E}$ can be expressed as mixed-integer linear inequalities.*

In Section 3.2 we have presented an algorithm to construct a *big enough* convex terminal set. For this type of terminal sets the optimal problem **QI**(N) is non-convex (except $N = 1$), therefore difficult to solve on-line. If we construct a polyhedral terminal set, the optimal problem becomes a mixed-integer quadratic programming (also very demanding computationally), but using branch-and-bound methods is more tractable that the non-convex problem.

From Section 3.2 we obtained a convex terminal set $\mathcal{E}_{\mathrm{L}}$. According to Lemma 3.8 we can use also polyhedral or union of polyhedral sets: $\cup_{i\in\mathcal{I}_0}\mathcal{E}(i)$ with $\mathcal{E}(i) = \{x \in \mathbb{R}^n : H_i x \leq h_i\} \subseteq \mathcal{P}_i$ as a positive invariant terminal set. One way of obtaining such a union of polyhedral sets is:

$$\{x : x^T P_{i,L-1}x \leq 1\} \cap \mathcal{P}_i \subseteq \mathcal{E}(i) \subseteq \{x : x^T P_{i,L}x \leq 1\} \cap \mathcal{P}_i$$

and then use $\cup_{i\in\mathcal{I}_0}\mathcal{E}(i)$ as a terminal set, and as terminal cost $P(x) = x^T P_{i,L}x$ if $x \in \mathcal{P}_i$, where $P_{i,L}$ are given by the LMIs (47). Finding such a set $\mathcal{E}(i)$ is an LMI problem. Proposition 3.3 still holds, but this time the optimal problem is a mixed-integer quadratic programming.

Another way to construct a terminal convex set is presented in [8, 17]. Moreover, in [8] sufficient conditions are given for this set to be a polytope. $\diamondsuit$

Example 1: We consider the PWL system (41) with system matrices given by:

$$A_1 = \begin{bmatrix} 0.35 & -0.6062 \\ 0.6062 & 0.35 \end{bmatrix}, A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ |x_1| \leq 5,\ |x_2| \leq 5,\ |u| \leq 1,$$
$$E1 = [1\ \ 0],\ E2 = [-1\ \ 0],\ Q = I,\ R = 0.1.$$

taken from [1].

For this system the LMIs from Step 1 has a solution for a common P:

$$P_{1,1} = P_{2,1} = P_1 = \begin{bmatrix} 1.3593 & 0 \\ 0 & 1.967 \end{bmatrix},$$
$$F_{1,1} = [-0.4646\ \ -0.1423],\ F_{2,1} = [0.4646\ \ -0.1423].$$

Iterating Step 2 for $L = 3$ we obtain the following terminal set (positive invariant set):

$$\mathcal{E}_3 = \{x \in \mathbb{R}^2 : x^T \begin{bmatrix} 0.0441 & 0 \\ 0 & 0.0627 \end{bmatrix} x \leq 1\}$$

and applying then Step 3 we obtain the following quadratic terminal cost:

$$P(x) = x^T \begin{bmatrix} 6.7534 & 0 \\ 0 & 9.2863 \end{bmatrix} x.$$

If we apply the MPC scheme **QI**(N) for the terminal ellipsoidal set given by P_1 we need at least $N = 4$ in order to have feasibility of the optimization problem for any $x \in [-5\ 5]\times[-5\ 5]$. Therefore we have to solve on-line a non-convex optimization problem

