

Technical report 06-002

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I. Necoara, E.C. Kerrigan, B. De Schutter, and T.J.J. van den Boom, "Finite-horizon min-max control of max-plus-linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 1088–1093, June 2007.

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Finite-Horizon Min-Max Control of Max-Plus-Linear Systems

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Abstract

We provide a solution to a class of finite-horizon min-max control problems for uncertain max-plus-linear systems where the uncertain parameters are assumed to lie in a given convex and compact set, and it is required that the closed-loop input and state sequence satisfy a given set of linear inequality constraints for all admissible uncertainty realizations. We provide sufficient conditions such that the value function is guaranteed to be convex and continuous piecewise affine, and such that the optimal control policy is guaranteed to be continuous and piecewise affine on a polyhedral domain.

Index Terms

Optimal control, discrete event systems, max-plus-linear systems, min-max control.

I. INTRODUCTION

Discrete-event systems (DES) are event-driven dynamic systems, i.e. the state transitions are driven by events, rather than a clock. In the last couple of decades there has been an increase in the amount of research on DES that can be modeled as max-plus-linear (MPL)

This research was performed while the first author was visiting the Control Group of the University of Cambridge, Cambridge, UK. Research supported by the STW-MPC project “Model predictive control for hybrid systems” (DMR.5675), the European Network of Excellence HYCON (FP6-IST-511368), and the Royal Academy of Engineering, UK.

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Manuscript received December 7, 2005; revised January 28, 2007.

systems. MPL systems are *nonlinear* dynamic systems that are “linear” in the max-plus-algebra [1] and often arise in the context of manufacturing systems, railway networks, parallel computing, etc.

Although there are some papers on optimal control for MPL systems (see [2], [3] and the references therein), the literature on robust control for this class of systems is relatively sparse. Some of the contributions include open-loop min-max model predictive control [4] and closed-loop control based on residuation theory [3], [5], [6]. This paper considers the MPL versions of the finite-horizon robust optimal control problem [7] for uncertain dynamic systems using the min-max paradigm. The main advantage of this paper compared to existing results on robust control of MPL systems [3]–[6] is the fact that we optimize over feedback policies, rather than open-loop input sequences, and that we incorporate state and input constraints directly into the problem formulation. In general, this results in increased feasibility and a better performance.

We use a dynamic programming approach similar to the one used in [8], [9] for finite-horizon min-max control of uncertain linear systems with constraints. One of the key contributions of this paper is to provide sufficient conditions such that we can employ results from convex analysis to compute robust optimal controllers for MPL systems. Note that we require the stage cost to have a particular representation in which the coefficients corresponding to the state vector are non-negative, *and* that the matrix associated with the state constraints should also be non-negative. However, these conditions are not restrictive for practical applications (see Section I-B).

This section proceeds by introducing some notation and defining the min-max control problem of interest. In Section II we present the exact solution of the problem via parametric programming based on a dynamic programming approach. We conclude with an example in Section III.

A. Definitions and Notation

Define $\varepsilon := -\infty$ and $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic (MPA) addition (\oplus) and multiplication (\otimes) are defined as [1]: $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$ for $x, y \in \mathbb{R}_\varepsilon$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$ we define: $[A \oplus B]_{ij} := A_{ij} \oplus B_{ij}$, $[A \otimes C]_{ij} := \bigoplus_{k=1}^n A_{ik} \otimes C_{kj}$ for all i, j . For a positive integer N , let $\underline{N} := \{1, 2, \dots, N\}$. Given a matrix $H = [H_{ij}]$, by $H \geq 0$ we mean that $H_{ij} \geq 0$ for all i, j . Given a set $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, let $\text{Proj}_n \mathcal{Z} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ s. t. } (x, y) \in \mathcal{Z}\}$. A polyhedron is the intersection of a finite

