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On MPC for max-plus-linear systems: Analytic solution and stability

T.J.J. van den Boom, B. De Schutter and I. Necoara

Abstract— **In previous work we have extended the popular Model Predictive Control (MPC) design technique to a class of discrete event systems that can be described by a model that is "linear" in the max-plus algebra. In this paper we study the analytic solution of the MPC controller for these max-plus-linear (MPL) systems and we compare this solution with other well-known control schemes. Furthermore, we derive tuning rules for the controller parameters that guarantee us a stable feasible and stable operation of the controller.**

I. INTRODUCTION

Model predictive control (MPC) [6], [8] is a proven technology for the control of multivariable systems in the presence of input, output and state constraints and is capable of tracking pre-scheduled reference signals. These attractive features make MPC widely accepted in the process industry. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to discrete event systems (DES). In this paper we consider the class of DES with synchronization but no concurrency. Such DES can be described by models that are "linear" in the max-plus algebra [1], [3], and therefore they are called max-plus-linear (MPL) DES.

In [4] we have extended MPC to MPL systems. In [13] we have studied stability and tuning of MPC controllers for MPL systems and observed that for MPL systems, stability is not an intrinsic feature of the system, but it also depends on the input and the due dates (i.e., the reference signal) of the system. In [14] we derived a max-plus (and so event-driven) equivalent of the conventional end-point constraint, which is in this case an inequality constraint. In this paper we show that by a proper tuning of the design parameters this end-point constraint is not needed, and stability can still be guaranteed. Furthermore we derive an analytic solution of the MPL-MPC controller and show the relation to well-known max-plus control schemes from literature [1], [2], [7], [9], [10], [11].

II. MAX-PLUS-LINEAR SYSTEMS

In this section we give the basic definition of the max-plus algebra and we present some results on max-plus-linear systems.

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows [1], [3]: $x \oplus y = \max(x, y)$, $x \otimes y = x + y$ for elements $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \ ,$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n} (a_{ik} + c_{kj}) \ ,$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$. The matrix E_n is the $n \times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. Finally, the max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n\times n}$ is defined as follows: $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ for $k = 1, 2, \ldots$.

In [4], [5], [13] we have studied MPC for DES in which there is synchronization but no concurrency. It has been shown [1], [3] that these systems can be described by a model of the form[1]

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \ , \tag{1}$$
$$y(k) = C \otimes x(k) \ , \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$. DES that can be described by this model will be called max-plus-linear (MPL). The index k is called the event counter. The state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instant at which the input event occurs for the kth time, and the output $y(k)$ contains the time instant at which the output event occurs for the kth time.

III. THE MPC PROBLEM FOR MAX-PLUS-LINEAR SYSTEMS

In [4] we have shown that prediction of future values of $y(k)$ for the system (1)–(2) can be done by successive substitution, leading to the expression

$$\tilde{y}(k) = \tilde{C} \otimes x(k-1) \ \oplus \ \tilde{D} \otimes \tilde{u}(k) \ , \tag{3}$$

where $\tilde{C}$ and $\tilde{D}$ are given by

$$\tilde{C} = \begin{bmatrix} C \otimes A \\ C \otimes A^{\otimes^2} \\ \vdots \\ C \otimes A^{\otimes^{N_\mathrm{p}}} \end{bmatrix} ,$$

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[1] In this paper we will restrict ourselves to single-input-single-output (SISO) systems for the sake of simplicity. All results can be extended to multivariable systems.

$$\tilde{D}=\begin{bmatrix} C\otimes B & \varepsilon & \cdots & \varepsilon \\ C\otimes A\otimes B & C\otimes B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C\otimes A^{\otimes^{N_\mathrm{p}-1}}\otimes B & C\otimes A^{\otimes^{N_\mathrm{p}-2}}\otimes B & \dots & C\otimes B \end{bmatrix},$$

and $\tilde{u}(k)$, $\tilde{y}(k)$ are defined as:

$$\tilde{y}(k)=\begin{bmatrix} \hat{y}(k|k) \\ \hat{y}(k+1|k) \\ \vdots \\ \hat{y}(k+N_\mathrm{p}-1|k) \end{bmatrix},\ \ \tilde{u}(k)=\begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N_\mathrm{p}-1|k) \end{bmatrix},$$

where $\hat{y}(k+j|k)$ denotes the prediction of $y(k+j)$ based on knowledge at event step k, $u(k+j|k)$ denotes the future input sequence based on knowledge at event step k, and N_p is the prediction horizon.

The MPC problem for MPL systems is formulated as follows [4], given $x(k-1)$:

$$\min_{\tilde{u}(k),\tilde{y}(k)} J(\tilde{u}(k),\tilde{y}(k)) = \min_{\tilde{u}(k),\tilde{y}(k)} J_\mathrm{out}(\tilde{y}(k)) + \beta J_\mathrm{in}(\tilde{u}(k)) \quad (4)$$

subject to (3) and

$$u(k) \geq u(k-1), \qquad \text{for } j=0,\ldots,N_\mathrm{p}-1\,, \quad (5)$$

where J_out is the output cost criterion, J_in the input cost criterion and β is a trade-off variable with $0<\beta<1$.

