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Robustly stabilizing MPC for perturbed PWL systems: Extended Report

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Abstract— **In this paper we derive two robustly stable model predictive control (MPC) schemes for the class of piecewise linear (PWL) and hybrid systems. We assume that the plant model is subject to unknown but bounded disturbances and the states of the system can be measured or estimated. We derive a piecewise feedback controller based on linear matrix inequalities (LMI) that stabilizes the nominal system. Further we develop an algorithm for constructing a convex robustly positively invariant (RPI) set for the system. Using this convex RPI set as a terminal set we propose first a min-max feedback MPC scheme with known mode based on a dual-mode approach that stabilizes the system. The second robustly stable MPC scheme is based on a semi-feedback controller, but this time the mode of the system is unknown. Extension of the results from this paper to hybrid systems is also discussed.**

I. Introduction

A. Overview

In recent years, the study of hybrid systems has received a growing attention in control theory. Model predictive control (MPC) is applied to hybrid systems due to its ability to handle hard input, state, and/or output constraints. MPC is a control scheme in which the current input is computed by solving, at each sample step, a optimal control problem; the optimization of the performance function over the prediction period yields an optimal input sequence and the current control action is chosen to be the first input in this sequence according to the receding horizon principle.

The theory of the MPC for linear systems and in particular for linear systems with disturbances is quite mature (see [1], [15], [16], [25] and the references therein), but its extension to hybrid systems is still an active area of research. Recently, research has been focused on developing stabilizing MPC schemes for hybrid systems and in particular for piecewise linear (PWL) and piecewise affine (PWA) systems [2], [13], [17], [21]–[23]. PWL systems are defined by partitioning the state space of the system in a finite number of polytopes and associating to each polytope a different linear dynamic.

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Since disturbances are always present, it is important that the MPC controller be robust. In order to guarantee constraint fulfillment for every possible disturbance realization within a certain set, it is clear that the control action has to be chosen safe enough to cope with the effect of the worst disturbance realization. This effect is typically evaluated by predicting the open-loop trajectory of the system driven by such a worst-case disturbance. As investigated in [18], [25], this inevitably leads to a conservative scheme, and therefore, those authors suggest to use closed-loop predictions.

Some of the contributions in the literature on optimal control for perturbed hybrid systems include optimal control of continuous piecewise affine (PWA) systems with bounded disturbances [7], [13], [23]. In [7] the stability and l_2 gain analysis for the class of PWA systems is discussed. In [13] robust control for the class of *continuous* PWA systems is considered in the min-max framework, the optimal problem being solved using dynamic programming. In [23] a min-max MPC scheme for the same class of systems is employed and the optimal problem is recast as a set of linear programming (LP) problems using the equivalent max-min canonical representation of a continuous PWA system.

In this paper we consider the class of PWL systems with additive disturbance. In Section II we derive a local controller for the nominal system based on linear matrix inequality (LMI) framework. We give a complete discussion for the solution of the LMIs. Different levels of conservatism from applying the S-procedure are discussed. In Section III we construct a convex robustly positively invariant set for the system. Conditions when this set is a polytope are also derived. We propose two MPC algorithms for stabilizing a perturbed PWL system. In first algorithm we assume the mode to be known. Under this assumption we derive a stable min-max feedback MPC scheme based on a dual-mode approach. The notion that feedback is present in the receding-horizon implementation of the scheme leads to improve the performance and also the feasibility difficulties that arise with open loop min-max MPC techniques. This MPC scheme is based on solving at each step a mixed-integer linear programming problem (MILP). The second MPC scheme assumes unknown mode. We use the so-called *closed-loop paradigm* [24] by considering a semi-feedback control which combines a local control law with an open-loop correction in order to guarantee the input-state constraints. This scheme therefore renounces some degrees of freedom which in principle are available within a general min-max formulation. On the other hand, it allows to well balance increased computational burden and reduction of conservativeness. This scheme consists in solving at each sample k a quadratic optimization problem. From computational point of view, the second scheme is less demanding (quadratic programming) that the first scheme (mixed-integer linear programming). Finally, in Section VI we discuss the possible extensions of these two MPC schemes to PWA and hybrid systems.

B. Definitions

We use the following notations: a PWA system with additive disturbance is defined as

$$x(k+1) = A_i x(k) + B_i u(k) + a_i + w(k), \text{ if } x(k) \in \mathcal{P}_i \quad (1)$$

x, u and w denote, respectively, the state, input and disturbance; $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ is a finite partition of $\mathbb{R}^n$ into a number of polyhedral cells (n is the number of states). The closure $\text{cl}(\mathcal{P}_i)$ is given by $\text{cl}(\mathcal{P}_i) = \{x : E_i x \geq e_i\}$. When $a_i = 0$, $e_i = 0$, $\forall\, i \in \mathcal{I}$, we get a piecewise linear (PWL) system:

$$x(k+1) = A_i x(k) + B_i u(k) + w(k), \text{ if } x(k) \in \mathcal{P}_i \quad (2)$$

It is assumed that the disturbance belongs to a bounded polyhedron $w \in W$, and that the control and state are required to satisfy the constraints $u \in U_\text{c}$ and $x \in X_\text{c}$; where X_c, U_c and W are all polytopes, with $0 \in U_\text{c}, W$ and $0 \in \text{int}(X_\text{c})$.

In the sequel we will use also the following definitions: given two sets $Y, Z \subset \mathbb{R}^n$, the Minkowski sum of Y and Z is defined as $Y \oplus Z = \{y + z : y \in Y, z \in Z\}$ and the Pontryagin difference as $Y \ominus Z = \{y \in \mathbb{R}^n : y \oplus Z \subseteq Y\}$.

Let $M^\perp$ denote the orthogonal complement of a matrix M ($M^\perp$ exists only if M has linear dependent rows). We have then $M^T M^\perp = 0$ and $[M\ M^\perp]$ is nonsingular. We use also the following lemma:

Lemma 1.1: (*Finsler's lemma* [4]) Let Q be a symmetric matrix and a matrix M of appropriate dimension. The following two relation are equivalent:

(i) $M^{\perp T} Q M^\perp < 0$

(ii) $Q < \sigma M M^T$, for some $\sigma \in \mathbb{R}$.

The objective of this paper is to design a state feedback $u_k = \mu(x_k)$, via predictive control, which steers the state of system (2) as close as possible to the origin while satisfying the state and input constraints for all admissible disturbances. Clearly, the presence of an additive disturbance acting on the system (2) means that it is not possible to guarantee asymptotic stability (i.e. $\lim_{k\to\infty} x_k = 0$), but rather we try to steer the initial state x_0 to a neighborhood of the origin $\mathcal{O}$.

We assume for simplicity of the presentation that from a certain mode $i \in \mathcal{I}$ all the transitions to any other mode are possible (the case when only some transitions are possible from a certain mode can be implemented straightforwardly).

II. Derivation of the nominal controller

We consider the nominal system associated to the perturbed PWL system (2):

$$x_{k+1} = A_i x_k + B_i u_k, \text{ if } x_k \in \mathcal{P}_i \quad (3)$$

We want to design a stabilizing PWL state feedback controller

$$u_k = F_i x_k, \text{ if } x_k \in \mathcal{P}_i \quad (4)$$

so as to provide a satisfactory, or even optimal in some sense (e.g. LQ, $\mathcal{H}_\infty$), control performance to system (3). For instance, we want to bound the infinite-horizon quadratic cost:

$$J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \quad (5)$$

with $Q = Q^T > 0$, $R = R^T > 0$ and $\mathbf{u} = (u_0, u_1, ...)$, using a piecewise quadratic Lyapunov function

$$V(x) = x^T P_i x, \text{ if } x \in \mathcal{P}_i. \quad (6)$$

Using an approach as in [15] we impose the constraint:

$$V(x_{k+1}) - V(x_k) \leq -l(x_k, u_k), \ \forall\, k \geq 0 \quad (7)$$

where $l(x, u) = x^T Q x + u^T R u$. Relation (7) can be written

$$\begin{aligned} & x_k^T (A_i + B_i F_i)^T P_j (A_i + B_i F_i) x_k - x_k^T P_i x_k \\ & \leq -x_k^T Q x_k - x_k^T F_i^T R F_i x_k, \ \forall\, x_k \in \mathcal{P}_i. \end{aligned} \quad (8)$$

Moreover. we can require that given $\epsilon > 0$

$$x^T P_i x > \epsilon x^T x, \ \text{ only for } x \in \mathcal{P}_i. \quad (9)$$

Because we need (8)–(9) to be valid only for $x \in \mathcal{P}_i$, we can use *S-procedure* [4] in order to reduce conservatism when we solve (8). We can relax the matrix inequalities (8)–(9) to: find P_i, F_i, U_{ij}, V_i $i, j \in \mathcal{I}$, such that U_{ij}, V_i has all entries non-negative that satisfies the following matrix inequalities:

$$\begin{cases} x^T (A_i + B_i F_i)^T P_j (A_i + B_i F_i) x - x^T P_i x \\ \leq -x^T Q x - x^T F_i^T R F_i x - x^T E_i^T U_{ij} E_i x, \\ x^T P_i x > x^T E_i^T V_i E_i x, \ \forall\, x \in \mathbb{R}^n, \ \forall\, i, j \in \mathcal{I}. \end{cases}$$

In conclusion, we obtain the following matrix inequalities in P_i, F_i, U_{ij}, V_i (with all entries of U_{ij}, V_i non-negative):

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i \leq 0, \ \forall\, i, j \in \mathcal{I} \quad (10)$$

$$P_i > E_i^T V_i E_i, \ \forall\, i \in \mathcal{I}. \quad (11)$$

In the sequel the symbol * is used to induce a symmetric structure in an LMI. The following proposition give a solution to (10)–(11):

Theorem 2.1: The matrix inequalities (10)–(11) have a solution if and only if the following matrix inequalities have a solution

$$\begin{bmatrix} B_i^T P_j B_i + \theta R - I & B_i^T P_j A_i + F_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i + \theta Q - F_i^T F_i \end{bmatrix} < 0$$
$$P_i > E_i^T V_i E_i \tag{12}$$

where U_{ij}, V_i have all entries non-negative and θ is a scalar such that $\theta > 0$.