number of closed half-spaces. A closed half-space is defined by $\{x \in \mathbb{R}^n : \alpha_i^T x \leq \beta_i\}$ for some $\alpha_i \in \mathbb{R}^n, \beta_i \in \mathbb{R}$. A function $g: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is *proper* if $\{x \in \mathbb{R}^n : -\infty < g(x) < +\infty\} \neq \emptyset$ [10]. The epigraph of a function $g: X \rightarrow \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ is defined as $\text{epi } g := \{(x, t) \in X \times \mathbb{R} : g(x) \leq t\}$. A function $g(\cdot)$ is piecewise affine (PWA) if its epigraph is a finite union of polyhedra [10]. Let \mathcal{F}_{mps} denote the set of *max-plus-scaling* functions, i.e. $g: X \rightarrow \mathbb{R}, x \mapsto g(x) = \max_{j \in \underline{l}} \{\alpha_j^T x + \beta_j\}$, where $X \subseteq \mathbb{R}^q, \alpha_j \in \mathbb{R}^q$ and $\beta_j \in \mathbb{R}$. Note that a max-plus-scaling function is another representation for a convex PWA function [10]. Let $\mathcal{F}_{\text{mps}}^+$ denote the set of *max-plus-non-negative-scaling* functions, i.e. functions $g(x) = \max_{j \in \underline{l}} \{\alpha_j^T x + \beta_j\}$ with $\alpha_j \geq 0$ for all $j \in \underline{l}$.

B. Problem Definition

The system matrices of a DES modeled as an MPL system usually consist of sums or maximization of internal process times, transportation times, etc. Therefore, we consider the following uncertain MPL system [4]:

$$x(k) = A(w(k)) \otimes x(k-1) \oplus B(w(k)) \otimes u(k), \quad y(k) = C(w(k)) \otimes x(k), \quad (1)$$

where $A(\cdot) \in \mathcal{F}_{\text{mps}}^{n \times n}, B(\cdot) \in \mathcal{F}_{\text{mps}}^{n \times m}$ and $C(\cdot) \in \mathcal{F}_{\text{mps}}^{p \times n}$ (note that these matrix functions are nonlinear). Here, k is an event *counter* while $x(k) \in \mathbb{R}_e^n, y(k) \in \mathbb{R}_e^p$, and $u(k) \in \mathbb{R}_e^m$ represent event occurrence *times* [1]. For a manufacturing system, $u(k)$ would represent the time instants at which raw material is fed to the system for the k^{th} time, $x(k)$ the time instants at which the machines start processing the k^{th} batch of intermediate products, and $y(k)$ the time instants at which the k^{th} batch of finished products leaves the system. We gather in the vector $w(k) \in \mathbb{R}^q$ all the uncertainty caused by disturbances and errors in the estimation of process and event times. The (unknown) disturbance signal $w(k)$ is assumed to be time-varying and to take on values from a compact and convex set $W \subset \mathbb{R}^q$. For a manufacturing system the components of $w(k)$ would represent the uncertain transportation and processing times of the plant (see e.g. [4]). We also consider a reference signal $\{r(k) \in \mathbb{R}^p\}_{k \geq 0}$ which the system (1) is required to track.

We frequently use the short-hand notation $f(x, u, w) := A(w) \otimes x \oplus B(w) \otimes u$, and it is easy to verify that $f(\cdot) \in \mathcal{F}_{\text{mps}}^n$ and $f(\cdot, \cdot, w) \in (\mathcal{F}_{\text{mps}}^+)^n$ for any fixed w (because $[f(x, u, w)]_i = [\max_{j,l} \{A_{ij}(w) + x_j, B_{il}(w) + u_l\}]_i$ according to Lemma 1 below). Since $f(\cdot, w)$ is a max expression of affine terms in (x, u) , each component of $f(\cdot, w)$ is convex [10].