Remark 1: In [4] we have taken constraints on inputs and outputs into account of the form $E(k)\,\tilde{u}(k) + F(k)\,\tilde{y}(k) \leq h(k)$. In this paper we restrict ourselves only to constraint (5).

Equation (5) guarantees a non-decreasing input signal[2]. The above problem will be called the MPL-MPC problem.

MPC uses a receding horizon strategy. So after computation of the optimal control sequence $u^*(k|k),\ldots,u^*(k+N_\mathrm{p}-1|k)$, only the first control sample $u(k) = u^*(k|k)$ will be implemented, subsequently the horizon is shifted and the model and the initial state estimate can be updated if new measurements are available, then the new MPL-MPC problem is solved, etc.

In the remainder of this paper we consider the following output and input objective functions:

$$J_\mathrm{out}(\tilde{y}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \max\Big(\hat{y}(k+j|k) - r(k+j), 0\Big), \quad (6)$$

$$J_\mathrm{in}(\tilde{u}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \Big(r(k+j) - u(k+j|k)\Big). \quad (7)$$

The criteria can be interpreted as follows: J_out measures the tracking error or tardiness of the system, which is equal to the delay between the output date $\hat{y}(k+j|k)$ and due date $r(k+j)$ if $\hat{y}(k+j|k) - r(k+j) > 0$, and zero otherwise; J_in intends to maximize the input dates[3] $u(k+j|k)$.

We can rewrite $J_\mathrm{out}(\tilde{y}(k))$ as a function of $\tilde{u}$ by substitution of (3) in (6) and we obtain:

$$J_\mathrm{out}(\tilde{u}(k)) = \sum_{i=0}^{N_\mathrm{p}-1} \max\Big((\tilde{C}\otimes x(k-1) \oplus \tilde{D}\otimes\tilde{u}(k))_i - \tilde{r}_i(k), 0\Big). \quad (8)$$

This $J_\mathrm{out}(\tilde{u}(k))$ is a convex function of the variable $\tilde{u}(k)$. The term $r(k)$ in J_in has been added to obtain a bounded objective function J_in. It has no influence on the optimization.

Lemma 1: Assume $\beta < 1/N_\mathrm{p}$, and define

$$\bar{u}(k) = \begin{bmatrix} u(k-1) & u(k-1) & \cdots & u(k-1) \end{bmatrix}^T, \quad (9)$$

$$\begin{aligned} \tilde{z}(k) &= \begin{bmatrix} z^T(k|k) & z^T(k+1|k) & \ldots & z^T(k+N_\mathrm{p}-1|k) \end{bmatrix}^T, \\ &= \tilde{C}\otimes x(k-1) \;\oplus\; \tilde{D}\otimes\bar{u}(k) \;\oplus\; \tilde{r}(k), \end{aligned} \quad (10)$$

and consider the maximization problem

$$\tilde{u}^*(k) = \arg\max_{\tilde{u}(k)} \sum_{\ell=1}^{N_\mathrm{p}} \tilde{u}_\ell(k),$$

subject to

$$\tilde{D}_j \otimes \tilde{u}(k) \leq z(k+j|k), \quad \textit{for } j=0,\ldots,N_\mathrm{p}-1, \quad (11)$$

$$u(k+j) \geq u(k+j-1), \quad \textit{for } j=0,\ldots,N_\mathrm{p}-1, \quad (12)$$

where $\tilde{D}_j$ denotes the jth row of $\tilde{D}$. Then $\tilde{u}^*$ is the optimal solution of the original MPL-MPC problem.
The optimal output cost is $J_\mathrm{out}(\tilde{u}^*(k)) = \sum_{j=1}^{N_\mathrm{p}} (z(k+j|k) - r(k+j))$, and the optimal input cost is $J_\mathrm{in}(\tilde{u}^*(k)) = \sum_{j=1}^{N_\mathrm{p}} \Big(r(k+j) - u^*(k+j|k)\Big)$.