Proof: It is easy to see that (10) can be written as

$$\begin{bmatrix} F_i \\ I \end{bmatrix}^T \begin{bmatrix} B_i^T P_j B_i + R & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + Q + E_i^T U_{ij} E_i \end{bmatrix} \begin{bmatrix} F_i \\ I \end{bmatrix} < 0$$

We have the relation $\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp = \begin{bmatrix} F_i \\ I \end{bmatrix}^T$ since $\begin{bmatrix} F_i \\ I \end{bmatrix}$ is a basis of *ker*($\begin{bmatrix} -I \\ F_i^T \end{bmatrix}$) (where *ker*(A) denotes the kernel of the matrix A). Therefore, the previous formula can be written

$$\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp Q_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp T} < 0 \tag{13}$$

where $Q_{ij} = \begin{bmatrix} B_i^T P_j B_i + R & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i + Q \end{bmatrix}$. Using now the Finsler's lemma we obtain that (13) is equivalent with

$$Q_{ij} < \sigma_{i,j} \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \tag{14}$$

with $\sigma_{ij} \in \mathbb{R}$. Of course (14) has a solution if and only if

$$Q_{ij} < \sigma \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \tag{15}$$

with $\sigma > 0$ has a solution (Take $\sigma > \max_{i,j}\{0, \sigma_{ij}\}$ for the implication "(14) $\Rightarrow$ (15)". The other implication is obvious). Now if we divide (15) with $\sigma > 0$ and denote with $P_i \to 1/\sigma P_i$, $U_{ij} \to 1/\sigma U_{ij}$, $V_i \to 1/\sigma V_i$ and $\theta \to 1/\sigma$ we obtain (12). $\diamondsuit$

The matrix inequalities (12) are not LMIs due to the term $F_i^T F_i$. Therefore we have to use standard algorithms for solving bilinear matrix inequalities (BMI) such as the path-following algorithm [8].

Now we discuss some possible relaxations for (10)–(11). First relaxation is to replace the constraint (11) with $P_i > 0$. In this case we can apply the Schur complement to (10).

Theorem 2.2: With the relaxation $P_i > 0$ the matrix inequalities (10) have a solution if and only if the following matrix inequalities have a solution

$$\begin{bmatrix} P_i - Q - E_i^T U_{ij} E_i & * & * \\ A_i + B_i F_i & S_j & 0 \\ F_i & 0 & R^{-1} \end{bmatrix} > 0 \tag{16}$$
$$0 < P_j \le S_j^{-1}, \ \forall i,j \in \mathcal{I}. \tag{17}$$

Proof: Note that (10) is equivalent with

$$(A_i + B_i F_i)^T S_j^{-1} (A_i + B_i F_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i < 0, \qquad \forall\, i,j \in \mathcal{I} \tag{18}$$
$$0 < P_j \le S_j^{-1}, \qquad \forall\, j \in \mathcal{I} \tag{19}$$

In this way we take into account also the case $S_j = P_j^{-1}$. Indeed, it is clear that if (10) has a solution then there exists an $\epsilon > 0$ such that

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i < -\epsilon (A_i + B_i F_i)^T (A_i + B_i F_i).$$

Then, we can take $S_j^{-1} = P_j + \epsilon I > P_j$ and thus we obtain (18)–(19). The other implication is obvious.

Now, using the Schur complement (see [4]), the matrix inequalities (18)–(19) are equivalent with (16)–(17). $\diamondsuit$

We give here an algorithm for finding a feasible solution of matrix inequalities (16)–(17), using an approach as in [10]. We want to solve the feasibility problem: find $\{P_i, S_i, F_i\}_{i \in \mathcal{I}}$ that satisfy the following matrix inequalities

$$LMI(S_i, P_i, F_i) < 0 \tag{20}$$
$$0 < P_i \le S_i^{-1}, \text{ for all } i \in \mathcal{I}, \tag{21}$$

where $LMI(S_i, P_i, F_i) < 0$ are LMIs as in (16). It is clear that $0 < P_i \le S_i^{-1}$ is equivalent with $0 < S_i \le P_i^{-1}$ or $\lambda_{\max}(PS) \le 1$ ($\lambda_{\max}$ denotes the maximum eigenvalue). We take $0 < \theta < 1$. The algorithm consist in three steps.

Step 1

Solve $LMI(S_i, P_i, F_i) < 0$, for all $i \in \mathcal{I}$. Therefore we have available $\{P_i^0, S_i^0, F_i^0\}_{i \in \mathcal{I}}$. If $P_i^0 \le (S_i^0)^{-1}$ then we stop, because we found a solution. Otherwise, choose $\beta_i^0 > \lambda_{\max}(P_i^0 S_i^0)$.

Step 2

For all $k \ge 0$. Fix P_i^k. Solve the following LMIs:

$$LMI(S_i, P_i^k, F_i) < 0$$
$$0 < S_i < \beta_i^k (P_i^k)^{-1}, \text{ for all } i \in \mathcal{I},$$

We obtain $\{S_i^{k+1}\}_{i \in \mathcal{I}}$ and we define $\alpha_i^k = (1 - \theta)\lambda_{\max}(S_i^{k+1} P_i^k) + \theta \beta_i^k$.

Step 3

Fix S_i^{k+1}. Solve the following LMIs:

$$LMI(S_i^{k+1}, P_i, F_i) < 0$$
$$0 < P_i < \alpha_i^k (S_i^{k+1})^{-1}, \text{ for all } i \in \mathcal{I},$$

We obtain $\{P_i^{k+1}, F_i^{k+1}\}_{i \in \mathcal{I}}$ and we define $\beta_i^{k+1} = (1 - \theta)\lambda_{\max}(P_i^{k+1} S_i^{k+1}) + \theta \alpha_i^k$.

Properties of the algorithm:

1) If Step 1 is feasible then Step 2 and 3 are feasible for all $k \ge 0$.
2) If there exists k such that $\alpha_i^k \le 1$ in Step 2 or $\beta_i^k \le 1$ in Step 3 for all $i \in \mathcal{I}$, then we stop the algorithm. We found a solution.
3) $0 < \beta_i^{k+1} < \alpha_i^k < \beta_i^k$ for all $i \in \mathcal{I}$. Therefore there exists $\beta_i^* = \lim_{k \to \infty} \beta_i^k$ for all $i \in \mathcal{I}$. If $\beta_i^* < 1$ for all $i \in \mathcal{I}$, then the algorithm gives us a solution.

We propose now a second relaxation. If we do not apply the S-procedure for (8)–(9), i.e. we replace the condition "$x \in \mathcal{P}_i$", with $x \in \mathbb{R}^n$, then (8) becomes:

$$(A_i+B_iF_i)^T P_j(A_i+B_iF_i)-P_i+Q+F_i^T RF_i \leq 0 \quad (22)$$

for all $i, j \in \mathcal{I}$ and $P_i > 0$. When we do not take the piecewise linear structure of the system into account, then we can see such a system as a linear system with polytopic uncertainty. There are two well-known linearization methods in the literature for (22): one uses the linearizing change of variables by introducing: $S_i = P_i^{-1}, F_i = Y_i S_i$ (see [4], [15], [17], [21]) and the second one is $P_i = S_i^{-1}, F_i = Y_i G^{-1}$ (see [6], [22]).

Proposition 2.3: If the following LMIs in G, Y_i, S_i

$$\begin{bmatrix} G+G^T-S_i & * & * & * \\ A_iG+B_iY_i & S_j & * & * \\ Q^{1/2}G & 0 & I & * \\ R^{1/2}Y_i & 0 & 0 & I \end{bmatrix} > 0 \quad (23)$$

for all $i, j \in \mathcal{I}$ have a solution then $F_i = Y_i G^{-1}, P_i = S_i^{-1}$ are solutions of (22).

Proof: From (23) using the Schur complement, we observe first that G is a nonsingular matrix because

$$G + G^T > S_i$$

and also

$$0 < S_i \Rightarrow (S_i - G)^T S_i^{-1}(S_i - G) \geq 0$$

therefore we get the following relation:

$$G + G^T - S_i \leq G^T S_i^{-1} G$$

and

$$\begin{aligned} 0 <& G + G^T - S_i - (A_iG + B_iY_i)^T S_j^{-1}(*) - G^T QG \\ &- Y_i^T RY_i \leq G^T S_i^{-1}G - (A_iG + B_iY_i)^T S_j^{-1}(*) \\ &- G^T QG - Y_i^T RY_i \\ &= G^T (S_i^{-1} - (A_i + B_iY_iG^{-1})^T S_j^{-1}(*) \\ &- Q - G^{-T} Y_i^T RY_iG^{-1})G \end{aligned}$$

Taking $F_i = Y_i G^{-1}, P_i = S_i^{-1}$ we obtain from the last relation the matrix inequalities (22). ◊

If we are interested only in stability and we do not apply the S-procedure we must solve the following LMIs:

$$(A_i + B_iF_i)^T P_j(A_i + B_iF_i) - P_i < 0 \quad (24)$$

for all $i, j \in \mathcal{I}$.

Proposition 2.4: The following LMIs in G_i, Y_i, S_i

$$\begin{bmatrix} G_i + G_i^T - S_i & * \\ A_iG_i + B_iY_i & S_j \end{bmatrix} > 0 \quad (25)$$

for all $i, j \in \mathcal{I}$ have a solution if and only if $F_i = Y_i G_i^{-1}, P_i = S_i^{-1}$ are solutions of (24).