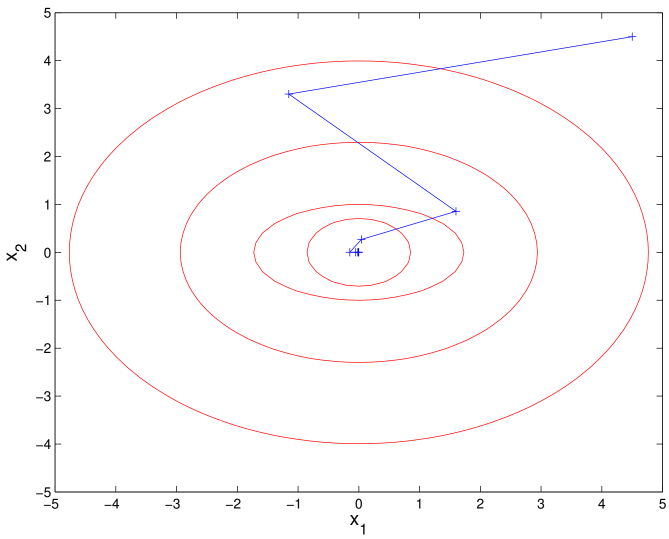

Figure 1: Enlargement of ellipsoidal terminal set and the trajectory corresponding to MPC scheme QI(1).

(computationally very demanding), but using the terminal set and cost given by this algorithm for $N = 1$ the optimization problem is feasible for any $x \in [-5\ 5] \times [-5\ 5]$. Therefore, at each step we have to solve a convex optimization (which actually is a optimization problem with quadratic objective function subject to linear and quadratic constraints), see also Figure 1.

4 Appendix

Fact 1

Let Q be a $(n+1) \times (n+1)$ symmetric matrix. Then $Q \geq 0$ if and only if $\begin{bmatrix} x \\ 1 \end{bmatrix}^T Q \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0,$ for any $x \in \mathbb{R}^n$.

Fact 2 (*Finsler's lemma*) Let Q be a symmetric matrix and a matrix B of appropriate dimension. The following two relation are equivalent:
(i) $B^{\perp T} Q B^{\perp} < 0$
(ii) $Q < \sigma B^T B$, for some $\sigma \in \mathbb{R}$.

Fact 3 (Proof Proposition 2.6) The first case $i, j \in \mathcal{I}_1$ was already proved. For each of the three remaining cases we will give the corresponding LMIs and BMIs. Afterward, we apply the same steps as in the proof for the first case.

Recall that we have defined:

$$P_i = \begin{bmatrix} \tilde{P}_i & 0 \\ 0 & 0 \end{bmatrix}, \text{ if } i \in \mathcal{I}_0; P_i = \begin{bmatrix} \tilde{P}_i & p_i \\ p_i^T & p_{ii} \end{bmatrix}, \text{ if } i \in \mathcal{I}_1.$$

We also define:

$$\tilde{S}_i, \tilde{F}_i \text{ if } i \in \mathcal{I}_0; S_i = \begin{bmatrix} \tilde{S}_i & s_i \\ s_i^T & s_{ii} \end{bmatrix}, F_i = [\tilde{F}_i\ f_i] \text{ if } i \in \mathcal{I}_1.$$

Case 2: $i, j \in \mathcal{I}_0$. Then (8)–(9) become:

$$\left[\begin{array}{ccc} \tilde{P}_i - Q - \tilde{E}_i^T U_{ij} \tilde{E}_i & * & * \\ \tilde{A}_i + \tilde{B}_i \tilde{F}_i & \tilde{S}_j & 0 \\ \tilde{F}_i & 0 & R^{-1} \end{array}\right] \geq 0,\ \tilde{S}_i \tilde{P}_i + \tilde{P}_i \tilde{S}_i \leq 2I.$$

Case 3: $i \in \mathcal{I}_0, j \in \mathcal{I}_1$. We define:

$$A_i(1) = \left[\begin{array}{cc} \tilde{A}_i + \tilde{B}_i \tilde{F}_i & 0 \\ 0 & 1 \end{array}\right], F_i = [\tilde{F}_i\ 0] \text{ if } i \in \mathcal{I}_0,$$

then (8) becomes:

$$\left[\begin{array}{ccc} P_i - \bar{Q} - E_i^T U_{ij} E_i & * & * \\ A_i(1) & S_j & 0 \\ F_i & 0 & R^{-1} \end{array}\right] \geq 0.$$