Proof: Define

$$\tilde{y}^0(k) = \tilde{C}\otimes x(k-1) \;\oplus\; \tilde{D}\otimes\bar{u}(k), \quad (13)$$

then

$$\tilde{z}(k) = \tilde{y}^0(k) \;\oplus\; \tilde{r}(k). \quad (14)$$

We will prove this Lemma by contradiction.
First let us consider a $\tilde{u}^\sharp$ that satisfies (11) and (12) but for which $\sum_{\ell=1}^{N_\mathrm{p}} \tilde{u}^\sharp_\ell(k) < \sum_{\ell=1}^{N_\mathrm{p}} \tilde{u}^*_\ell(k)$. Define $\tilde{y}^\sharp(k) = \tilde{C}\otimes x(k-1) \;\oplus\; \tilde{D}\otimes\tilde{u}^\sharp(k)$, then from (14) it follows that

$$\begin{aligned} \max(\tilde{y}^\sharp_i(k), \tilde{r}_i(k)) &= \max(\max(\tilde{y}^0_i(k), \tilde{D}_i\otimes\tilde{u}^\sharp(k)), \tilde{r}_i(k)) \\ &= \max(\tilde{y}^0_i(k), \tilde{D}_i\otimes\tilde{u}^\sharp(k), \tilde{r}_i(k)) \\ &= \max(\tilde{z}_i(k), \tilde{D}_i\otimes\tilde{u}^\sharp(k)), \end{aligned}$$

[2]Note that the input sequences correspond to occurrence times of consecutive events, and so $u(k)$ should be nondecreasing.

[3]For a manufacturing system, this corresponds to a scheme in which raw material is fed to the system as late as possible. Note that this implies that the internal buffer levels are kept as low as possible.

Using (11) we find $\max(\tilde{y}_i^\sharp(k), \tilde{r}_i(k)) = \tilde{z}_i(k)$ and so $J_{\rm out}(\tilde{u}^\sharp(k)) = \sum_{i=1}^{N_{\rm p}} \tilde{z}_i(k) - \tilde{r}_i(k) = J_{\rm out}(\tilde{u}^*(k))$.
On the other hand

$$\begin{aligned} J_{\rm in}(\tilde{u}^\sharp(k)) &= \sum_{j=0}^{N_{\rm p}-1} \Big(r(k+j) - u^\sharp(k+j|k) \Big) \\ &> \sum_{j=0}^{N_{\rm p}-1} \Big(r(k+j) - u^*(k+j|k) \Big) = J_{\rm in}(\tilde{u}^*(k)) \end{aligned}$$

and so $J(\tilde{u}^\sharp(k)) > J(\tilde{u}^*(k))$ which means that $\tilde{u}^\sharp(k)$ cannot be the optimal value.

Next let us consider a vector $\tilde{u}'(k)$ that satisfies (12) but does not satisfy (11). Let $\alpha > 0$ be such that

$$\max_j (\tilde{D}_j \otimes \tilde{u}'(k) - z(k+j|k)) = \alpha$$

then there exists a $j' \in \{1, \ldots, N_{\rm p}-1\}$ such that $y'(k+j'|k) = \tilde{D}_{j'} \otimes \tilde{u}'(k) = z(k+j'|k) + \alpha$ and so

$$\begin{aligned} J_{\rm out}(\tilde{u}'(k)) &= \sum_{j=0}^{N_{\rm p}-1} \max\Big(y'(k+j|k) - r(k+j), 0 \Big) \\ &\geq \sum_{j=0}^{N_{\rm p}-1} \max\Big(z(k+j|k) - r(k+j), 0 \Big) + \alpha \\ &= J_{\rm out}(\tilde{u}^*(k)) + \alpha \end{aligned}$$

Now define $\tilde{u}''(k) = (\tilde{u}'(k) - \alpha) \oplus \bar{u}(k)$. Then $\tilde{D}_i \otimes \tilde{u}''(k) \leq \tilde{z}_i(k)$ and so $J_{\rm out}(\tilde{u}''(k)) = J_{\rm out}(\tilde{u}^*(k)) \leq J_{\rm out}(\tilde{u}'(k)) - \alpha$.
On the other hand

$$\begin{aligned} J_{\rm in}(\tilde{u}''(k)) &= \sum_{j=0}^{N_{\rm p}-1} \Big(r(k+j) - u''(k+j|k) \Big) \\ &\leq \sum_{j=0}^{N_{\rm p}-1} \Big(r(k+j) - u'(k+j|k) \Big) + N_{\rm p}\,\alpha \\ &= J_{\rm in}(\tilde{u}'(k)) + N_{\rm p}\,\alpha \end{aligned}$$

and so

$$\begin{aligned} J(\tilde{u}''(k)) &= J_{\rm out}(\tilde{u}''(k)) + \beta J_{\rm in}(\tilde{u}''(k)) \\ &\leq J_{\rm out}(\tilde{u}'(k)) - \alpha + \beta J_{\rm in}(\tilde{u}'(k)) + \beta N_{\rm p}\,\alpha \\ &= J(\tilde{u}'(k)) + (\beta N_{\rm p} - 1)\,\alpha \\ &\leq J(\tilde{u}'(k)) \end{aligned}$$

which means that $\tilde{u}'(k)$ cannot be the optimal value.
This proves that $\tilde{u}^*$ is the optimal solution of the original MPC problem. ∎

Lemma 2: Let

$$u^*(k+j|k) = \begin{cases} \min\limits_i (z(k+i|k) - \tilde{D}_{ij}) \\ \qquad\qquad \text{for } \ j = N_{\rm p}-1 \\ \min\Big(\min\limits_i (z(k+i|k) - \tilde{D}_{ij}), u^*(k+j+1|k) \Big) \\ \qquad\qquad \text{for } \ j = 1, \ldots, N_{\rm p}-2 \end{cases} \tag{15}$$

Then $\tilde{u}^*(k)$ is the optimal solution of the MPL-MPC problem.