Proof: We prove the sufficiency first. From (25) using Schur complement arguments, we observe first that G_i is a nonsingular matrix because $G_i + G_i^T > S_i$ and also

$$(S_i - G_i)^T S_i^{-1}(S_i - G_i) \geq 0 \ \text{ since } \ S_i > 0.$$

Therefore, we get the following relation:

$$G_i + G_i^T - S_i \leq G_i^T S_i^{-1} G_i.$$

Now using Schur complement formula in (25), we get:

$$\begin{aligned} 0 &< G_i + G_i^T - S_i - (A_iG_i + B_iY_i)^T S_j^{-1}(*) \\ &\leq G_i^T S_i^{-1}G_i - (A_iG_i + B_iY_i)^T S_j^{-1}(*) \\ &= G_i^T (S_i^{-1} - (A_i + B_iY_iG_i^{-1})^T S_j^{-1}(*) \\ &G_i^T (S_i^{-1} - (A_i + B_iY_iG_i^{-1})^T S_j^{-1}(*))G_i \end{aligned}$$

Taking $F_i = Y_i G_i^{-1}, P_i = S_i^{-1}$, and using the fact that G_i is invertible, we obtain from the last relation the LMIs (24):

$$(A_i + B_iF_i)^T P_j(A_i + B_iF_i) - P_i < 0, \text{ for any } i, j \in \mathcal{I}.$$

We can prove that the converse is also true by taking $S_i = P_i^{-1}, G_i = P_i^{-1} + g_i I$ with $g_i > 0$ a positive scalar and $Y_i = F_i G_i$. Using again Schur complement arguments and taking g_i small enough, we can prove that if the LMIs (24) have a solution, then with the above notations, S_i, G_i, Y_i is also a solution of (25) (see also [6]). ◊

Now let us assume that we have found $P_i, F_i, \ i \in \mathcal{I}$ with one of the methods proposed before. In that case the nominal system (3) in closed-loop with the controller $u = F_i x$ if $x \in \mathcal{P}_i$ is asymptotically stable, having $V(x)$ as a Lyapunov function (due to (7)). For any initial state $x_0 \in \mathcal{P}_i$, applying the input sequence $u_k = F(k)x_k = F_j x_k$ if $x_k \in \mathcal{P}_j$ we have that $V(x_\infty) = 0$. Then, summing up the relation (7)) from $k = 0, ..., \infty$, we obtain the following bound on the infinite-horizon quadratic cost:

$$J_\infty(x_0, F(\cdot)x) \leq x_0^T P_i x_0.$$

The control law $u_k = F_i x_k$ if $x_k \in \mathcal{P}_i$ does not depend on the initial state and is regarded as fixed a-priori and referred to as the nominal feedback.

III. Convex robustly positively invariant set

In the sequel we assume that we have determined a state feedback controller $u(k) = F_i x(k)$ if $x(k) \in \mathcal{P}_i$ that stabilizes the nominal system (3) as we discussed in Section II. We denote with $A_{F_i} = A_i + B_i F_i$ for all $i \in \mathcal{I}$. Then the PWL system with additive disturbance (2) becomes:

$$x(k+1) = A_{F_i} x(k) + w(k), \text{ if } x(k) \in \mathcal{P}_i. \quad (26)$$

We define the following set:

$$X_F = \cup_{i \in \mathcal{I}} \{x \in \mathcal{P}_i : x \in X_c, F_i x \in U_c\}$$

Definition 3.1 ([3], [14]): (i) Given a perturbed system $x(k+1) = f(x(k), w(k))$, with $w(k) \in W$. The set Ω is a *robustly positively invariant* (RPI) set for this system if $f(x, w) \in \Omega$ for any $x \in \Omega$ and $w \in W$.

(ii) A set $\Omega \subseteq X_F$ is an RPI set for system (26) if for any $x \in \Omega \cap \mathcal{P}_i$ with $i \in \mathcal{I}$, we have $A_{F_i}x + w \in \Omega$ for all $w \in W$. The maximal (minimal) RPI set for system (26) is defined as the largest (smallest, non-empty) with respect to inclusion, RPI set for (26). $\diamondsuit$

It can be easily seen that both the minimal and the maximal RPI set associated to system (26) is in general not a convex set (it is a union of polyhedral sets [11]). Our aim is to compute a polyhedral RPI set, since we want to obtain only linear constraints for the MPC schemes that we propose in the sequel. For system (26) the evolution of the mode $i = i(k)$ depends on the state $x(k)$. Nevertheless, for ease of computation of a convex (polyhedral) RPI set for (26), this relation mode-state will be disregarded and we will consider that $i(k)$ evolves independently of $x(k)$ (i.e. any mode $i(k) \in \mathcal{I}$ can be active at any sample step k)

$$\begin{cases} x_{k+1} = A_{F_{i(k)}}x_k + w_k, \\ i(k+1) \in \mathcal{I} \end{cases} \tag{27}$$

where $i(\cdot)$ is a switching signal in $\mathcal{I}^{\mathbb{N}}$. Note that this relaxation is considered only in this section, in the next section where we present the MPC scheme we consider again the standard PWL system (2). This type of relaxation was used also in [5], [17] in the context of MPC for deterministic systems.

Definition 3.2: A set Ω is an RPI set for system (27) if for any $x \in \Omega$ we have that $A_{F_i}x + w \in \Omega$, for any possible switching $i \in \mathcal{I}$ and any admissible disturbance $w \in W$. $\diamondsuit$

In the sequel we construct the maximal RPI set for system (27). Let X_{F_i} denote the set of states that satisfy the state-input constraints:

$$X_{F_i} = \{x \in \mathbb{R}^n : x \in X_c, F_i x \in U_c\}. \text{ Then } \bigcap_{i \in \mathcal{I}} X_{F_i} \subseteq X_F.$$

We define the following set recursion:

$$\begin{aligned} &\mathcal{O}_0^i = X_{F_i}, \\ &\mathcal{O}_t^i = \{x \in X_{F_i} : A_{F_i}x \oplus W \subseteq \cap_{j \in \mathcal{I}} \mathcal{O}_{t-1}^j\} \end{aligned} \tag{28}$$

for any $i \in \mathcal{I}$ and $t = 1, 2, ...$

The set $\mathcal{O}_t^i$ represents the set of initial states $x(0)$, for which under the closed-loop dynamics (27) the state-input constraints are satisfied up to sample step t assuming that initially $i(0) = i$. It is clear from the definition (28) that $\mathcal{O}_{t+1}^i \subseteq \mathcal{O}_t^i$, and therefore $\mathcal{O}_t^i$ converges to $\mathcal{O}_\infty^i$. We define:

$$\begin{aligned} &\mathcal{O}_\infty^i = \lim_{t \to \infty} \mathcal{O}_t^i = \cap_{t \geq 0} \mathcal{O}_t^i, \\ &\mathcal{O}_\infty = \cap_{i \in \mathcal{I}} \mathcal{O}_\infty^i. \end{aligned} \tag{29}$$

Theorem 3.3: (i) The maximal RPI set included in $\cap_{i \in \mathcal{I}} X_{F_i}$ for the system (27) is the *convex* set $\mathcal{O}_\infty$.

(ii) Any RPI set for the system (27) is also an RPI set for the PWL system (26). In particular $\mathcal{O}_\infty$ is an RPI set for the PWL system (26).

Proof: (i) It is easy to observe that since the sets X, U, and W are polytopes (described by linear inequalities), all the sets $\mathcal{O}_t^i$ are described by a finite number of linear inequalities. Therefore, all $\mathcal{O}_t^i$ are convex sets for all $i \in \mathcal{I}$ and $t \geq 0$. Since $\mathcal{O}_\infty$ is the intersection of convex sets, $\mathcal{O}_\infty$ is also convex.

For any $x \in \mathcal{O}_\infty$ we have $x \in \mathcal{O}_{t+1}^i$ for all $i \in \mathcal{I}$ and $t \geq 0$. According to (28) we have $A_{F_i}x \oplus W \subseteq \cap_{j \in \mathcal{I}} \mathcal{O}_t^j$ for all $i \in \mathcal{I}$ and $t \geq 0$. Hence $A_{F_i}x \oplus W \subseteq \mathcal{O}_\infty$ for all $i \in \mathcal{I}$. Therefore $\mathcal{O}_\infty$ is an RPI set for system (27).

It is well-known [3], [14] that the maximal RPI set for a system is the set of all initial states in $\cap_{i \in \mathcal{I}} X_{F_i}$ for which the evolution of the system remain in $\cap_{i \in \mathcal{I}} X_{F_i}$. Due to the recursion (28) it is clear that $\mathcal{O}_\infty$ is the maximal RPI set for system (27) included in $\cap_{i \in \mathcal{I}} X_{F_i}$. Indeed, let $T \subseteq \cap_{i \in \mathcal{I}} X_{F_i}$ be an RPI set for the system (27) and let $x \in T$. Then from the definition of an RPI set for the system (27) (see Definition 3.1) we have $A_{F_i}x \oplus W \subseteq T \subseteq \cap_{i \in \mathcal{I}} X_{F_i} = \cap_{i \in \mathcal{I}} \mathcal{O}_0$ for all $i \in \mathcal{I}$. This implies that $x \in \mathcal{O}_1^i$ for all $i \in \mathcal{I}$ (according to the recursion (28)). Therefore, $T \subseteq \mathcal{O}_1^i$ for all $i \in \mathcal{I}$. By iterating this procedure we obtain that $T \subseteq \mathcal{O}_t^i$ for all $t \geq 0$ and $i \in \mathcal{I}$. In conclusion $T \subseteq \mathcal{O}_\infty$, i.e. $\mathcal{O}_\infty$ is maximal.