Case 4: $i \in \mathcal{I}_1, j \in \mathcal{I}_0$. We define:

$$A_i(2) = [\tilde{A}_i\ \ \tilde{a}_i] + \tilde{B}_i [\tilde{F}_i\ \ f_i],$$

then (8) becomes:

$$\left[\begin{array}{ccc} P_i - \bar{Q} - E_i^T U_{ij} E_i & * & * \\ A_i(2) & \tilde{S}_j & 0 \\ F_i & 0 & R^{-1} \end{array}\right] \geq 0.$$

Fact 4 (Proof Proposition 2.9)
We have considered the general PW quadratic Lyapunov function:

$$V(x) = \begin{cases} \left[\begin{array}{c} x \\ 1 \end{array}\right]^T \left[\begin{array}{cc} \tilde{P}_i & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ 1 \end{array}\right] & \text{if } i \in I_0 \\ \left[\begin{array}{c} x \\ 1 \end{array}\right]^T P_i \left[\begin{array}{c} x \\ 1 \end{array}\right] & \text{if } i \in I_1 \end{cases}$$

But from LMIs (35) $S_i^{-1} = P_i$, therefore to get $\tilde{P}_i,\ i \in I_0$ we should declare

$$S_i = \left[\begin{array}{cc} \tilde{S}_i & 0 \\ 0 & 1 \end{array}\right],\ i \in I_0$$

and then $\tilde{P}_i = \tilde{S}_i^{-1},\ \forall\, i \in I_0$.

Due to degree of freedom introduced by G, when we have a PWA system we should proceed as follows: take

$$G = \left[\begin{array}{cc} \tilde{G} & 0 \\ 0 & \tilde{g} \end{array}\right],\ Y_i = [\tilde{Y}_i\ \ \tilde{y}_i]$$

with $\tilde{y}_i = 0$ if $i \in I_0$, therefore

$$F_i = [\tilde{Y}_i \tilde{G}^{-1}\ \ \tilde{y}_i \tilde{g}^{-1}],\ i \in I_1,\ F_i = [\tilde{Y}_i \tilde{G}^{-1}\ \ 0],\ i \in I_0$$

In this case the feedback controller has the general form:

$$u = \tilde{Y}_i \tilde{G}^{-1} x + \tilde{y}_i \tilde{g}^{-1},\ \ x \in \mathcal{P}_i,\ \ i \in I_1,\ u = \tilde{Y}_i \tilde{G}^{-1} x,\ \ x \in \mathcal{P}_i,\ \ i \in I_0$$

We have to discuss the other three cases as we did in **Fact 2**.

Fact 5 (Algorithm for computing $0 < P \leq S^{-1}$) We want to solve the feasibility problem: find $\{P_i, S_i, F_i\}_{i\in\mathcal{I}}$ that satisfy the following matrix inequalities

$$LMI(S_i, P_i, F_i) < 0 \tag{49}$$

$$0 < P_i \leq S_i^{-1}, \text{ for all } i \in \mathcal{I}, \tag{50}$$

where $LMI(S_i, P_i, F_i) < 0$ are LMIs as in (27). It is clear that $0 < P_i \leq S_i^{-1}$ is equivalent with $0 < S_i \leq P_i^{-1}$ or $\lambda_{\max}(PS) \leq 1$ ($\lambda_{\max}$ denotes the maximum eigenvalue). We take $0 < \theta < 1$. The algorithm consist in three steps (see also [13]).

Step 1

Solve $LMI(S_i, P_i, F_i) < 0$, for all $i \in \mathcal{I}$. Therefore we have available $\{P_i^0, S_i^0, F_i^0\}_{i\in\mathcal{I}}$. If $P_i^0 \leq (S_i^0)^{-1}$ then we stop, because we found a solution. Otherwise, choose $\beta_i^0 > \lambda_{\max}(P_i^0 S_i^0)$.