Proof: The proof is straightforward:
Let $\tilde{u}^\sharp(k)$ be such that for some j' there holds $u^\sharp(k+j'|k) > u^*(k+j'|k)$, then either $u^\sharp(k+j'|k) > \min_i(z(k+i|k) - \tilde{D}_{ij'})$ or $u^\sharp(k+j'|k) > u^*(k+j'+1)$. In the first case, there will be an i' such that $[\tilde{D} \otimes \tilde{u}^\sharp(k)]_{i'} > \tilde{z}_{i'}(k)$ and so constraint (11) will not be satisfied. In the second case, (12) will not be satisfied. So $\tilde{u}^\sharp$ cannot be the optimal solution.
Now let $\tilde{u}^\flat(k)$ be such that for some j' there holds $\tilde{u}^\flat_{j'}(k) < \tilde{u}^*_{j'}(k)$. Then constraints (11) and (12) are both satisfied, but

$$\sum_{j=0}^{N_{\rm p}-1} u^\flat(k+j|k) < \sum_{j=0}^{N_{\rm p}-1} u^*(k+j|k),$$

and so $\tilde{u}^\flat$ cannot be the optimal solution. ∎

Qualitative comparison with other existing max-plus control approaches

The max-plus control approaches proposed in [1], [7], [9], [10] typically involve an open-loop control approach over a given horizon $N_{\rm p}$ and for a given due date signal r such that the output y of the system satisfies $y(k+i) \leq r(k+i)$ for $k = 1, \ldots, N_{\rm p}$. The solution of this optimal control problem is computed using residuation [3], resulting in a just-in-time control input. The disadvantage of this approach is that the resulting control input sequence is sometimes decreasing, and thus not physically feasible. This issue is overcome in [2], [11] by considering residuation-based approaches that result in non-decreasing input sequences.

IV. THE SHIFTED SYSTEM

Let us assume that the system is strongly connected[4], which means that there are no internal buffers that are not (indirectly) coupled to the output of the system (observability). Let λ be the unique max-plus algebraic eigenvalue of A, so there exists an eigenvector v_λ such that $A \otimes v_\lambda = v_\lambda \otimes \lambda$. Let c be the cycle length, so $A^{\otimes^c} = A + c\,\lambda$. The eigenvalue λ gives a minimum for the average duration of a system cycle. If the asymptotic slope of the due date signal $r(k)$ is smaller than λ, the system cannot complete tasks in time and $y(k) - r(k)$ will grow unbounded in time. Therefore in this paper we choose

$$r(k) = r_0 + \rho\,k. \tag{16}$$

where we assume that $\rho > \lambda$. Now define $u(k) = \rho\,k + u_\rho(k)$, $x(k) = \rho\,k + x_\rho(k)$, $y(k) = r_0 + \rho\,k + y_\rho(k)$, and A_ρ and C_ρ are matrices with $[A_\rho]_{ij} = [A]_{ij} - \rho$ and $[C_\rho]_{ij} = [C]_{ij} - [r_0]_i$. To every ρ we can then associate a shifted system

$$\begin{aligned} x_\rho(k) &= A_\rho \otimes x_\rho(k-1) \ \oplus \ B \otimes u_\rho(k), \\ y_\rho(k) &= C_\rho \otimes x_\rho(k). \end{aligned}$$

[4]This means that the precedence graph of the system is strongly connected [1], or equivalently, the system matrix A is irreducible.

The MPL-MPC problem for the shifted system means that we aim to minimize

$$
\begin{aligned}
J &= \min_{\tilde{u}} \sum_{j=0}^{N_\mathrm{p}-1} \max(y(k+j|k) - r(k+j), 0) \\
&\quad + \beta(r(k+j) - u(k+j|k) \\
&= \min_{\tilde{u}_\rho} \sum_{j=0}^{N_\mathrm{p}-1} \max(y_\rho(k+j|k), 0) + \beta(r_0 - u_\rho(k+j|k)),
\end{aligned}
$$

subject to

$$
\begin{aligned}
&u(k+j|k) - u(k+j-1|k) = \\
&\qquad = u_\rho(k+j|k) - u_\rho(k+j-1|k) + \rho \geq 0
\end{aligned}
$$

or $\qquad u_\rho(k+j|k) - u_\rho(k+j-1|k) \geq -\rho.$

V. STABILITY

In this paper we adopt the notion of stability for DES from [12], in which a DES is called stable if all its buffer levels remain bounded.