(ii) First we have that $\mathcal{O}_\infty \subseteq \cap_{i \in \mathcal{I}} X_{F_i} \subseteq X_F$. If $x \in \mathcal{O}_\infty \cap \mathcal{P}_i$ then $A_{F_j}x \oplus W \subseteq \mathcal{O}_\infty$ for all $j \in \mathcal{I}$. In particular for $j = i$ we have $A_{F_i}x \oplus W \subseteq \mathcal{O}_\infty$. Therefore, $\mathcal{O}_\infty$ is an RPI set for the system (26). For a general RPI set the reasoning is similar. $\diamondsuit$

Remark 3.4 A larger RPI set for system (26) is the set $\cup_{i \in \mathcal{I}} (\mathcal{O}_\infty^i \cap \mathcal{P}_i)$, but it is not convex (it is a union of convex sets). However since the additional uncertainty is inherently introduced in the extended dynamics (27) with respect to (26), this set is not the maximal RPI set for (26). $\diamondsuit$

Because the sets $\mathcal{O}_t^i$ are described by a finite number of linear inequalities, it is important to know whether the set $\mathcal{O}_\infty$ can be *finitely determined*, i.e. whether there exists a finite t^* such that $\mathcal{O}_{t^*}^i = \mathcal{O}_{t^*+1}^i$ for all $i \in \mathcal{I}$ (therefore $\mathcal{O}_\infty = \cap_{i \in \mathcal{I}} \mathcal{O}_{t^*}^i$ is a polyhedral set). In the sequel we give necessary conditions for finite determination. Using the recursion (28) and the commutativity property of intersection, we have:

$$\begin{aligned} &\mathcal{O}_0 = \cap_{i \in \mathcal{I}} \mathcal{O}_0^i, \ \mathcal{O}_t = \cap_{i \in \mathcal{I}} \mathcal{O}_t^i \text{ for all } t \geq 1 \Rightarrow \\ &\mathcal{O}_{t+1} \subseteq \mathcal{O}_t, \text{ and therefore, } \mathcal{O}_\infty = \cap_{t \geq 0} \mathcal{O}_t. \end{aligned}$$

Now, $\mathcal{O}_t$ can be written in terms of Pontryagin differences:

$$\begin{aligned} &Y_0 = \cap_{i \in \mathcal{I}} X_{F_i}, \ \mathcal{O}_0 = Y_0; \\ &Y_1 = Y_0 \ominus W, \ \mathcal{O}_1 = \cap_{i \in \mathcal{I}} \{x \in \mathcal{O}_0 : A_{F_i}x \in Y_1\}; \\ &Y_t = \cap_{(i_1, \dots i_{t-1}) \in \mathcal{I} \times \dots \times \mathcal{I}} (Y_{t-1} \ominus A_{F_{i_1}} \dots A_{F_{i_{t-1}}} W), \\ &\mathcal{O}_t = \cap_{(i_1, \dots i_t) \in \mathcal{I} \times \dots \times \mathcal{I}} \{x \in \mathcal{O}_{t-1} : A_{F_{i_1}} \dots A_{F_{i_t}} x \in Y_t\}. \end{aligned} \tag{30}$$

It is clear that $Y_{t+1} \subseteq Y_t$ (because $0 \in W$). Therefore, the limit of this sequence $Y_\infty = \cap_{t \geq 0} Y_t$ exists. We have the following proposition:

Theorem 3.5: If the system (26) is asymptotically stable and if there exists an index $t_0 \geq 0$ such that $\mathcal{O}_{t_0}$ is bounded and $0 \in \text{int}(Y_\infty)$, then $\mathcal{O}_\infty$ is finitely determined and therefore also a polyhedral set.

Proof: Since (26) is asymptotically stable, then for any $(i_1,...,i_t) \in \mathcal{I} \times ... \times \mathcal{I}$ we have

$$\begin{cases} A_{F_{i_1}}...A_{F_{i_t}}x \to 0, \text{when } t \to \infty, \text{ for all } x \in \mathbb{R}^n \\ \mathcal{O}_{t_0} \text{ bounded} \\ 0 \in \text{int}(Y_\infty) \end{cases}$$

implies that there exists a $t^* \geq t_0$ such that for all $(i_1,...,i_{t^*+1}) \in \mathcal{I} \times ... \times \mathcal{I}$:

$$A_{F_{i_1}}...A_{F_{i_{t^*+1}}}x \in Y_\infty \subseteq Y_{t^*+1}, \text{ for all } x \in \mathcal{O}_{t_0}$$

Since $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t_0}$ we have :

$$A_{F_{i_1}}...A_{F_{i_{t^*+1}}}x \in Y_{t^*+1}, \text{ for all } x \in \mathcal{O}_{t^*}$$

Therefore, according to the recursion (30), $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t^*+1}$. But $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*}$. In conclusion we have the equality $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$ and $\mathcal{O}_\infty = \mathcal{O}_{t^*}$. Since $\mathcal{O}_{t^*}$ is described by a finite number of linear inequalities, $\mathcal{O}_\infty$ is a polyhedral set. $\diamondsuit$

Remark 3.6 The conditions from Proposition 3.5 are similar with those corresponding to linear case (see [14]). Also, according to Section II the matrices A_{F_i} will be asymptotically stable. The algorithm for computing $\mathcal{O}_\infty$ stops once the following condition is met: there exists an index t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$. $\diamondsuit$

If t^* is large the algorithm might require too many iterations. We propose an alternative check whether or not a given polyhedral set is an RPI set for the system (26). Let $\Omega = \{x \in \mathbb{R}^n : h_j^T x \leq 1, \; j = 1,...,K\} \subseteq X_F$ be a polytope that contains the origin in the interior. Then according to Definition 3.1, Ω is an RPI set for the system (26) if for all $x \in \Omega \cap \mathcal{P}_i$ and for all $i \in \mathcal{I}$ we have $A_{F_i}x \oplus W \subseteq \Omega$. This condition can be translated in terms of computing some linear programming (LP) problems. We denote with $h_j(W) = \max_{w \in W} h_j^T w$ (this is an LP problem, because we assumed that W is a polyhedral set) for all $j = 1,...,K$. For all $i \in \mathcal{I}$ and $j = 1,...,K$ we consider the following LP problem:

$$\sigma_i^j = \max_x \; h_j^T A_{F_i} x + h_j(W) - 1$$
$$\begin{cases} h_k^T x \leq 1, \; k = 1,...,K \\ E_i x \geq 0 \end{cases} \tag{31}$$

From the above discussion we have the following:

Proposition 3.7: If for all $i \in \mathcal{I}$ and $j = 1,...,K$ the optimal values satisfy $\sigma_i^j \leq 0$, then Ω is an RPI set for system (26).

Proof: For a fixed i, the condition $\sigma_i^j \leq 0$ for all $j = 1,...,K$ expresses the fact that $A_{F_i}x \oplus W \subseteq \Omega$ for any $x \in \Omega \cap \mathcal{P}_i$. Therefore, Ω is a RPI set for the system (26). $\diamondsuit$

Remark 3.8 If after a certain number of iterations $t_{\max}$, the algorithm does not stop, then we have available the set $\mathcal{O}_{t_{\max}} = \{x : H_{t_{\max}} x \leq h_{t_{\max}}\}$. Then, a starting point in searching for a set Ω in Proposition 3.7 can be to take the set $\Omega = \{x : H_{t_{\max}} x \leq h\}$, and h should be chosen appropriately, i.e. $\sigma_i^j \leq 0$.

While the test for invariance from Proposition 3.7 can be applied also for PWA systems (i.e. when not all a_i are zero), the computation of $\mathcal{O}_\infty$ cannot be done straightforwardly for PWA systems. In the case when we have also affine dynamics, we can compute $\mathcal{O}_\infty$ associated to the PWL dynamics only, using the above set recursions. $\diamondsuit$

IV. Feedback min-max MPC scheme

In the sequel we develop a stable MPC scheme for the system (2) with additive disturbance, based on feedback min-max approach. For deterministic systems, almost all MPC schemes contain two ingredients: a terminal set and a terminal cost (see also [19] for a survey). If the system is uncertain, the stability and also the feasibility may be lost. In order to achieve robustness, the controller must stabilize the system for all possible realizations of the disturbance along the prediction horizon. Different robust MPC scheme have been proposed for linear systems: some of them are based on a nominal prediction (see [20]), other are based on the worst case disturbance as in feedback min-max MPC formulation (see [12], [25]). In this paper we use a dual-mode MPC formulation. We assume that we have computed a stabilizing controller for the nominal system $u = F_i x$ if $x \in \mathcal{P}_i$, according to Section II and also we have available a polyhedral RPI set T_{set}, obtained using the techniques derived in Section III.

In order to determine a suitable control law, an optimal control problem $\mathcal{V}_N(.)$ with horizon N is solved. The standard feedback min-max MPC is defined as follows [12], [25] : let $\mathbf{w} = (w(0),...,w(N-1))$ be a possible realization of the disturbance over the interval 0 to $N-1$. Efficient control in the presence of the disturbance requires state feedback [18]; therefore, the decision variable (for a given initial state x) in the optimal control problem is a control policy defined as:

$$\pi = (u(0), \mu_1(\cdot), ..., \mu_{N-1}(\cdot)),$$

where $u(0) \in U_{\text{c}}$ and $\mu_k : X_{\text{c}} \to U_{\text{c}}, \; k = 1,...,N-1$ is a state feedback control law. Let $x(k; x, \pi, \mathbf{w})$ denote the solution to (2) at time k. The feedback min-max optimization problem is defined as:

$$\min_\pi \max_{\mathbf{w} \in W^N} \sum_{k=0}^{N-1} l(x_k, u_k) \tag{32}$$
$$\begin{cases} x_k = x(k; x, \pi, \mathbf{w}) \in X_{\text{c}}, \; \forall k = 1,...,N-1 \\ u_k = \mu_k(x(k; x, \pi, \mathbf{w})) \in U_{\text{c}}, \; \forall k = 0,...,N-1 \\ x_N = x(N; x, \pi, \mathbf{w}) \in T_{\text{set}}, \; \forall \mathbf{w} \in W^N, \end{cases}$$

where $l(x,u)$ is defined as follows:

$$l(x,u) = \begin{cases} \|Qx\| + \|Ru\|, \text{ if } x \notin T_{\text{set}} \\ 0, \text{ if } x \in T_{\text{set}}, \end{cases} \tag{33}$$

with the matrices $Q \geq 0, R > 0$ and where $\|x\|$ represents a PWA norm (for instance 1-norm or ∞-norm). The constraint

$x_N \in T_{\rm set}$ is the stability condition and is necessary in order to guarantee stability as we will see in the sequel.