A typical stage cost used in the optimal control literature for MPL systems is [4]:

$$\ell(x, u, r, w) = \sum_{j=1}^p \max\{[C(w) \otimes x - r]_j, 0\} - \gamma \sum_{j=1}^m u_j, \quad (2)$$

where $0 \leq \gamma$. For a manufacturing systems this stage cost has the interpretation that the first term penalizes the delay of the finishing times $y = C(w) \otimes x$ with respect to the due date signal r , while the second term tries to maximize the feeding times u . Typical constraints for a DES are of the form:

$$y(k) \leq r(k) + h^{yt}(k), \quad u_i(k) - u_j(k) \leq h_{ij}^u(k), \quad x_i(k) - u_j(k) \leq h_{ij}^{xu}(k), \quad u(k+1) - u(k) \geq 0, \quad (3)$$

where the h variables define bounds on the differences in times. Since the input represents times, the signal $u(\cdot)$ should be non-decreasing, i.e. $u(k+1) - u(k) \geq 0$. However, by remodeling the system as in Appendix A, this constraint can be removed.

For the min-max problem to be defined below, the constraints (3) and the stage cost (2) considered will be generalized. We therefore assume that the system is subject to input and state constraints over a finite horizon of length N :

$$H_k x(k) + G_k u(k) + F_k r(k) + E_k w(k) \leq h_k, \quad k = 1, 2, \dots, N \quad (4)$$

where $H_k \in \mathbb{R}^{n_{H_k} \times n}$, $G_k \in \mathbb{R}^{n_{H_k} \times m}$, $F_k \in \mathbb{R}^{n_{H_k} \times p}$, $E_k \in \mathbb{R}^{n_{H_k} \times q}$ and $h_k \in \mathbb{R}^{n_{H_k}}$.

Effective control in the presence of disturbance requires optimization over *feedback policies* [7], [11], rather than open-loop input sequences. Therefore, we will define the decision variable in the optimal control problem for a given initial condition x and the reference signal $\mathbf{r} := [r_1^T, r_2^T, \dots, r_N^T]^T$, as a control policy $\pi := (\mu_1(\cdot), \mu_2(\cdot), \dots, \mu_N(\cdot))$, where each $\mu_i : \mathbb{R}^n \times \mathbb{R}^{pN} \rightarrow \mathbb{R}^m$ is a state feedback control law (recall that p is the dimension of the output and N is the prediction horizon). Let $\mathbf{w} := [w_1^T, w_2^T, \dots, w_N^T]^T$ denote a realization of the disturbance over the horizon $k = 1, \dots, N$. Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the state solution of (1) at step i when the initial state is x , the control is determined by the policy π , i.e. $u(i) = \mu_i(\phi(i-1; x, \pi, \mathbf{w}), \mathbf{r})$, and the disturbance sequence is \mathbf{w} . By definition, $\phi(0; x, \pi, \mathbf{w}) := x$. The cost is defined as:

$$V_N(x, \pi, \mathbf{r}, \mathbf{w}) := \sum_{i=1}^N \ell_i(x_i, u_i, r_i, w_i), \quad (5)$$

where $x_i := \phi(i; x, \pi, \mathbf{w})$ (and thus $x_0 = x$), $u_i := \mu_i(x_{i-1}, \mathbf{r})$, and ℓ_i is the stage cost. The following assumptions will be used throughout the paper:

A1: The matrices H_k in (4) are non-negative for all $k \in \underline{N}$ (recall that $\underline{N} = \{1, 2, \dots, N\}$).

A2: The stage costs $\ell_i(\cdot)$ satisfy: $\ell_i(\cdot) \in \mathcal{F}_{\text{mps}}$ and $\ell_i(\cdot, u, r, w) \in \mathcal{F}_{\text{mps}}^+$ for all (u, r, w) .

Note that the constraints (3) and the stage cost (2) satisfy Assumptions **A1** and **A2**.