In [13] we showed that for a strongly connected system all the buffer levels are bounded if the dwelling times of the parts or batches in the system remain bounded. This implies that for an observable strongly connected DES with due date $r(k)$ closed-loop stability is achieved if there exist finite constants M_{yr}, M_{ry} and M_{yu} such that

$$y(k) - r(k) \leq M_{yr}, \tag{17}$$

$$r(k) - y(k) \leq M_{ry}, \tag{18}$$

$$y(k) - u(k) \leq M_{yu} \ . \tag{19}$$

Condition (17) means that the delay between the actual output date $y(k)$ and the due date $r(k)$ remains bounded. Condition (18) implies that the stock time will remain bounded. Finally, condition (19) means that the time between the starting date $u(k)$ and the output date $y(k)$ (i.e., the throughput time) is bounded. For due date (16) with $\rho > 0$ this implies finite buffer levels.

We will proof stability for the shifted system. Note that if the shifted system is stable, also the original system will be stable. First we need the following lemma:

Lemma 3: Assume a finite impulse response, so

$$C_\rho \otimes {A_\rho}^{\otimes^k} \otimes B > \varepsilon \quad \text{for all } k \geq 0.$$

Then for any $\mu \geq 0$ there exist an N_p such that for all $\ell \geq N_\mathrm{p} - 1$ there holds:

$$\max_i [C_\rho \otimes (A_\rho)^{\otimes^\ell}]_i - \max_j [C_\rho \otimes A_\rho]_j \leq -\mu, \tag{20}$$

and

$$C_\rho \otimes \Big({A_\rho}^{\otimes^{j-i}} - (i+1)\,\rho \Big) \otimes B - C_\rho \otimes B \leq 0. \tag{21}$$

Proof: Note that there exists an ℓ_0 such that ${A_\rho}^{\otimes^{\ell+c}} = {A_\rho}^{\otimes^\ell} + c(\lambda - \rho)$, for all $\ell > \ell_0$, where c is the cycle length, λ is the max-plus-algebraic eigenvalue of A, and $\rho > \lambda$ is the asymptotic slope of due date signal $r(k)$. Let

$$a_{\max} = \max_{\ell=0,\ldots,\ell_0+c-1} \max_{i,j} [{A_\rho}^{\otimes^\ell}]_{i,j},$$

and let m_0 be the smallest integer satisfying $m_0 \geq (a_{\max} + \mu)/c(\rho - \lambda)$ and let N_p satisfy $N_\mathrm{p} > \ell_0 + m_0\, c$. Consider $\ell = \ell_0 + m_1\, c + m_2$ where $m_1, m_2 \in \mathbb{Z}$, $m_1 \geq m_0$, $0 \leq m_2 \leq c - 1$, then

$$
\begin{aligned}
&[(A_\rho)^{\otimes^{\ell_0 + m_1\, c + m_2}}]_{i,j} = [(A_\rho)^{\otimes^{\ell_0 + m_2}}]_{i,j} + m_1 c(\lambda - \rho) \\
&\leq a_{\max} - m_1 c(\rho - \lambda) \leq a_{\max} - (a_{\max} + \mu) \leq -\mu \ .
\end{aligned}
$$

Now for $\ell > N_\mathrm{p}$ we find

$$
\begin{aligned}
&\max_i [C_\rho \otimes (A_\rho)^{\otimes^\ell}]_i - \max_p [C_\rho \otimes A_\rho]_p \\
&= \max_i [C_\rho \otimes A_\rho \otimes (A_\rho)^{\otimes^{\ell-1}}]_i - \max_p [C_\rho \otimes A_\rho]_p \\
&= \max_i [C_\rho \otimes A_\rho]_i + \max_{i,j} [(A_\rho)^{\otimes^{\ell-1}}]_{i,j} - \max_p [C_\rho \otimes A_\rho]_p \\
&= \max_{i,j} [(A_\rho)^{\otimes^{\ell-1}}]_{i,j} \\
&\leq -\mu \ ,
\end{aligned}
$$

This proofs that there exists an N_p such that (20) holds.

Define $\nu = \max_i [C_\rho]_i + \max_p [B]_p - C_\rho \otimes B$, and let n_0 be the smallest integer satisfying $n_0 \geq (a_{\max} + \nu)/c(\rho - \lambda)$. Now define $i_0 = (\nu + a_{\max})/\rho - 1$ and let N_p satisfy $N_\mathrm{p} > \ell_0 + i_0 + n_0\, c$. Consider $\ell = \ell_0 + i_0 + n_1\, c + n_2$ where $n_1, n_2 \in \mathbb{Z}$, $n_1 \geq n_0$, $0 \leq n_2 \leq c - 1$, then for $0 \leq i \leq i_0$ there holds:

$$
\begin{aligned}
&[{A_\rho}^{\otimes^{\ell-i}} - (i+1)\,\rho]_{i,j} \leq [(A_\rho)^{\otimes^{\ell_0 + i_0 + n_1 c + n_2 - i}}]_{i,j} \\
&= [(A_\rho)^{\otimes^{\ell_0 + i_0 + n_2 - i}}]_{i,j} - n_1 c(\rho - \lambda) \leq a_{\max} - n_1 c(\rho - \lambda) \\
&\leq a_{\max} - (a_{\max} + \nu) \leq -\nu,
\end{aligned}
$$

and for $i_0 \leq i \leq \ell$ there holds:

$$
\begin{aligned}
&[{A_\rho}^{\otimes^{\ell-i}} - (i+1)\,\rho]_{i,j} \leq a_{\max} - (i_0+1)\,\rho \\
&\leq a_{\max} - (a_{\max} + \nu) \leq -\nu \ ,
\end{aligned}
$$

So we can derive

$$
\begin{aligned}
&C_\rho \otimes \Big({A_\rho}^{\otimes^{\ell-i}} - (i+1)\,\rho \Big) \otimes B - C_\rho \otimes B \leq \\
&\leq \max_q [C_\rho]_q + \max_{j_1,j_2} [{A_\rho}^{\otimes^{\ell_0 + i_0 + n_1\, c + n_2 - i}} - (i+1)\,\rho]_{j_1,j_2} \\
&\quad + \max_p [B]_p - C_\rho \otimes B \\
&\leq \max_q [C_\rho]_q - \nu + \max_p [B]_p - C_\rho \otimes B \\
&\leq \max_q [C_\rho]_q - \Big(\max_i [C_\rho]_i + \max_p [B]_p - C_\rho \otimes B \Big) \\
&\quad + \max_p [B]_p - C_\rho \otimes B \\
&\leq 0 \ .
\end{aligned}
$$

This proofs that there exists an N_p such that (21) holds. So if we choose $N_\mathrm{p} = \max\Big(\ell_0 + m_0\, c, \ell_0 + i_0 + n_0\, c\Big)$, we are sure that both (20) and (21) are satisfied. ■

Theorem 4: Assume that for the finite values of $x_\rho(k-1)$ there hold

$$\max_{i,j} |\, [x_\rho(k-1)]_i - [x_\rho(k-1)]_j \,| \leq \mu \;, \tag{22}$$

where $\mu > 0$ is constant. Now let N_p satisfy conditions (20) and (21) from lemma 3, and let β be such that

$$\beta < 1/N_\mathrm{p} \;, \tag{23}$$

then the closed loop will be stable.

Proof: For any initial $x_\rho(k-1)$ satisfying (22), we have

$$\begin{aligned} &C_\rho \otimes A_\rho \otimes x_\rho(k-1) = \\ &\quad = \max_m([C_\rho \otimes A_\rho]_m + [x_\rho(k-1)]_m) \\ &\quad \geq \max_m [C_\rho \otimes A_\rho]_m + [x_\rho(k-1)]_j - \mu \end{aligned}$$

for any $j = 1, \ldots, n$. Now we derive for $\ell \geq N_\mathrm{p} - 1$:

$$\begin{aligned} &C_\rho \otimes (A_\rho)^{\otimes^\ell} \otimes x_\rho(k-1) - C_\rho \otimes A_\rho \otimes x_\rho(k-1) \leq \\ &\leq \max_i \Big([C_\rho \otimes (A_\rho)^{\otimes^\ell}]_i + [x_\rho(k-1)]_i\Big) \\ &\quad - \max_j \Big(\max_m [C_\rho \otimes A_\rho]_m + [x_\rho(k-1)]_j - \mu\Big) \\ &\leq \max_i \Big([C_\rho \otimes (A_\rho)^{\otimes^\ell}]_i + [x_\rho(k-1)]_i - [x_\rho(k-1)]_i\Big) \\ &\quad - \max_m [C_\rho \otimes A_\rho]_m + \mu \\ &= \max_i [C_\rho \otimes (A_\rho)^{\otimes^\ell}]_i - \max_m [C_\rho \otimes A_\rho]_m + \mu \\ &\leq 0 \;, \end{aligned}$$

because of (20).
Furthermore for any $i = 1, \ldots, \ell$ there holds:

$$\begin{aligned} &C_\rho \otimes B \;\oplus\; u_\rho(k-1) = \\ &= \max\Big(C_\rho \otimes B, u_\rho(k-1)\Big) \\ &\geq \max\Big(C_\rho \otimes \Big({A_\rho}^{\otimes^{j-i}} - (i+1)\,\rho\Big) \otimes B, u_\rho(k-1)\Big) \\ &= C_\rho \otimes \Big({A_\rho}^{\otimes^{j-i}} - (i+1)\,\rho\Big) \otimes B \;\oplus\; u_\rho(k-1) \;, \end{aligned}$$

