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MPC FOR MAX-PLUS-LINEAR SYSTEMS WITH GUARANTEED STABILITY

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Abstract: Model predictive control (MPC) is a popular controller design technique in the process industry. Conventional MPC uses linear or nonlinear discrete-time models. Previously, we have extended MPC to a class of discrete event systems that can be described by a model that is "linear" in the max-plus algebra. In this paper we consider the stability of MPC for these max-plus linear (MPL) systems, and we derive an MPL-MPC equivalent of the conventional end-point constraint. We show that with this end-point constraint the optimized cost function can be seen as a Lyapunov function for the system and can thus be used to prove stability.

Keywords: discrete event systems, model predictive control, max-plus-linear systems, Lyapunov stability.

1. INTRODUCTION

Model predictive control (MPC) (Garcia *et al.* 1989, Maciejowski 2002) is a proven technology for the control of multivariable systems in the presence of input, output and state constraints and is capable of tracking pre-scheduled reference signals. These attractive features make MPC widely accepted in the process industry. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to discrete event systems. Typical examples of discrete event systems (DES) are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, and logistic systems. In this paper we consider the class of DES with synchronization but no concurrency. Such DES can be described by models that are "linear" in the max-plus algebra (Baccelli *et al.* 1992, Cuninghame-Green 1979), and therefore they are called max-plus-linear (MPL) DES.
In (De Schutter and van den Boom 2001) we have extended MPC to MPL systems. In (van den Boom and De Schutter 2002) we have studied stability and tuning of MPC controllers for MPL systems and observed that for MPL systems, stability is not an intrinsic feature of the system, but it also depends on the input and the due dates (i.e., the reference signal) of the system. Some guidelines for an initial tuning of the MPL-MPC controller were derived, but still no stability could be guaranteed. In this paper we will derive a max-plus equivalent of the conventional end-point constraint (which is only appropriate for time-driven systems, (Rawlings and Muske 1993)), which in our case will be an inequality constraint.

2. MAX-PLUS ALGEBRA AND MAX-PLUS-LINEAR SYSTEMS

In this section we give the basic definition of the max-plus algebra and we present some results on a class of max-plus functions.

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows (Baccelli *et al.* 1992, Cuninghame-Green 1979): $x \oplus y = \max(x, y)$, $x \otimes y = x + y$ for numbers $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\dots,n} (a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$. The matrix E_n is the $n\times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The matrix $\varepsilon_{m\times n}$ is the $m\times n$ max-plus-algebraic zero matrix: $(\varepsilon_{m\times n})_{ij} = \varepsilon$ for all i, j. Finally, the max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n\times n}$ is defined as follows: $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ for $k = 1, 2, \dots$.

In (De Schutter and van den Boom 2001, De Schutter and van den Boom June, 2000, van den Boom and De Schutter 2002) we have studied MPC for DES in which there is synchronization but no concurrency. It has been shown (Baccelli *et al.* 1992) that these systems can be described by a model of the form

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \qquad (1)$$
$$y(k) = C \otimes x(k)\ . \qquad (2)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$. DES that can be described by this model will be called max-plus-linear (MPL). The index k is called the event counter. The state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instants at which the input events occur for the kth time, and the output $y(k)$ contains the time instants at which the output events occur for the kth time.

3. THE MPC PROBLEM FOR MAX-PLUS-LINEAR SYSTEMS

In (De Schutter and van den Boom 2001) we have shown that prediction of future values of $y(k)$ for the system (1)–(2) can be done by successive substitution, leading to the expression

$$\tilde{y}(k) = \tilde{C} \otimes x(k-1) \ \oplus \ \tilde{D} \otimes \tilde{u}(k) \qquad (3)$$

where $\tilde{C}$ and $\tilde{D}$ are given by

$$\tilde{C} = \begin{bmatrix} C \otimes A \\ C \otimes A^{\otimes^2} \\ \vdots \\ C \otimes A^{\otimes^{N_\mathrm{p}}} \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} C \otimes B & \varepsilon & \dots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{N_\mathrm{p}-1}} \otimes B & C \otimes A^{\otimes^{N_\mathrm{p}-2}} \otimes B & \dots & C \otimes B \end{bmatrix},$$

and $\tilde{u}(k)$, $\tilde{y}(k)$ are defined as:

$$\tilde{y}(k) = \begin{bmatrix} \hat{y}(k|k) \\ \hat{y}(k+1|k) \\ \vdots \\ \hat{y}(k+N_\mathrm{p}-1|k) \end{bmatrix}, \quad \tilde{u}(k) = \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N_\mathrm{p}-1|k) \end{bmatrix}$$

where $\hat{y}(k+j|k)$ denotes the prediction of $y(k+j)$ based on knowledge at event step k, $u(k+j|k)$ denotes the future input sequence based on knowledge at event step k, and N_p is the prediction horizon.