For linear systems problem (32) can be solved efficiently [12], [25], using the extreme disturbance realizations. In our settings, due to the nonlinearities of the system, this approach cannot be applied directly. To overcome this problem, we propose to restrict the admissible control policies π to only those that guarantee that, for every value of the disturbance, the mode of the system $i(k)$ is unique at each sample step k:

$$x(k; x, \pi, \mathbf{w}) \in \mathcal{P}_{i(k)},\ \forall \mathbf{w} \in W^N. \tag{34}$$

Therefore, we restrict the system to the admissible control policies only that guarantee the mode of the system is "certain" at sample step k, but the state is not known. This extra constraint (34), which expresses the fact that $i(k)$ is independent of the disturbance realization is not too restrictive since a cautious action may avoid uncertainty in the mode (at least in the case where the disturbances are not too large and the control inputs are not constrained too much). It can be easily observed that imposing (34) to the system (2) the state set generated by the disturbance at each sample step k is a *convex* set:

$$\begin{aligned} x(k; x, \pi, W^k) = &\, x(k; x, \pi, \mathbf{0}) \\ &+ X(k; i(0), ..., i(k-1), W^k) \end{aligned} \tag{35}$$

where first term expresses the nominal trajectory corresponding to the system (3) and the second term represents a convex uncertainty set associated with the state, which depends on the switching mode sequence $i(0), ..., i(k-1)$ and on the set W^k.

In this new settings, i.e. with the extra constraints (34), where x is a given state, the optimization problem (32) becomes :

$$\mathcal{V}_N(x) = \min_{\pi} \max_{\mathbf{w} \in W^N} \sum_{k=0}^{N-1} l(x_k, u_k) \tag{36}$$

$$\begin{cases} \text{constraint (34)},\ x_0 = x \\ x_k \in X_{\rm c}, \qquad\qquad k = 1, ..., N-1 \\ u_k = \mu_k(x_k) \in U_{\rm c},\ \ k = 0, ..., N-1 \\ x_N \in T_{\rm set},\ \ \forall \mathbf{w} \in W^N. \end{cases}$$

In this form, the optimization problem (36) has infinite dimension, but in the sequel we will show that (36) can be reduced to a finite-dimensional optimization problem. Using the constraint (34) and the fact that W is a bounded polyhedron with v vertices, let $\mathcal{L}_v^N$ denote the set of indexes ℓ such that $\mathbf{w}^\ell = (w(0)^\ell, ..., w(N-1)^\ell)$ takes values only on the vertices of W. It is clear that $\mathcal{L}_v^N$ is a finite set with the cardinality $V_N = v^N$. Further, let $\mathbf{u}^\ell = (u_0^\ell, ..., u_{N-1}^\ell)$ denote a control sequence associated with the ℓ-th disturbance realization $\mathbf{w}^\ell$ and let $x_k^\ell = x(k; x_0, \mathbf{u}^\ell, \mathbf{w}^\ell)$ be the solution of the PWL model (2) with the additional constraint (34). Therefore, given the current state x, let $\mathbf{u} = \{\mathbf{u}^1, ..., \mathbf{u}^{V_N}\}$, we want to find the solution of the following finite-dimensional optimization problem:

$$\mathcal{V}_N(x) = \min_{\mathbf{u}, N} \max_{\ell \in \mathcal{L}_v^N} \sum_{k=0}^{N-1} l(x_k^\ell, u_k^\ell) \tag{37}$$

$$\begin{cases} \text{constraint (34)}, x_0^\ell = x, \quad \forall \ell \in \mathcal{L}_v^N \\ x_k^\ell \in X_{\rm c},\ \ k = 1, ..., N-1,\ \forall \ell \in \mathcal{L}_v^N \\ u_k^\ell \in U_{\rm c},\ \ k = 0, ..., N-1,\ \forall \ell \in \mathcal{L}_v^N \\ x_N^\ell \in T_{\rm set},\ \forall \ell \in \mathcal{L}_v^N \\ x_k^{\ell_1} = x_k^{\ell_2} \ \Rightarrow\ u_k^{\ell_1} = u_k^{\ell_2},\ \forall \ell_1, \ell_2 \in \mathcal{L}_v^N \end{cases}$$

The last constraint is the well-known *causality constraint* [25] and expresses the fact that the control law at sample step k for the state x_k^ℓ is independent of the control and disturbance sequence taken to reach that state. The causality constraint can be posed in linear terms (see [12], [25]).

The optimization problem to be solved at step k is:

Robust feedback min-max optimization problem:

$$\mathcal{V}_{N_k}(x_k) = \min_{\mathbf{u}, N_k} \max_{\ell \in \mathcal{L}_v^{N_k}} \sum_{j=0}^{N_k-1} l(x_{k+j|k}^\ell, u_{k+j|k}^\ell) \tag{38}$$

$$\begin{cases} \text{constraint (34)},\ x_{k|k}^\ell = x_k, \qquad \forall \ell \in \mathcal{L}_v^{N_k} \\ x_{k+j|k}^\ell \in X_{\rm c},\ \ j = 1, ..., N_k - 1,\ \forall \ell \in \mathcal{L}_v^{N_k} \\ u_{k+j|k}^\ell \in U_{\rm c},\ \ j = 0, ..., N_k - 1,\ \forall \ell \in \mathcal{L}_v^{N_k} \\ x_{N_k|k}^\ell \in T_{\rm set},\ \forall \ell \in \mathcal{L}_v^{N_k}, N_k \in \{1, \cdots, N_{\max}\} \\ x_{k+j|k}^{\ell_1} = x_{k+j|k}^{\ell_2} \Rightarrow u_{k+j|k}^{\ell_1} = u_{k+j|k}^{\ell_2}, \forall \ell_1, \ell_2 \in \mathcal{L}_v^{N_k} \end{cases}$$

where $x_{k+j|k}^\ell$ is the prediction of the state at step $k+j$ given by the model (2), corresponding to the ℓ-th disturbance realization $(w(0), ..., w(N_k - 1))$ and applying the input sequence $u_{k|k}^\ell, ..., u_{N_k-1|k}^\ell$. The constraint (34) is imposed only to the states $x_{k+j|k}^\ell$ with $j = 1, ..., N_k - 1$ and not to $x_{N_k|k}^\ell$. The only constraint on the state $x_{N_k|k}^\ell$ is the terminal constraint: $x_{N_k|k}^\ell \in T_{\rm set}$.

The feedback min-max MPC controller is based on a dual-mode approach. For any $k \geq 0$, given the current state x_k, the algorithm is formulated as follows:

Algorithm 1

- if $x_k \in T_{\rm set} \cap \mathcal{P}_i$ then $u^{\rm RH}(x_k) = F_i x_k, \quad \forall i \in \mathcal{I}$
- otherwise, solve (38) and set $u^{\rm RH}(x_k)$ to the first control in the optimal solution computed: $u_{k|k}^\ell$,

where $u^{\rm RH}(x)$ is the control input applied to the system according to the receding horizon strategy.

A. Stability

We give first some definitions taken from [12]: a set $T_{\rm set}$ is *robustly stable* if and only if (iff) for all $\epsilon > 0$, there exists a $\gamma > 0$ such that $d(x_0, T_{\rm set}) < \gamma$ implies $d(x(k), T_{\rm set}) < \epsilon$ for all $k \geq 0$ and all admissible disturbance sequences. The set $T_{\rm set}$ is *robustly finite-time attractive* with domain of attraction X iff for all $x_0 \in X$ there exist a finite-time M such that $x(k) \in T_{\rm set}$ for all $k \geq M$. The set $T_{\rm set}$ is *robustly finite-time stable* with the domain of attraction

X iff it is robustly stable and robustly finite-time attractive with domain of attraction X. We define also the set:

$$X_N = \{x \in \mathbb{R}^n : \text{for which (37) has solution}\}$$

Because X, U, T_{set} and W are all polyhedral sets and $l(.,.)$ is a convex function, we can prove that:

Theorem 4.1: If the optimization problem $\mathcal{V}_{N_0}(x_0)$ is feasible (hence has an optimum), then all subsequent optimization problems $\mathcal{V}_{N_k}(x_k)$ are feasible. Moreover, there exists finite K such that $x_K \in T_{\text{set}}$.