For each initial condition x and due dates \mathbf{r} we define the set of feasible policies π :

$$\Pi_N(x, \mathbf{r}) := \{\pi : H_i \phi(i; x, \pi, \mathbf{w}) + G_i \mu_i(\phi(i-1; x, \pi, \mathbf{w}), \mathbf{r}) + F_i r_i + E_i w_i \leq h_i, \forall \mathbf{w} \in \mathcal{W}, i \in \underline{N}\}, \quad (6)$$

where $\mathcal{W} := W^N$. Also, let X_N denote the set of initial states *and* reference signals for which a feasible policy exists, i.e. $X_N := \{(x, \mathbf{r}) : \Pi_N(x, \mathbf{r}) \neq \emptyset\}$. The following min-max problem will be referred to as the *finite-horizon robust optimal control* problem:

$$\mathbb{P}_N(x, \mathbf{r}) : \quad V_N^0(x, \mathbf{r}) := \inf_{\pi \in \Pi_N(x, \mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{r}, \mathbf{w}). \quad (7)$$

Let $\pi_N^0(x, \mathbf{r}) =: (\mu_1^0(x, \mathbf{r}), \mu_2^0(\cdot), \dots, \mu_N^0(\cdot))$ denote a minimizer of the worst-case problem $\mathbb{P}_N(x, \mathbf{r})$ whenever the infimum is attained, i.e. $\pi_N^0(x, \mathbf{r}) \in \arg \min_{\pi \in \Pi_N(x, \mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{r}, \mathbf{w})$.

Remark 1: MPL systems are DES and they thus differ from conventional time-driven systems in the sense that the event counter k is not directly related to a specific time where in practice the optimization problem $\mathbb{P}_N(x, \mathbf{r})$ has to be solved at some given time t . For a detailed description of how to deal with these timing issues the interested reader is referred to [12].

II. DYNAMIC PROGRAMMING SOLUTION VIA PARAMETRIC PROGRAMMING

Dynamic programming (DP) [7], [11], [13] is a well-known method for solving sequential, or multi-stage, decision problems. More specifically, for the control problem considered in this paper the problem is split into N stages and for each stage s (starting from the last stage and going backwards to the first stage, i.e. $s \in \underline{N}$ denotes the “time-to-go”) we can compute sequentially the partial return functions $V_s^0(\cdot)$, the associated set-valued optimal control laws $\kappa_s(\cdot)$ (such that $\mu_{N-s+1}^0(x, \mathbf{r}) \in \kappa_s(x, \mathbf{r})$ for any $(x, \mathbf{r}) \in X_s$) and their domains X_s . If we define

$$J_s(x, \mathbf{r}, u) := \max_{w \in W} \{\ell_{N-s+1}(f(x, u, w), u, r_{N-s+1}, w) + V_{s-1}^0(f(x, u, w), \mathbf{r})\}, \forall (x, \mathbf{r}, u) \in Z_s, \quad (8a)$$

$$Z_s := \{(x, \mathbf{r}, u) : H_{N-s+1} f(x, u, w) + G_{N-s+1} u + F_{N-s+1} r_{N-s+1} + E_{N-s+1} w \leq h_{N-s+1}, \\ (f(x, u, w), \mathbf{r}) \in X_{s-1}, \forall w \in W\}, \quad (8b)$$

then using the optimality principle we can compute $\{V_s^0(\cdot), \kappa_s(\cdot), X_s\}_{s=1}^N$ recursively:

$$V_s^0(x, \mathbf{r}) = \min_u \{J_s(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z_s\}, \quad \forall (x, \mathbf{r}) \in X_s, \quad (8c)$$

$$\kappa_s(x, \mathbf{r}) = \arg \min_u \{J_s(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z_s\} \quad \forall (x, \mathbf{r}) \in X_s, \quad X_s = \text{Proj}_{n+pN} Z_s, \quad (8d)$$

with the boundary conditions $X_0 = \mathbb{R}^n \times \mathbb{R}^{pN}$, $V_0^0(x, \mathbf{r}) = 0$, $\forall (x, \mathbf{r}) \in \mathbb{R}^n \times \mathbb{R}^{pN}$.