The MPC problem for MPL systems is formulated as follows (De Schutter and van den Boom 2001):

$$\min_{\tilde{u}(k),\tilde{y}(k)} J(\tilde{u}(k), \tilde{y}(k)) = \\ = \min_{\tilde{u}(k),\tilde{y}(k)} J_\mathrm{out}(\tilde{y}(k)) + \beta J_\mathrm{in}(\tilde{u}(k)) \qquad (4)$$

subject to (3) and

$$\Delta u(k+j) \geqslant 0 \qquad \text{for } j = 0, \dots, N_\mathrm{p}-1 \qquad (5)$$

where J_out is the output cost criterion, J_in the input cost criterion and β is a trade-off variable with $0 < \beta < 1$. Equation (5) guarantees a non-decreasing input signal [1]. The above problem will be called the MPL-MPC problem.

MPC uses a receding horizon strategy. So after computation of the optimal control sequence $u(k|k), \dots, u(k+N_\mathrm{p}-1|k)$, only the first control sample $u(k) = u(k|k)$ will be implemented, subsequently the horizon is shifted and the model and the initial state estimate can be updated if new measurements are available, then the new MPC problem is solved, etc.

In the remainder of this paper we consider the following output and input objective functions:

$$J_\mathrm{out}(\tilde{y}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{m} \max(\hat{y}_i(k+j|k) - \\ r_i(k+j), 0) \qquad (6)$$

$$J_\mathrm{in}(\tilde{u}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{l} \max(\mu_i(k+j) - \\ u_i(k+j|k), 0) \qquad (7)$$

where the signal $\mu(k)$ is related to the steady-state behavior, and will be defined in Section 4. The criteria can be interpreted as follows: J_out measures the tracking error or tardiness of the system, which is equal to the delay between the output date $\hat{y}_i(k+j|k)$ and due date $r_i(k+j)$ if $\hat{y}_i(k+j|k) - r_i(k+j) > 0$, and zero otherwise; J_in intends to maximizes the input dates $u_i(k+j|k)$.

If we replace (3) by the following (convex) inequality:

$$\tilde{y}(k) \geq \tilde{C} \otimes x(k-1) \ \oplus \ \tilde{D} \otimes \tilde{u}(k) \qquad (8)$$

we obtain the relaxed MPL-MPC problem, which is defined by the optimization of (4) subject to (5), and (8).

---

[1] Note that the input sequences correspond to occurrence times of consecutive events, and so $u(k)$ should be nondecreasing.

Proposition 1. (De Schutter and van den Boom 2001) *Let* $(\tilde{u}^*, \tilde{y}^*)$ *be an optimal solution of the relaxed MPL-MPC problem. If we define* $\tilde{y}^\sharp = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^*$ *then* $(\tilde{u}^*, \tilde{y}^\sharp)$ *is an optimal solution of the original MPL-MPC problem.*

So the MPL-MPC problem can be recast as a convex problem. Moreover, by introducing some additional dummy variables the problem can even be reduced to a linear programming problem (De Schutter and van den Boom 2001).

4. STABILITY AND STEADY-STATE BEHAVIOR

Stability in conventional system theory is concerned with boundedness of the states. In MPL systems however, k is an event counter and $x_i(k)$ refers to the occurrence time of an event. So the sequence $x_i(k), x_i(k+1), \ldots$ should always be nondecreasing, and for $k \to \infty$ the event time $x_i(k)$ will usually grow unbounded. We therefore adopt the notion of stability for DES from (Passino and Burgess 1998), in which a DES is called stable if all its buffer levels remain bounded.