Proof: Denote $N_0 = N$. At step $k = 0$, with the initial state $x_0 = x \in \mathcal{P}_{i_0}$, let $(u^{*\ell}_{0|0}, ..., u^{*\ell}_{N-1|0})$ be the optimal solution corresponding to the ℓ-th disturbance realization, satisfying the constraints (34), therefore producing the "certain" switching sequence $i_0, i_1, ..., i_{N-1}$. Let $x^{\ell}_{0|0}, ..., x^{\ell}_{N-1|0}$ be the corresponding state trajectories. From the causality constraints we have: $u^{*\ell_1}_{0|0} = u^{*\ell_2}_{0|0} = u^*_0$ for any $\ell_1 \neq \ell_2 \in \mathcal{L}^N_v$. Now, according to the receding horizon principle the input u^*_0 is applied and the disturbance takes a certain value $w_0 = \sum_{\ell \in \mathcal{L}^N_v} \mu_\ell w^\ell_0 \in W$, where w^ℓ_0 is a vertex of W and μ_ℓ are appropriate convex scalar weights. Therefore,

$$x_1 = A_{i_0} x + B_{i_0} u^*_0 + w_0 = \sum_{\ell \in \mathcal{L}^N_v} \mu_\ell x^\ell_1,$$

with $x^\ell_1 = A_{i_0} x + B_{i_0} u^*_0 + w^\ell_0$, i.e. x_1 lies in the convex hull: $\text{co}\{x^\ell_1 : \ell \in \mathcal{L}^N_v\}$. Now, the following prediction horizon $N_1 = N - 1$ and the control sequence defined as:

$$\sum_{\ell \in \mathcal{L}^{N-1}_v} \mu_\ell u^{*\ell}_{1|0}, ..., \sum_{\ell \in \mathcal{L}^{N-1}_v} \mu_\ell u^{*\ell}_{N-1|0} \qquad (39)$$

is feasible and the state predictions at step $k = 1$ evolve in convex hull of the predictions at step $k = 0$: $x_{1+j|1} \in \text{co}\{x^\ell_{1+j|0}, \ \ell \in \mathcal{L}^{N-1}_v\}$, where $x_{1+j|1}$ with $j = 1, ..., N-1$ is the state prediction at step $k = 1$, applying the input sequence (39) and an arbitrary disturbance sequence. Moreover, the switching sequence is certain: $i_1, ..., i_{N-1}$ (we used here that all sets X, U, T_{set} are convex). In conclusion the problem $\mathcal{V}_{N_1}(x_1)$ is feasible and has an optimum. By induction, we can prove that all subsequent optimization problems $\mathcal{V}_{N_k}(x_k)$ are feasible.

Moreover, since T_{set} is bounded, there exists a $\mu > 0$ such that $\|x\| \geq \mu$ for all $x \in X_{\text{c}} \setminus T_{\text{set}}$. Then

$$\mathcal{V}_{N_{k+1}}(x_{k+1}) - \mathcal{V}_{N_k}(x_k) \leq -\|x_k\| \leq -\mu$$

if $x_k \notin T_{\text{set}}$. Now, assume that for $k \to \infty$, $x_k \notin T_{\text{set}}$. Then, $0 \leq \mathcal{V}_{N_k}(x_k) \leq \mathcal{V}_{N_0}(x_0) - k\mu \to -\infty$ as $k \to \infty$, i.e. we get a contradiction. Therefore, the state x_k enters T_{set} in finite time and then it remains there for all subsequent steps. ♢

Using Theorem 4.1, we can establish the following stability result:

Theorem 4.2: The feedback min-max MPC law $u^{\text{RH}}(.)$ given by the *Algorithm 1* makes T_{set} robustly finite-time stable for the system (2) in closed-loop with $u^{\text{RH}}(x)$ with a region of attraction X_N.

Proof: From previous theorem it follows that the state reaches T_{set} in finite number of steps. Inside T_{set} we apply the controller $F_i x$ if $x \in \mathcal{P}_i$, which keeps the state in this set T_{set}, because it is a RPI set, regardless the values of the disturbance from W. ♢

B. Computational complexity

The mixed logical dynamical framework (MLD) represents one of the main tools for computing optimal control for PWA systems [2]. From a computational point of view, the optimization problem (37) can be recast as a mixed-integer linear programming problem (MILP), using standard "tricks" [12]. An alternative formulation of (37) is :

$$\min_{\mathbf{u},\eta} \{\eta : \sum_{k=0}^{N-1} l(x^\ell_k, u^\ell_k) \leq \eta, \ \forall \ell \in \mathcal{L}^N_v, \ \mathbf{u} \in C(x)\}$$

where in $C(x)$ we gathered all the constraints from optimization problem (37), and are all described by linear inequalities. Therefore, we have obtained a feedback min-max MPC scheme that is based on solving at each sample step an MILP. The result is not surprising, since in [23] the min-max MPC scheme is also computed by solving a sequence of linear programming problems (that can be seen as a MILP). With this approach the number of decision variables and constraints grows exponentially with the length of the control horizon N. Therefore, the optimization problem (37) is computationally intensive for large N. But this is a drawback also for linear systems, as the authors of [12] remark.

V. Robust MPC with unknown mode

A. Robust MPC using quadratic optimization problems

The maximal RPI set $\tilde{\mathcal{O}}_\infty$ included in X_F, associated to system (26) is (in general) not a convex set. Given any initial state $x_0 \in \tilde{\mathcal{O}}_\infty$, we are sure that applying the nominal controller the trajectory of the system (26) remains in this set, as close as possible to the origin. The maximal RPI set $\tilde{\mathcal{O}}_\infty$, for which the nominal controller $u = F_i x$ is feasible, is in general small. Now we derive a robustly stable MPC scheme that uses prediction control trajectories which do not correspond to fixed state feedback control laws. Therefore, we enlarge the set of initial states that can be steered to a target set, close to the origin. We introduce a new control variable c_k such that the new input applied to the system is

$$u_k = F_i x_k + c_k, \text{ if } x_k \in \mathcal{P}_i. \qquad (40)$$

Let N be the control horizon, then $c_k, ..., c_{k+N-1}$ represent degrees of design freedom and $c_{k+N+j} = 0, \ \forall j \geq 0$. In this case the PWL system (26) becomes

$$x_{k+1} = A_{F_i} x_k + B_i c_k + w_k, \text{ if } x_k \in \mathcal{P}_i. \qquad (41)$$

Employing a reasoning similar to [16], the dynamics of (41) can be described by the autonomous PWL system

$$z_{k+1} = \mathcal{A}_i z_k + \mathcal{D} w_k, \text{ if } z_k \in \tilde{\mathcal{P}}_i \qquad (42)$$

where $z \in \mathbb{R}^{n+Nm}$, $z = \begin{bmatrix} x \\ f \end{bmatrix}$, $f = [c_k^T, ..., c_{k+N-1}^T]^T$, $\mathcal{D} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\mathcal{A}_i = \begin{bmatrix} A_{F_i} & \begin{bmatrix} B_i & 0 & ... & 0 \end{bmatrix} \\ 0 & M \end{bmatrix}$, $M = \begin{bmatrix} 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$, $\mathrm{cl}(\tilde{\mathcal{P}}_i) = \{z : [E_i \;\; 0]z \geq 0\}$.

Clearly the stability properties of the matrices $\mathcal{A}_i$ depend only on the matrices A_{F_i}. We denote

$$X_{0,z}^i = \{z = [x^T \; f^T]^T : x \in X_{\mathrm{c}} \cap \mathcal{P}_i, F_i x + c_k \in U_{\mathrm{c}}\} \\ = \{z : [I \;\; 0]z \in X_{\mathrm{c}} \cap \mathcal{P}_i, [F_i \;\; I \;\; 0]z \in U_{\mathrm{c}}\}.$$

Remark 5.1 If there exists an RPI set $\mathcal{O}$ for system (26), then there must exist at least one RPI set $\mathcal{O}_z \subseteq \cup_{i\in\mathcal{I}} X_{0,z}^i$ for system (42). Indeed, from the definition of the system (42), it is clear that $\mathcal{O}_z = \{z = [x^T \; 0]^T : x \in \mathcal{O}\}$ is an RPI set for this system. ♢

So, if the maximal RPI set $\tilde{\mathcal{O}}_\infty$ for (26) exists, then there exists also a maximal RPI set $\mathcal{O}_{\infty,z}$ for the augmented system (42) and the projection of $\mathcal{O}_{\infty,z}$ into the state space $\mathbb{R}^n$ (denoted with $\mathcal{O}_{\infty,zx}$) contains $\tilde{\mathcal{O}}_\infty$. Therefore, the benefits of using free control moves are clear now. The robust MPC algorithm is defined as follows:

Algorithm (II)

1) Off-line step
   - compute F_i according to Section II
   - compute the maximal RPI set $\mathcal{O}_{\infty,z}$ for (42)
2) On-line step: each step k, given x_k solve

$$J_N^*(k) = \min_f f^T f, \text{ s.t. } z = [x_k^T \; f^T]^T \in \mathcal{O}_{\infty,z} \quad (43)$$

Implement the controller $u_k = F_i x_k + c_k$.

The maximal RPI set for (42) included in $\cup_{i\in\mathcal{I}} X_{0,z}^i$ is in general a union of polyhedral sets: $\mathcal{O}_{\infty,z} = \cup_{j=1}^q \mathcal{O}_z^j$, where $\mathcal{O}_z^j$ are polytopes. Therefore, at step 2 of Algorithm (II) we have to solve q quadratic programming (QP) problems, and then to choose f for which $f^T f$ is the smallest one. The distance of a point x to the closed, convex set $\mathcal{O}$ is defined as $d(x, \mathcal{O}) = \min_{x^o \in \mathcal{O}} \|x - x^o\|$. We consider $\|X\|$ as the norm 2 ($\|X\|_2$) for vectors and matrices.

Theorem 5.2: Given $x_0 \in \mathcal{O}_{\infty,zx}$, the receding horizon implementation of the Algorithm (II) asymptotically steers the trajectory of (41) to $\tilde{\mathcal{O}}_\infty$.