because of (21).
Now define $z_\rho(k+j|k) = z(k+j|k) - r(k+j)$, then with

$$\begin{aligned} z(k+j|k) = {}& C \otimes A^{\otimes^{j+1}} \otimes x(k-1) \oplus \\ & \bigoplus_{i=0}^{j} C \otimes \Big(A^{\otimes^{j-i}}\Big) \otimes B \otimes u(k-1) \;\oplus\; r(k+j) \;, \end{aligned}$$

we obtain

$$\begin{aligned} &z_\rho(k+j|k) = z(k+j|k) - r_0 - (k+j)\rho \\ &= (C - r_0) \otimes (A^{\otimes^{j+1}} - (j+1)\rho) \otimes \\ &\;(x(k-1) - (k-1)\rho) \oplus \\ &\;\bigoplus_{i=0}^{j} (C - r_0) \otimes \Big(A^{\otimes^{j-i}} - (j-i)\rho - (i+1)\,\rho\Big) \otimes \\ &\;B \otimes (u(k-1) - (k-1)\rho) \;\oplus\; (r(k+j) - r(k+j)) \\ &= C_\rho \otimes {A_\rho}^{\otimes^{j+1}} \otimes x_\rho(k-1) \oplus \\ &\;\bigoplus_{i=0}^{j} C_\rho \otimes \Big({A_\rho}^{\otimes^{j-i}} - (i+1)\,\rho\Big) \otimes B \otimes u_\rho(k-1) \;\oplus\; 0. \end{aligned}$$

Now for any $j \geq N_\mathrm{p} - 1$ we have

$$\begin{aligned} &C_\rho \otimes B \;\oplus\; u_\rho(k-1) \\ &\geq C_\rho \otimes \Big({A_\rho}^{\otimes^{j-i}} - (i+1)\,\rho\Big) \otimes B \;\oplus\; u_\rho(k-1), \end{aligned}$$

and $C_\rho \otimes A_\rho \otimes x_\rho(k-1) \leq C_\rho \otimes (A_\rho)^{\otimes^j} \otimes x_\rho(k-1)$, and so we find $(z_\rho(k|k) - z_\rho(k+j|k)) \geq 0$ for $j \geq N_\mathrm{p} - 1$. Now we can write $u_\rho^*(k|k) \leq z_\rho(k) - C_\rho \otimes B$, and $u_\rho^*(k+N_\mathrm{p}|k+1) = z_\rho(k+N_\mathrm{p}) - C_\rho \otimes B$. Define

$$\begin{aligned} &\delta u_\rho(k+j|k+1) = u_\rho^*(k+j|k) - u_\rho^*(k+j|k+1), \\ &\qquad\qquad \text{for } j = 1, \ldots, N_\mathrm{p} - 1, \\ &u'_\rho(k|k) = z_\rho(k) - C_\rho \otimes B, \\ &\delta_z = \max\Big(z_\rho(k+N_\mathrm{p}-1|k) - z_\rho(k+N_\mathrm{p}|k+1), 0\Big), \end{aligned}$$

then $u_\rho^*(k|k) \leq u'_\rho(k|k)$ and

$$\begin{aligned} &u_\rho^*(k{+}N_\mathrm{p}{-}1|k) - u_\rho^*(k{+}N_\mathrm{p}|k{+}1) = \\ &\qquad = \Big(z_\rho(k{+}N_\mathrm{p}{-}1|k) {-} C_\rho {\otimes} B\Big) \\ &\qquad\quad - \Big(z_\rho(k{+}N_\mathrm{p}|k{+}1) {-} C_\rho {\otimes} B\Big) \\ &\qquad = z_\rho(k{+}N_\mathrm{p}{-}1|k) - z_\rho(k{+}N_\mathrm{p}|k{+}1) \\ &\qquad \leq \delta_z \;, \end{aligned}$$

and so we find:

$$\begin{aligned} \delta u_\rho(k{+}N_\mathrm{p}|k{+}1) = {}& u_\rho^*(k{+}N_\mathrm{p}{-}1|k) \\ & - u_\rho^*(k{+}N_\mathrm{p}{-}1|k{+}1) \leq \delta_z \;. \end{aligned}$$

Similarly we find for $j = 1, \ldots, N_\mathrm{p}$–1: $\delta u_\rho(k{+}j|k{+}1) \leq \delta_z$.
Now we derive:

$$\begin{aligned} &J(k) - J(k{+}1) = \\ &\quad = z_\rho(k|k) - \beta u_\rho^*(k|k) - z_\rho(k{+}N_\mathrm{p}|k+1) \\ &\qquad + \beta u_\rho^*(k{+}N_\mathrm{p}|k{+}1) - \beta \sum_{j=1}^{N_\mathrm{p}-1} \delta u_\rho(k+j|k{+}1) \\ &\quad \geq z_\rho(k|k) - \beta u'_\rho(k|k) - z_\rho(k{+}N_\mathrm{p}|k+1) \\ &\qquad + \beta u_\rho^*(k{+}N_\mathrm{p}|k{+}1) - \beta \sum_{j=1}^{N_\mathrm{p}-1} \delta u_\rho(k+j|k{+}1) \\ &\quad = (1{-}\beta) z_\rho(k|k) - (1{-}\beta) z_\rho(k{+}N_\mathrm{p}) \end{aligned}$$