Note that in our case we have due dates and that we assume that finished parts are removed from the output buffer at the due dates (provided that the parts are present). This means that there are delays if the parts are not produced before the due date. These delays should also remain bounded. Therefore, we add as an additional condition for stability that all delays between due dates and actual output dates remain bounded as well. If there are no internal buffers that are not (indirectly) coupled to the output of the system (observability), and if the system cannot be partitioned in independent subsystems (i.e. the system is strongly connected[2]), then it is easy to verify that all the buffer levels are bounded if the dwelling times of the parts or batches in the system remain bounded. This implies that for an observable strongly connected DES with due date $r(k)$ closed-loop stability is achieved if there exist finite constants M_{yr}, M_{ry} and M_{yu} such that

$$y_i(k) - r_i(k) \le M_{yr}, \quad \forall i \tag{9}$$
$$r_i(k) - y_i(k) \le M_{ry}, \quad \forall i \tag{10}$$
$$y_i(k) - u_j(k) \le M_{yu}, \quad \forall i, j \ . \tag{11}$$

Condition (9) means that the delay between the actual output date $y(k)$ and the due date $r(k)$ remains bounded. Condition (10) implies that the number of parts in the output buffer will remain bounded. Finally, condition (11) means that the time between the starting date $u(k)$ and the output date $y(k)$ (i.e., the throughput time) is bounded.

An important observation is that stability is not an intrinsic feature of the system, but it also depends on the input and the due dates (i.e., the reference signal) of the system. More precisely, it depends on the asymptotic slope of the input and due date sequence.

In (van den Boom and De Schutter 2002) we already observed that the max-plus-algebraic eigenvalue of the system matrix A plays a crucial role in stability[3]. We assume A is irreducible. This eigenvalue λ gives a minimum for the average duration of a system cycle. If the asymptotic slope of the due date signal $r(k)$ is smaller than λ, the system cannot complete tasks in time and $y(k) - r(k)$ will grow unbounded in time. So assume

$$r(k) = \rho\, k + d(k), \quad \text{where } |d(k)| \le d_{\max} \tag{12}$$

then a necessary condition for stability is that

$$\lambda < \rho$$

With due date signal (12) we can study steady-state behavior. Because of stability conditions (9),(10),(11) there have to exist finite values $v_{\max}$, $z_{\max}$, $w_{\max}$, such that

$$\begin{aligned} u(k) &= \rho\, k + v(k), \quad \text{where } |v(k)| \le v_{\max} \\ x(k) &= \rho\, k + z(k), \quad \text{where } |z(k)| \le z_{\max} \\ y(k) &= \rho\, k + w(k), \quad \text{where } |w(k)| \le w_{\max} \end{aligned}$$

To every ρ we can associate a shifted system

$$\begin{aligned} z(k) &= \bar{A} \otimes z(k-1) \ \oplus \ B \otimes v(k) \\ w(k) &= C \otimes z(k) \end{aligned}$$

where $[\bar{A}]_{ij} = [A]_{ij} - \rho$. Stability means that all signals in this system should remain bounded. We now consider the steady-state behavior of this shifted system. First note that the steady-state in MPL systems shows a cyclic behavior, and there exist sequences $v_{ss}(j)$, $z_{ss}(j)$ and $w_{ss}(j)$, $j = 1, \ldots, c$ such that

$$\begin{aligned} z_{ss}(j+1) &= \bar{A} \otimes z_{ss}(j) \oplus B \otimes v_{ss}(j) \\ &\qquad\qquad \text{for } j = 1, \ldots, c-1 \\ z_{ss}(1) &= \bar{A} \otimes z_{ss}(c) \oplus B \otimes v_{ss}(c) \\ w_{ss}(j) &= C \otimes z_{ss}(j) \ \text{ for } j = 1, \ldots, c \end{aligned}$$

where c is called the steady-state cycle length. We denote the steady-state by the quadruple $(v_{ss}, z_{ss}, w_{ss}, c)$ (Note that in general there exist several steady-states).

Now consider $d(k)$ to be fixed beyond some point: $d(k+j) = \bar{d}$ for $j > j_0$. We are now particularly

[2] This means that the precedence graph of the system is strongly connected (Baccelli *et al.* 1992), or equivalently, the system matrix A is irreducible.