Proof: If $x_0 \in \mathcal{O}_{\infty,zx}$, then (43) has a solution at $k = 0$, $f_0^* = [c_0^{*T} ... c_{N-1}^{*T}]^T$. Moreover, there exists an $i_0 \in \mathcal{I}$ such that $x_0 \in \mathcal{P}_{i_0} \cap \mathcal{O}_{\infty,zx}$. Let us denote with $f_1^{\mathrm{feas}} = [c_1^{*T} ... c_{N-1}^{*T} \; 0]^T$. Applying the feedback input $u_0 = F_{i_0} x_0 + c_0^*$ to the system (41), and keeping in mind that $\mathcal{O}_{\infty,z}$ is an RPI set for (42), then we obtain $[x_1^T \; f_1^{\mathrm{feas}T}]^T \in \mathcal{O}_{\infty,z}$. Therefore, f_1^{feas} is feasible at $k = 1$.

By induction, we can prove that given $x(k)$, for all $k \geq 1$ the optimization problem (43) has an optimal solution $f_k^* = [c_k^{*T}, ..., c_{k+N-1}^{*T}]^T$ and at sample step $k+1$ we have a feasible solution $f_{k+1}^{\mathrm{feas}} = [c_{k+1}^{*T} ... c_{k+N-1}^{*T} 0]^T$. In conclusion

$$J_N^*(k+1) - J_N^*(k) \leq -\|c_k^*\|^2. \quad (44)$$

So, the sequence $\{J_N^*(k)\}_{k\geq 0}$ is non-increasing and bounded below by 0. Therefore, it converges to $J_N^\infty < \infty$. Summing the relation (44) from 0 to ∞ we obtain: $0 \leq \sum_{k\geq 0} \|c_k^*\|^2 \leq J_N^*(0) - J_N^\infty < \infty$. In conclusion, the series $\sum_{k\geq 0} \|c_k^*\|$ is convergent. We conclude that $c_k^* \to 0$ as $k \to \infty$. Therefore, $\lim_{k\to\infty} d(x_k, \tilde{\mathcal{O}}_\infty) = 0$, because $\tilde{\mathcal{O}}_\infty$ is the maximal set of states for which the controller $u = F_i x$ if $x \in \mathcal{P}_i$ is feasible. ♢

B. Robust MPC using a single QP problem

In this section we develop an new MPC scheme, such that we solve on-line a single quadratic optimization problem.

Off-line step

In this step, we compute *off-line* the set of initial states and input correction sequences that steer these states to the RPI set $\mathcal{O}_\infty$ (defined in (29)) in N steps, using the controller (40), where N is the prediction horizon. This set is obtained recursively as follows:

$$\mathcal{X}_0^i = \mathcal{O}_\infty^i, \; \forall i \in \mathcal{I}, \quad (45)$$
$$\mathcal{X}_{k+1}^i = \left\{ \begin{bmatrix} x \\ c \\ \tilde{c} \end{bmatrix} : \begin{matrix} \begin{bmatrix} A_{F_i} x + B_i c \oplus W \\ \tilde{c} \end{bmatrix} \in \bigcap_{j\in\mathcal{I}} \mathcal{X}_k^j \\ x \in X_{\mathrm{c}}, \; F_i x + c \in U_{\mathrm{c}} \end{matrix} \right\}$$

$k = 0, ..., N-1$ and $i \in \mathcal{I}$. Note that a similar recursion was proposed also in [5] in the context of gain scheduling for nonlinear systems. The dimension of the sets $\mathcal{X}_k^i$ increases as k increases. Clearly $\mathcal{X}_N^i \subseteq \mathbb{R}^{n+mN}$. We denote with X_k^i the projection of $\mathcal{X}_k^i$ into the state space $\mathbb{R}^n$. In conclusion the set of initial states that can be steered to $\mathcal{O}_\infty$ in N steps, using the semi-feedback controller (40) is:

$$X_N = \cup_{i\in\mathcal{I}} (X_N^i \cap \mathcal{P}_i).$$

Because $X_{\mathrm{c}}, U_{\mathrm{c}}$ and W are polytopes and initially $\mathcal{X}_0^i = \mathcal{O}_\infty^i$, with $\mathcal{O}_\infty^i$ a polytope, we obtain the $\mathcal{X}_N^i$'s are polytopic sets as well. As a consequence X_N^i is a polytope, for any $i \in \mathcal{I}$. Therefore, X_N is a union of polytopes.

Proposition 5.3: The set $\cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i)$ is an RPI set for the augmented system (42).

Proof: Let $z_0 = [x_0^T \;\; f_0^T]^T \in \cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i)$. There exists an $i_0 \in \mathcal{I}$ such that $x_0 \in X_N^{i_0}$. From the definition of $X_N^{i_0}$, applying the feedback input $u(k) = F_{i_0} x_0 + c_0$ to the system (41), according to the set recursion (45), we obtain that there exists a mode $j \in \mathcal{I}$ such that $x_1 \in X_{N-1}^j$. Moreover, let us denote with $f_1^{\mathrm{feas}} = [c_1^T ... c_{N-1}^T \;\; 0]^T$. Then $z_1 = [x_1^T \;\; f_1^{\mathrm{feas}T}]^T \in (\mathcal{X}_N^j \cap \tilde{P}_j)$. In conclusion $z_1 \in \cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i)$.

By induction we can prove $z_k \in \cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i)$ for all $k \geq 0$ i.e. the set $\cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i)$ is an RPI set for the PWL system (42). ♢

It is clear that $\cup_{i\in\mathcal{I}} (\mathcal{X}_N^i \cap \tilde{P}_i) \subseteq \mathcal{O}_{\infty,z}$. Moreover, the evolution of (41) under the input sequence (40), with the initial state x_0 is given by:

$$x_{k+1} = A_{F_{i(k)}} ... A_{F_{i(0)}} x_0 \quad (46)$$
$$+ \sum_{j=1}^{k+1} A_{F_{i(k+1)}} ... A_{F_{i(j)}} (B_{i(j-1)} c_{j-1} + w_{j-1})$$

where $A_{F_{i(k+1)}} = I$ and $i(0), ..., i(k)$ is any feasible switching sequence corresponding to state sequence $x_0, ..., x_k$.

On-line step

Assume $x(k) \in \mathcal{P}_i$. At this stage, we compute *on-line*, at each step k, the following QP problem:

$$J_N^*(k) = \min_f f^T f, \text{ s.t. } [x_k^T \ f^T]^T \in \mathcal{X}_N^i \quad (47)$$

Then, according to the receding horizon strategy, the input applied to the system at step k is given by: $u_k = F_i x_k + c_k^*$. Once $x_k \in \mathcal{O}_\infty$, the MPC law is given by the local controller $u_k = F_i x_k$, which has the property that it keeps the state inside this RPI set for any disturbance in W.

We assume that the matrices A_{F_i} are asymptotically stable for all $i \in \mathcal{I}$. In this case, we have the following stability result for the MPC scheme presented in this section.

Theorem 5.4: Provided that the initial state $x_0 \in X_N$ then the feedback MPC law $u_k = F_i x_k + c_k^*$ drives the state x_k asymptotically to the RPI set $\mathcal{O}_\infty$.

Proof: Using similar arguments as in Theorem 5.2 we conclude that

$$c_k^* \to 0 \text{ as } k \to \infty. \quad (48)$$

Let us now prove that $d(x_k, \mathcal{O}_\infty) \to 0$ as $k \to \infty$. Given $x_0 \in X_N$ there exists an $x_0^o \in \mathcal{O}_\infty$ such that $d(x_0, \mathcal{O}_\infty) = \|x_0 - x_0^o\|$ (since $\mathcal{O}_\infty$ is a closed, convex set). Now $x_1 = A_{F_{i(0)}} x_0 + B_{i(0)} c_0^* + w_0$. Let us define $x_1^o = A_{F_{i(0)}} x_0^o + w_0$. From the definition of $\mathcal{O}_\infty$ it is clear that $x_1^o \in \mathcal{O}_\infty$. Therefore, we obtain:

$$d(x_1, \mathcal{O}_\infty) \le \|x_1 - x_1^o\| \le \|A_{F_{i(0)}}\| \ \|x_0 - x_0^o\| + \|B_{i(0)} c_0^*\|$$

By induction, using (46), we can prove that

$$\begin{aligned} d(x_{k+1}, \mathcal{O}_\infty) &\le \|x_{k+1} - x_{k+1}^o\| \\ &\le \|A_{F_{i(k)}}\| ... \|A_{F_{i(0)}}\| \ \|x_0 - x_0^o\| \\ &+ \sum_{j=1}^{k+1} \|A_{F_{i(k+1)}}\| ... \|A_{F_{i(j)}}\| \ \|B_{i(j-1)} c_{j-1}^*\|, \end{aligned} \quad (49)$$

for any feasible sequence of switches $i(0), ..., i(k)$, where $x_{k+1}^o = A_{F_{i(k)}} x_k^o + w_k \in \mathcal{O}_\infty$. Since A_{F_i} are asymptotically stable for any $i \in \mathcal{I}$, there exists a constant $0 < \delta < 1$ and $L > 0$ such that

$$\|A_{F_{i(k)}}\| ... \|A_{F_{i(j)}}\| \le L \delta^{k-j}. \quad (50)$$

Using now (50) and (48) in (49), we obtain

$$d(x_k, \mathcal{O}_\infty) \to 0 \text{ as } k \to \infty. \qquad \diamondsuit$$

VI. Extension to PWA and hybrid systems

Let us now discuss the possible extensions of the previous results to PWA and hybrid systems. We consider the PWA system (1), such that the origin is an equilibrium point. Therefore, the system (1) is described by PWL dynamics around the origin: $a_i = 0$ for all $i \in \mathcal{I}_0$. In this case we can derive a local controller corresponding to the PWL dynamics of the system according to Section II. The robustly positively invariant set $\mathcal{O}_\infty$ corresponds also to the PWL dynamics of the system (that are defined around the origin) and is constructing according to Section III. Therefore, for the PWA system (1) we have available a local controller $u = F_i x$ if $x \in \mathcal{P}_i$, with $i \in \mathcal{I}_0$ and a corresponding terminal set $\mathcal{O}_\infty = \cap_{i \in \mathcal{I}_0} \mathcal{O}_\infty^i$. Both, the controller and the terminal set are defined around the origin, where the PWL dynamics of (1) can be active.