$$
\begin{aligned}
&- \beta \sum_{j=1}^{N_\mathrm{p}-1} \delta u_\rho(k+j|k{+}1) \\
&= (1{-}\beta)(z_\rho(k|k) - \max(z_\rho(k{+}N_\mathrm{p}{-}1), z_\rho(k{+}N_\mathrm{p}))) \\
&\quad + (1{-}\beta)\max(z_\rho(k{+}N_\mathrm{p}{-}1) - z_\rho(k{+}N_\mathrm{p}), 0) \\
&\quad - \beta \sum_{j=1}^{N_\mathrm{p}-1} \delta u_\rho(k+j|k{+}1) \\
&\geq (1{-}\beta)(z_\rho(k|k) - \max(z_\rho(k{+}N_\mathrm{p}{-}1), z_\rho(k{+}N_\mathrm{p}))) \\
&\quad + (1{-}\beta)\delta_z - \beta(N_\mathrm{p} - 1)\delta_z \\
&= \underbrace{(1{-}\beta)}_{>0} \underbrace{(z_\rho(k) - \max(z_\rho(k{+}N_\mathrm{p}{-}1), z_\rho(k{+}N_\mathrm{p})))}_{\geq 0} + \\
&\quad + \underbrace{(1{-}\beta N_\mathrm{p})}_{>0} \underbrace{\delta_z}_{\geq 0} \\
&\geq 0\,.
\end{aligned}
$$

And so the cost function $J(k)$ will be a decreasing function for an increasing event step. This means that $J_\mathrm{out}(k)$ and $J_\mathrm{in}(k)$ will be bounded for $k \to \infty$. This implies that there exists an upper bound for $y(k) - r(k)$ and that $r(k) - u(k)$ will have both an upper and lower bound. With the property that $y(k) - u(k) \geq C \otimes B$ we also prove that $r(k){-}y(k)$ has an upper bound, which proves that the closed-loop system is stable. ■

VI. WORKED EXAMPLE

Consider an MPL system (1)-(2) with

$$
A = \begin{bmatrix} \varepsilon & 0 & \varepsilon & 9 \\ 4 & 3 & 4 & 5 \\ 8 & \varepsilon & 1 & 8 \\ 0 & 0 & \varepsilon & \varepsilon \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 5 \\ 2 \\ 8 \end{bmatrix},
$$
$$
C = \begin{bmatrix} 7 & 5 & 8 & \varepsilon \end{bmatrix}.
$$

We find a lower bound for $N_\mathrm{p} \geq 5$, and an upper bound for $\beta < 0.2$. In a first simulation we choose $N_\mathrm{p} = 5$ and $\beta = 0.18$. We find a stable operation and the cost-function $J(k)$ remains bounded. In a second simulation we choose $N_\mathrm{p} = 2$ and $\beta = 0.18$, and in a third simulation we choose $N_\mathrm{p} = 5$ and $\beta = 0.65$. In both cases the closed loop will become unstable and the cost function $J(k)$ will grow unboundedly. The evolution of cost function $J(k)$ for the three simulations is given in figure 1.

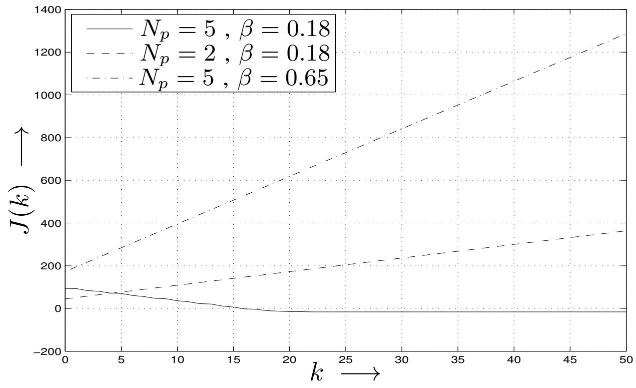

Fig. 1. $J(k)$ for various value of λ and N_p

VII. DISCUSSION

In this paper we have derived the analytic solution for the MPL-MPC control problem by studying the solution of the corresponding optimization problem. We compared the solution to other well-known control schemes for MPL systems. We derived bounds on the tuning variables, the prediction horizon N_p and the trade off constant β, such that the MPL-MPC now gives a guaranteed stable closed-loop, which means that all the buffer levels of the system remain bounded. A simulation example illustrates that a wrong choice of the tuning parameters may indeed destabilize the system.

Topics of future research are to incorporate general inequality constraints, perturbations and to extend the results to multivariable systems.

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