[3] For a strongly connected max-plus-linear system, the A-matrix only has one max-plus-algebraic eigenvalue λ and one max-plus-algebraic eigenvector v, such that $A \otimes v = v \otimes \lambda$.

interested in the steady-state that maximizes the input sequence $v_{ss}(j)$ without violating the due-date $(w_{ss}(j) - \bar{d} \leq 0)$. The steady-state $(\bar{v}, \bar{z}, \bar{w}, \bar{c})$ that realizes this is called the maximum steady-state, and is the solution of the following problem (with $c \in \{1, 2, 3, \ldots\}$):

$$(\bar{v}, \bar{z}, \bar{c}) = \arg\max_{v,z,c} \frac{1}{c} \sum_{j=1}^{c} \sum_{\ell=1}^{m} v_\ell(j) \tag{13}$$

subject to

$$z(j+1) = \bar{A} \otimes z(j) \oplus B \otimes v(j),\ j = 1, \ldots, c-1 \tag{14}$$

$$z(1) = \bar{A} \otimes z(c) \oplus B \otimes v(c) \tag{15}$$

$$\bar{d} \geq C \otimes z(j)\ ,\quad j = 1, \ldots, c \tag{16}$$

Lemma 2. Problem (13)-(16) can be rewritten as a maximization over an extended linear complementary problem (ELCP).

Proof: Equations (14)-(15) give:

$$\underbrace{\begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(c) \end{bmatrix}}_{\tilde{z}} = \underbrace{\begin{bmatrix} \epsilon & \cdots & \epsilon & \bar{A} \\ \bar{A} & \cdots & \epsilon & \epsilon \\ \vdots & \ddots & \vdots & \vdots \\ \epsilon & \cdots & \bar{A} & \epsilon \end{bmatrix}}_{H} \otimes \underbrace{\begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(c) \end{bmatrix}}_{\tilde{z}}$$

$$\oplus \underbrace{\begin{bmatrix} B & \epsilon & \cdots & \epsilon \\ \epsilon & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \epsilon \\ \epsilon & \cdots & \epsilon & B \end{bmatrix} \otimes \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(c) \end{bmatrix}}_{g}$$

or

$$\tilde{z} = H \otimes \tilde{z} \oplus g \tag{17}$$

Equation (17) can be rewritten as:

$$\tilde{z}_i(j+1) = \max\left(\max_{p=1,\ldots,nc} (\tilde{z}_p(j) + H_{ip}), g_i\right)$$
$$\text{for } i = 1, \ldots, n,\ j = 1, \ldots, nc$$

or equivalently

$$\tilde{z}_i \geq (\tilde{z}_p + H_{ip}),\quad \text{for } i = 1, \ldots, n,\ p = 1, \ldots, nc \tag{18}$$

$$\tilde{z}_i \geq g_i,\quad \text{for } i = 1, \ldots, n, \tag{19}$$

with the additional constraint that at least one inequality should hold with equality (i.e. at least one residual should be equal to 0):

$$(\tilde{z}_i - g_i) \prod_{p=1}^{nc} (\tilde{z}_i - \tilde{z}_p - H_{ip}) = 0 \tag{20}$$

Equation (16) can be written as a set of linear inequality constraints:

$$(z_p(j) + C_{\ell p} - \bar{d}_\ell) \leq 0 \tag{21}$$

for $\ell = 1, \ldots, l,\ j = 1, \ldots, c, p = 1, \ldots, n$. So problem (13)-(16) is equivalent of maximizing (13) subject to (18), (19), (20), and (21), which is a maximization over an ELCP (De Schutter and De Moor (De Schutter and De Moor 1999)). $\diamond$

Now define the signals

$$\mu(k) = \rho\, k + \bar{v}(\lfloor k, \bar{c} \rfloor) \tag{22}$$

$$\zeta(k) = \rho\, k + \bar{z}(\lfloor k, \bar{c} \rfloor) \tag{23}$$

where $\bar{v}$ and $\bar{z}$ are defined by the maximum steady-state and where $\lfloor k, \bar{c} \rfloor$ is a function which gives a value $b = \lfloor k, \bar{c} \rfloor$, such that there exists an integer m such that $k = \bar{c} \cdot m + b$ and $1 \leq b \leq \bar{c}$ (so $b = (k-1) \bmod \bar{c} + 1$). If we substitute the function $\mu(k)$ in criterion function (4) with (6),(7) we find that if the system is in maximum steady-state then we have: $J(k) = 0$.