The feedback min-max MPC scheme presented in Section IV can then be easily implemented for PWA system (1). Using the equivalence of a PWA systems with a MLD systems, the optimization problem (38) still remains a MILP. The stability results derived in Theorem 4.2 are still valid.

Remark 6.1 An interesting case is when $\mathcal{I}_0$ contains only one element $\mathcal{I}_0 = \{1\}$, i.e. the origin is contained in the interior of $\mathcal{P}_1$ (we have only one PWL dynamic that contains the origin and the rest of dynamics are PWA). As it is done in the linear case, we can construct a stabilizing controller for the PWL dynamic and a robustly positively invariant set. Then, we can formulate the feedback min-max MPC scheme (38) with a *fixed prediction horizon* N, since in that case the control sequence defined as:

$$\sum_{\ell \in \mathcal{L}_v^{N-1}} \mu_\ell u_{1|0}^{*\ell}, ..., \sum_{\ell \in \mathcal{L}_v^{N-1}} \mu_\ell u_{N-1|0}^{*\ell}, F_1 x_{N|0}^\ell \quad (51)$$

is feasible for the next step (and it keeps the next N modes fixed). The same stability properties are valid in this particular case. $\diamondsuit$

The second MPC scheme derived in Section V is more difficult to be implemented to other classes of hybrid systems, due to the special construction of the set X_N in the off-line step. It is easy to derive a local controllers $u = F_i x$ if $x \in \mathcal{P}_i$, for all $i \in \mathcal{I}$ that stabilizes the PWA system (1), using similar arguments as in Section II. Of course, if we are able to construct an RPI set $\mathcal{O}_\infty$ for all dynamics of the PWA system (not only for the PWL dynamics), then the stable MPC scheme presented in Section V can be implemented also for PWA systems (1). One way for constructing a RPI set for PWA system (1) corresponding to all dynamics of the systems is to use the Proposition 3.7. The stability result derived in Theorem 5.4 is valid also for this type of systems.

From [9], we know that the class of PWA systems is equivalent with some other important classes of hybrid systems like MLD systems, extended linear complementary systems, max-min-plus-scaling systems. Therefore, the stable MPC schemes derived in Section IV and V can be applied also to these classes of hybrid systems.

VII. Illustrative example

We consider the following example taken from [2], but this time with an additive term to take also into account the

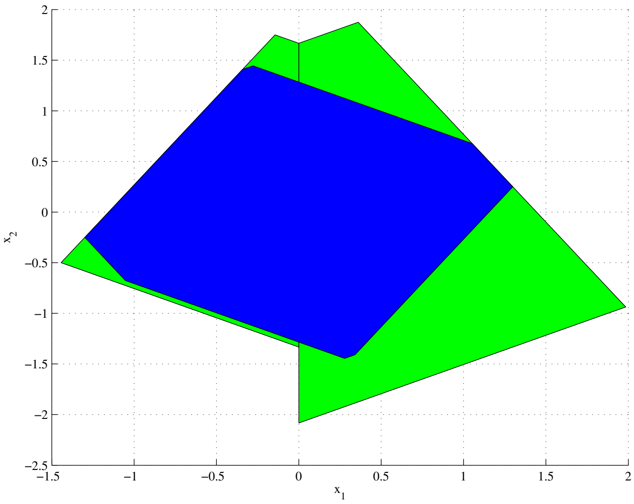

Fig. 1. The outer polyhedron represents the maximal RPI set $\tilde{\mathcal{O}_\infty}$ for the system while the inner polytope represents $\mathcal{O}_\infty$.

disturbance:

$$x(k+1) = 0.8 \begin{bmatrix} \cos\alpha(k) & -\sin\alpha(k) \\ \sin\alpha(k) & \cos\alpha(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + w(k),$$

$$\alpha(k) = \begin{cases} \pi/3 & \text{if } [1\ \ 0]x(k) \geq 0 \\ -\pi/3 & \text{if } [1\ \ 0]x(k) < 0 \end{cases}$$

with the following constraints:

$$X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 10\},\ U = \{u \in \mathbb{R} : |u| \leq 1\}.$$

We assume that the disturbance set is given by:

$$W = \{w \in \mathbb{R}^2 : w_1 = w_2,\ \|w\|_\infty \leq 0.1\}.$$

We get the following PWL feedback controller $u = F_i x,\ i = 1, 2$, according to Section II:

$$F_1 = [-0.692\ \ -0.4],\ F_2 = [0.866\ \ -0.5].$$

We see that the matrices A_{F_i} are strictly stable. Therefore, we can apply Theorem 3.5, the RPI set $\mathcal{O}_\infty$ being determined after 2 iterations (i.e. $t^* = 2$):

$$\mathcal{O}_\infty = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -0.866 & -0.5 \\ 0.866 & 0.5 \\ 0.866 & -0.5 \\ -0.866 & 0.5 \\ 0.499 & -0.866 \\ -0.499 & 0.866 \\ 0.500 & 0.866 \\ -0.500 & -0.866 \end{bmatrix} x \leq \begin{bmatrix} 1.25 \\ 1.25 \\ 1 \\ 1 \\ 1.3906 \\ 1.3906 \\ 1.1125 \\ 1.1125 \end{bmatrix} \right\}$$

which is a polytope that contains the origin in the interior (see also VII).

Applying the robust feedback min-max MPC scheme proposed in Section IV, with initial state $x_0 = [3\ \ 2.1]$, $Q = I_2, R = 1$, together with infinite norm $\|\cdot\|_\infty$, initial prediction horizon $N = 3$ and the terminal set being $\mathcal{O}_\infty$, we get the straight line in Figure VII. We also apply the MPC scheme proposed in Section V for the initial state $x_0 = [1.5\ \ 2.1]$ with the same prediction horizon $N = 3$. In the first plot we represented the trajectories of the system in closed-loop with the feedback min-max MPC scheme (straight line) and with the MPC scheme from Section V (dotted line). The inner polytope represents the set $\mathcal{O}_\infty$ and the outer polygon is the maximal RPI set associated to the PWL system. We remark that once the trajectory enters $\mathcal{O}_\infty$ it remains there in both schemes. In the second plot we have represented the optimal inputs given by our robust MPC schemes (straight line for the MPC scheme from Section IV and dotted line for the MPC scheme presented in Section V). Note that the constraints on input are satisfied. In order to compute the RPI set $\mathcal{O}_\infty$ we used the *Set invariance toolbox* from [11]. For solving the LMIs we used the Matlab LMI toolbox.

VIII. Conclusions

In this paper we have derived two stable MPC algorithms for the class of perturbed PWL systems with additive disturbance. We have derived LMIs conditions in order to find a PWL controller which stabilizes the nominal systems. We have taken into account the piecewise linear structure of the system, conservativeness being reduced using the S-procedure. We have computed a *convex* RPI set for the perturbed PWL system. We have derived conditions when this set is finitely determined and therefore is a polytope. Further we have proposed first a stable robust feedback min-max MPC scheme that uses the fact that the mode of the system is certain at each step k. We incorporate feedback in the control sequence, in order to increase the domain of the feasible control sequences. Our MPC scheme is based on solving at each step a MILP problem, but the computational complexity decreases at each step. The second stable MPC scheme is based on unknown mode, using a semi-feedback controller. For this scheme we have to solve on-line only a quadratic optimization problem. Therefore, this scheme is less demanding than the min-max MPC scheme. Extensions of these results to PWA and hybrid systems are also possible.

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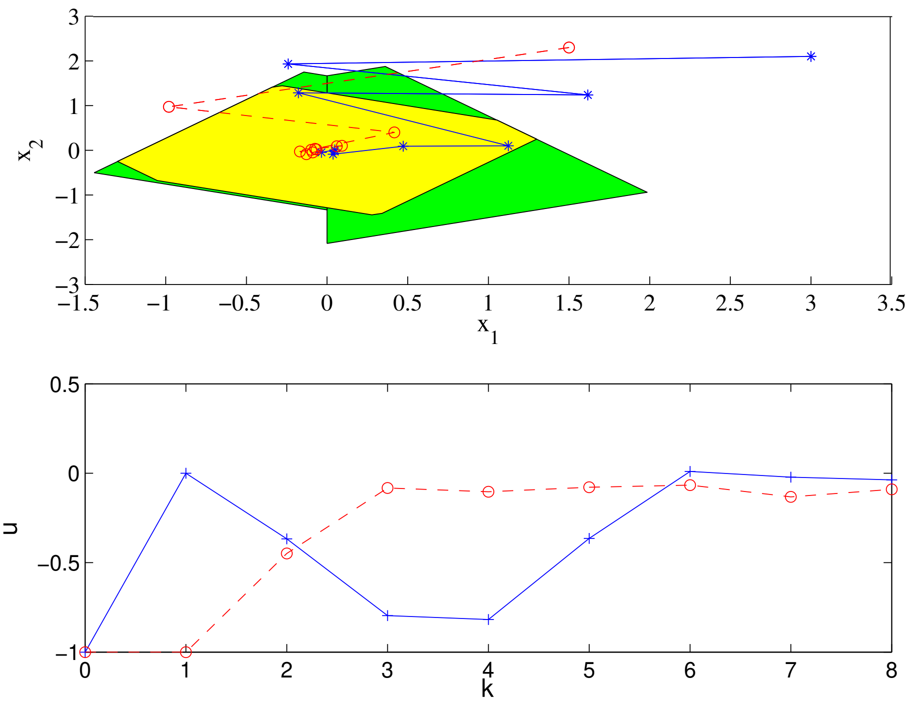

Fig. 2. Top: the trajectory of the system under the robust MPC control. Bottom: the optimal input given by the MPC.

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