Step 2

For all $k \geq 0$. Fix P_i^k. Solve the following LMIs:

$$LMI(S_i, P_i^k, F_i) < 0 \tag{51}$$

$$0 < S_i < \beta_i^k (P_i^k)^{-1}, \text{ for all } i \in \mathcal{I}, \tag{52}$$

We obtain $\{S_i^{k+1}\}_{i\in\mathcal{I}}$ and we define $\alpha_i^k = (1-\theta)\lambda_{\max}(S_i^{k+1}P_i^k) + \theta\beta_i^k$.

Step 3

Fix S_i^{k+1}. Solve the following LMIs:

$$LMI(S_i^{k+1}, P_i, F_i) < 0 \tag{53}$$

$$0 < P_i < \alpha_i^k (S_i^{k+1})^{-1}, \text{ for all } i \in \mathcal{I}, \tag{54}$$

We obtain $\{P_i^{k+1}, F_i^{k+1}\}_{i\in\mathcal{I}}$ and we define $\beta_i^{k+1} = (1-\theta)\lambda_{\max}(P_i^{k+1}S_i^{k+1}) + \theta\alpha_i^k$.

Properties of the algorithm:

1. If Step 1 is feasible then Step 2 and 3 are feasible for all $k \geq 0$.
2. If there exists k such that $\alpha_i^k \leq 1$ in Step 2 or $\beta_i^k \leq 1$ in Step 3 for all $i \in \mathcal{I}$, then we stop the algorithm. We found a solution.
3. $0 < \beta_i^{k+1} < \alpha_i^k < \beta_i^k$ for all $i \in \mathcal{I}$. Therefore there exists $\beta_i^* = \lim_{k\to\infty}\beta_i^k$ for all $i \in \mathcal{I}$. If $\beta_i^* < 1$ for all $i \in \mathcal{I}$, then the algorithm gives us a solution.

4.1 Example 2

We give now an example where the approach from Proposition 2.9 does not give a solution, while applying the Proposition 2.6 we obtain a solution for the matrix inequalities that we have to solve.

$$x(k+1) = \begin{cases} A_1x(k) + B_1u(k) \text{ if } E_1x(k) \geq 0 \\ A_2x(k) + B_2u(k) \text{ if } E_2x(k) \geq 0 \\ A_3x(k) + B_3u(k) \text{ if } E_3x(k) \geq 0 \\ A_4x(k) + B_4u(k) \text{ if } E_4x(k) \geq 0 \end{cases}$$

where the matrices of the system are given by

$$A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix},$$
$$A_3 = A_1,\ A_4 = A_2, B_i = [1 \quad 0]^T \text{ for all } i \in \{1,2,3,4\}.$$

The partitioning is given by:

$$E_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, E_3 = -E_1,\ E_4 = -E_2.$$

The tuning parameters Q and R are given by: $Q = 10^{-4} I_2,\ R = 10^{-3}$. We consider the following constraints: $|x_1| \leq 5,\ |x_2| \leq 5,\ |u| \leq 2$.

For this example applying the LMIs from Proposition 2.9 we do not get a feasible solution (using the Matlab LMI toolbox). We obtain conclusive results only if we are looking for a piecewise quadratic Lyapunov function and only if we apply the relaxations (S-procedure) from Proposition 2.6. We obtain as a feasible solution (applying the algorithm that we have proposed in Fact 5) the piecewise linear controllers $u = F_i x$ if $x \in \mathcal{P}_i$, with

$$F_1 = [-0.7162 \quad -0.9662],\ F_2 = [0.7657 \quad -0.4762],\ F_3 = F_1,\ F_4 = F_2,$$

and the piecewise quadratic Lyapunov function $P(x) = x^T P_i x$ if $x \in \mathcal{P}_i$ with

$$P_1 = \begin{bmatrix} 0.1589 & 0.1235 \\ 0.1235 & 0.1408 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0834 & -0.0207 \\ -0.0207 & 0.0815 \end{bmatrix}, P_3 = P_1, P_4 = P_2,$$