Lemma 3. Let the state for event step $(k-1)$ be in steady state (so $x(k-1) = \rho\,(k-1) + \bar{z}(\lfloor k, \bar{c} \rfloor)$), and let $u(k+j|k) = \mu(k+j)$ for $j = 0, \ldots, N_\mathrm{p} - 1$. Then $J(k) = 0$.

Proof: First note that because of (14)-(15) we find that $\hat{x}(k+j|k) = \rho\,(k+j) + \bar{z}(\lfloor k+j, \bar{c} \rfloor)$ for $j = 0, \ldots, N_\mathrm{p} - 1$, and with (16) we find:

$$\begin{aligned} \hat{y}(k+j|k) &= C \otimes \hat{x}(k+j|k) \\ &= \rho\,(k+j) + C \otimes \bar{z}(\lfloor k+j, \bar{c} \rfloor) \\ &\leq \rho\,(k+j) + C \otimes \bar{d} = r(k+j) \end{aligned}$$

for $j = 0, \ldots, N_\mathrm{p} - 1$. This makes the elements of J_out equal to $\max(\hat{y}_i(k+j|k) - r_i(k+j), 0) = 0$ for $j = 0, \ldots, N_\mathrm{p} - 1,\ i = 1, \ldots, m$ and the elements of J_in equal to $\max(\mu_i(k+j) - u_i(k+j|k), 0) = 0$ for $j = 0, \ldots, N_\mathrm{p} - 1,\ i = 1, \ldots, l$. Therefore $J(k) = J_\mathrm{out}(k) + J_\mathrm{in}(k) = 0$. $\diamond$

5. THE MPL-MPC END-POINT CONSTRAINT

In this section we introduce an end-point constraint that will make the closed-loop system asymptotically stable. The concept is analogous to the work of Kwon & Pearson (Kwon and Pearson 1979) for conventional systems. Similar to (Kwon and Pearson 1979) we will use the monotonicity of the performance index to prove stability. The main idea is to force the system to its steady state at the end of the prediction interval.

Theorem 4. Consider the MPL system given in the state space description for a due date signal (12) with $d(k) = \bar{d}$. Let $\mu(k)$ and $\zeta(k)$ be given by (22),(23), and consider a performance index by (4) with (6),(7). Now introduce the end-point constraint

$$x(k + N_\mathrm{p} - 1|k) \leq \zeta(k + N_\mathrm{p} - 1) \tag{24}$$

Then, the predictive control law, minimizing (4) subject to (5), (8) and (24) results in a stable closed loop.

Proof: Define

$$
\begin{aligned}
V(k) &= \min_{\tilde{u}(k),\tilde{y}(k)} J(\tilde{u}(k),\tilde{y}(k)) \\
&= \min_{\tilde{u}(k)} J(\tilde{u}(k), \tilde{C}\otimes x(k-1) \ \oplus\ \tilde{D}\otimes \tilde{u}(k)) \\
&= \min_{\tilde{u}(k)} \bar{J}(\tilde{u}(k)),
\end{aligned}
$$

and let

$$
\tilde{v}(k) = \begin{bmatrix} v(k|k) \\ v(k+1|k) \\ \vdots \\ v(k+N_\mathrm{p}-2|k) \\ v(k+N_\mathrm{p}-1|k) \end{bmatrix} = \tilde{u}(k) - \begin{bmatrix} \rho\cdot k \\ \rho\cdot(k+1) \\ \vdots \\ \rho\cdot(k+N_\mathrm{p}-2) \\ \rho\cdot(k+N_\mathrm{p}-1) \end{bmatrix}
$$
$$
= \tilde{u}(k) - \tilde{\rho}(k)
$$

The function $V(k)$ can be rewritten as

$$
\begin{aligned}
V(k) &= \min_{\tilde{u}(k)} \bar{J}(\tilde{u}(k)) = \min_{\tilde{v}(k)} \bar{J}(\tilde{v}(k)+\tilde{\rho}(k)) \\
&= \min_{\tilde{v}(k)} \sum_{j=0}^{N_\mathrm{p}-1}\sum_{i=1}^{m} \max\Big(w_i(k+j|k)-\bar{d}_i, 0\Big) \\
&\quad + \beta \sum_{j=0}^{N_\mathrm{p}-1}\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k,\bar{c}\rfloor) - v_i(k+j|k), 0\Big)
\end{aligned}
$$