$$S_1 = \begin{bmatrix} 19.5829 & -17.1677 \\ -17.1677 & 22.1358 \end{bmatrix}, S_2 = \begin{bmatrix} 12.1854 & 2.9662 \\ 2.9662 & 12.9486 \end{bmatrix}, S_3 = S_1, S_4 = S_2,$$

where the matrices U_{ij} obtained by applying the relaxations from Proposition 2.1 are given by:

$$U_{11} = \begin{bmatrix} 0.0046 & 0.0265 \\ 0.0265 & 0.0122 \end{bmatrix}, U_{12} = \begin{bmatrix} 0.0040 & 0.0301 \\ 0.0301 & 0.0065 \end{bmatrix},$$
$$U_{22} = \begin{bmatrix} 0.0001 & 0.0010 \\ 0.0010 & 0.0158 \end{bmatrix}, U_{21} = \begin{bmatrix} 0.0001 & 0.0022 \\ 0.0022 & 0.0154 \end{bmatrix}.$$

Using Remark 3.5 for Step 2 we obtain that the terminal set $\mathcal{E}_L = \{x \in \mathbb{R}^2 : x^T P_{i,L} x \leq 1,\ i = 1,2\}$ is given by:

$$P_{1,L} = \begin{bmatrix} 0.1405 & 0.1125 \\ 0.1125 & 0.1228 \end{bmatrix}, P_{2,L} = \begin{bmatrix} 0.0687 & -0.0292 \\ -0.0292 & 0.0689 \end{bmatrix}.$$

The terminal cost is obtained from Step 3: $P(x) = x^T P_{i,f} x$ if $x \in \mathcal{P}_i$:

$$P_{1,f} = P_{3,f} = \begin{bmatrix} 4.8284 & 1.5050 \\ 1.5050 & 0.8351 \end{bmatrix}, P_{2,f} = P_{4,f} = \begin{bmatrix} 4.4540 & 0.4351 \\ 0.4351 & 1.2127 \end{bmatrix}.$$

Applying the MPC for this terminal set and cost we obtain the trajectory from Figure 2.

For solving the LMIs we used the Matlab LMI toolbox.

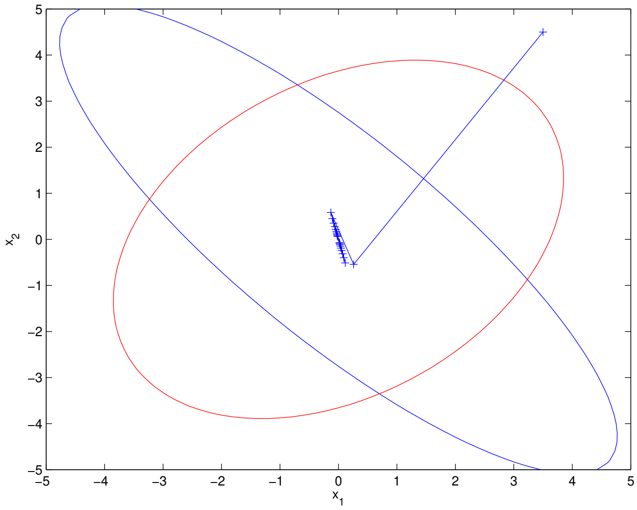

Figure 2: The ellipsoidal terminal set and the trajectory corresponding to MPC scheme QI(1).

5 Conclusions and Future Research

In this paper we have derived stabilization conditions for the class of PWA systems using the LMI framework. We consider the class of piecewise affine feedback controllers that guarantee stability of the closed-loop system. These controllers are derived from imposing that certain piecewise quadratic functions to be Lyapunov functions for the closed-loop system. Using LMIs arguments we have proved that the infinite-horizon quadratic cost is bounded if certain LMIs are satisfied. Using the upper bound of the infinite-horizon quadratic cost as a terminal cost and constructing also a convex terminal set, we show that the quasi-infinite receding horizon control stabilizes the PWA system. For future research we want to investigate stability of PWA with disturbances using a similar approach. Due to disturbance we do not have convergence to 0 but rather to a neighborhood of 0.

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