Let $\tilde{v}^*(k) = \arg\min_{\tilde{v}(k)} \bar{J}(\tilde{v}(k)+\tilde{\rho}(k))$, then

$$
\begin{aligned}
V(k) &= \bar{J}(\tilde{v}^*(k)+\tilde{\rho}(k)) \\
&= \sum_{j=0}^{N_\mathrm{p}}\sum_{i=1}^{m} \max\Big(w_i^*(k+j|k)-\bar{d}_i, 0\Big) \\
&\quad + \beta \sum_{j=0}^{N_\mathrm{p}}\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k+j,\bar{c}\rfloor) - v_i^*(k+j|k), 0\Big)
\end{aligned}
$$

where $w^*(k+j|k)$ is the output signal for event $k+j$ when the optimal input sequence $\tilde{v}^*(k)$ is applied. Now define for event step $(k+1)$ a suboptimal vector

$$
\tilde{v}^{\mathrm{sub}}(k+1) = \begin{bmatrix} v^*(k+1|k) \\ \vdots \\ v^*(k+N_\mathrm{p}-1|k) \\ \bar{v}(\lfloor k+N_\mathrm{p},\bar{c}\rfloor) \end{bmatrix}
$$

Applying this input sequence $\tilde{v}^{\mathrm{sub}}(k+1)$ to the system results in an output $w^{\mathrm{sub}}(k+j|k+1) = w^*(k+j|k)$ for $j=1,\ldots,N_\mathrm{p}-1$. Now we compute the values for $j=N_\mathrm{p}$. Note that (24) means that $z(k+N_\mathrm{p}-1|k+1) \le \bar{z}(\lfloor k+N_\mathrm{p}-1,\bar{c}\rfloor)$ and therefore:

$$
\begin{aligned}
&z^{\mathrm{sub}}(k+N_\mathrm{p}|k+1) \\
&= \bar{A}\otimes z(k+N_\mathrm{p}-1|k+1)\otimes B\otimes \bar{v}(\lfloor k+N_\mathrm{p}-1,\bar{c}\rfloor) \\
&\le \bar{A}\otimes \bar{z}(\lfloor k+N_\mathrm{p}-1,\bar{c}\rfloor)\otimes B\otimes \bar{v}(\lfloor k+N_\mathrm{p}-1,\bar{c}\rfloor) \\
&\le \bar{z}(\lfloor k+N_\mathrm{p},\bar{c}\rfloor)
\end{aligned}
$$

This gives:

$$
\begin{aligned}
w^{\mathrm{sub}}(k+N_\mathrm{p}|k+1) &= C\otimes z^{\mathrm{sub}}(k+N_\mathrm{p}|k+1) \\
&\le C\otimes \bar{z}(\lfloor k+N_\mathrm{p}-1,\bar{c}\rfloor) \\
&\le \bar{d}
\end{aligned}
$$

With this we get $\bar{J}(\tilde{v}^{\mathrm{sub}}(k+1)+\tilde{\rho}(k+1))$:

$$
\begin{aligned}
&\bar{J}(\tilde{v}^{\mathrm{sub}}(k+1)+\tilde{\rho}(k+1)) = \\
&= \sum_{j=0}^{N_\mathrm{p}-1}\sum_{i=1}^{m} \max\Big(w_i^{\mathrm{sub}}(k+1+j|k+1)-\bar{d}_i, 0\Big) \\
&+\beta \sum_{j=0}^{N_\mathrm{p}-1}\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k+1+j,\bar{c}\rfloor) \\
&\qquad\qquad - v_i^{\mathrm{sub}}(k+j|k+1), 0\Big) \\
&= \sum_{j=1}^{N_\mathrm{p}-1}\sum_{i=1}^{m} \max\Big(w_i^{\mathrm{sub}}(k+j|k+1)-\bar{d}_i, 0\Big) \\
&+\beta \sum_{j=1}^{N_\mathrm{p}-1}\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k+j,\bar{c}\rfloor) \\
&\qquad\qquad - v_i^{\mathrm{sub}}(k+j|k+1), 0\Big) \\
&+\sum_{i=1}^{m} \max\Big(w_i^{\mathrm{sub}}(k+N_\mathrm{p}|k+1)-\bar{d}_i, 0\Big) \\
&+\beta\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k+j,\bar{c}\rfloor) - v_i^{\mathrm{sub}}(k+N_\mathrm{p}|k+1), 0\Big) \\
&= \sum_{j=1}^{N_\mathrm{p}-1}\sum_{i=1}^{m} \max\Big(w_i^*(k+j|k)-\bar{d}_i, 0\Big) \\
&+\beta \sum_{j=1}^{N_\mathrm{p}-1}\sum_{i=1}^{l} \max\Big(\bar{v}_i(\lfloor k+j,\bar{c}\rfloor) - v_i^*(k+j|k), 0\Big) \\
&+\sum_{i=1}^{m} 0 + \beta\sum_{i=1}^{l} 0 \\
&\le V(k)
\end{aligned}
$$

Further, because of the receding horizon strategy, we do a new optimization

$$
\begin{aligned}
V(k+1) &= \min_{\tilde{v}(k+1)} \bar{J}(\tilde{v}(k+1)+\tilde{\rho}(k+1)) \\
&\le \bar{J}(\tilde{v}^{\mathrm{sub}}(k+1)+\tilde{\rho}(k+1))
\end{aligned}
$$

Because $V(k)\ge 0$ and $V(k+1)\le V(k)$ we find that $V(k)$ is a Lyapunov function, which proves closed-loop stability of the system in the sense of definition 2 as this implies that $y-r$ and $r-u$ are bounded. ◇

Note that (24) can be rewritten in the form

$$
\begin{aligned}
x(k+N_\mathrm{p}-1) &= A^{\otimes^{N_\mathrm{p}}}\otimes x(k-1) \\
&\oplus \bigoplus_{i=0}^{N_\mathrm{p}-1} A^{\otimes^{i}}\otimes B\otimes u(k+N_\mathrm{p}-1-i|k+1) \\
&\le \zeta(k+N_\mathrm{p}-1)
\end{aligned} \tag{25}
$$

In (De Schutter and van den Boom 2001) it was shown that constraints (25) can be rewritten as a linear constraint in $\tilde{u}$ and by doing so, the MPL-MPC problem of minimizing (4) subject to (5), (8) and (24) results in a convex optimization problem that can be solved using a linear programming algorithm.

Feasibility
The existence of a solution of the MPL-MPC problem at event step k problem can be verified by solving the system of (in)equalities (5), (8) and (25), which describes the feasible set of the problem. Infeasibility occurs when solving $\tilde{u}(k)$ from (5), (8) and (25) results in a solution set that is empty. This will happen when $A^{\otimes^{N_\mathrm{p}}} \otimes x(k-1) > \zeta(k+N_\mathrm{p}-1)$. In that case $\tilde{u}$ has to be chosen as small as possible (so $u(k+j+1) = u(k+j)$), such that $(x(k+N_\mathrm{p}-1) = A^{\otimes^{N_\mathrm{p}}} \otimes x(k-1)$ and x will grow with minimum increment λ, where ζ will grow with increment $\rho > \lambda$. There will always be a finite number of event steps such that $A^{\otimes^{N_\mathrm{p}}} \otimes x(k-1) \leq \zeta(k+N_\mathrm{p}-1)$ and we can switch to the MPL-MPC problem again.

6. DISCUSSION

Model predictive control (MPC) for max-plus-linear (MPL) systems is a practical approach to design optimal input sequences for a specific class of discrete event systems without concurrency or choice and in which only synchronization plays a role. In this paper we have studied stability of MPL-MPC. A discrete event system is called stable if all its buffer levels remain bounded. We therefore considered the steady-state properties of MPL systems in the case of a due date sequence with a constant slope. We derived a MPL-MPC equivalent of this conventional end-point constraint (Kwon and Pearson 1979), by which we now can guarantee closed-loop stability. We showed that because of the end-point constraint the optimized cost function is always decreasing. In this way the cost function can be seen as a Lyapunov function for the system and we have proven stability. The final MPC problem with end-point constraint still results in a linear programming problem.

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