To simplify notation in the rest of the paper, we first define two prototype problems. Next, we will study their properties. The prototype maximization problem $\mathbb{P}_{\max}(x, \mathbf{r}, u)$ is defined as:

$$\mathbb{P}_{\max}(x, \mathbf{r}, u): \quad J(x, \mathbf{r}, u) := \max_{w \in W} \{\ell(f(x, u, w), u, r, w) + V(f(x, u, w), \mathbf{r})\}, \quad \forall (x, \mathbf{r}, u) \in Z, \quad (9)$$

where $\ell : \mathbb{R}^{n+m+p+q} \rightarrow \mathbb{R}$, $V : \Omega \rightarrow \mathbb{R}$, r is a sub-block of \mathbf{r} (i.e. $\exists s \in \underline{N} : r_s = r$) and

$$Z := \{(x, \mathbf{r}, u) : Hf(x, u, w) + Gu + Fr + Ew \leq h, (f(x, u, w), \mathbf{r}) \in \Omega, \forall w \in W\}, \quad (10a)$$

$$X := \text{Proj}_{n+pN} Z. \quad (10b)$$

The prototype minimization problem $\mathbb{P}_{\min}(x, \mathbf{r})$ is defined as:

$$\mathbb{P}_{\min}(x, \mathbf{r}) : \quad V^0(x, \mathbf{r}) := \min_u \{J(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z\}, \quad \forall (x, \mathbf{r}) \in X \quad (11a)$$

$$\kappa(x, \mathbf{r}) := \arg \min_u \{J(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z\}, \quad \forall (x, \mathbf{r}) \in X. \quad (11b)$$

In terms of these prototype problems, it is easy to identify the DP recursion (8) by setting $r \leftarrow r_{N-s+1}$, $\ell \leftarrow \ell_{N-s+1}$, $V \leftarrow V_{s-1}^0$, $V^0 \leftarrow V_s^0$, $J \leftarrow J_s$, $X \leftarrow X_s$, $Z \leftarrow Z_s$, $\Omega \leftarrow X_{s-1}$, and by identifying H, G, F, E, h with $H_{N-s+1}, G_{N-s+1}, F_{N-s+1}, E_{N-s+1}, h_{N-s+1}$, respectively.

Clearly, we can now proceed to prove, via induction, that a certain set of properties is possessed by each element in the sequence $\{V_s^0(\cdot), \kappa_s(\cdot), X_s\}_{s=1}^N$ by showing that if $\{V(\cdot), \Omega\}$ has a given set of properties, then $\{V^0(\cdot), X\}$ also has these properties, with the properties of $\kappa(\cdot)$ being the same as those of each of the elements in the sequence $\{\kappa_s(\cdot)\}_{s=1}^N$. In the sequel, constructive proofs of the main results are presented, so that the reader can develop a prototype algorithm for computing the sequence $\{V_s^0(\cdot), \kappa_s(\cdot), X_s\}_{s=1}^N$.

A. Properties of X

The first result states that some properties of max-plus-scaling functions are preserved under addition, composition, and multiplication with a non-negative scalar.

Lemma 1: Suppose the functions g_1, g_2 and $g_3 = [g_{31}, \dots, g_{3n}]^T$ with g_1, g_2, g_{3j} of the form $g : Z \times W \rightarrow \mathbb{R} : (z, w) \mapsto g(z, w)$ have the property that for each $w \in W$, $g_i(\cdot, w), g_{3j}(\cdot, w) \in \mathcal{F}_{\text{mps}}^+$ and for each $z \in Z$, $g_i(z, \cdot), g_{3j}(z, \cdot) \in \mathcal{F}_{\text{mps}}$, for all i, j . Then, for any scalar $\lambda \geq 0$, $(\lambda g_1)(\cdot, w), (g_1 + g_2)(\cdot, w), g_1(g_3(\cdot, w), w) \in \mathcal{F}_{\text{mps}}^+$ for any fixed $w \in W$, and $(\lambda g_1)(z, \cdot), (g_1 + g_2)(z, \cdot), g_1(g_3(z, \cdot), \cdot) \in \mathcal{F}_{\text{mps}}$ for any fixed $z \in Z$.

Proof: This follows from the fact that $\lambda \max\{a, b\} = \max\{\lambda a, \lambda b\}$ if $\lambda \geq 0$, $\max\{a, b\} + \max\{c, d\} = \max\{a + c, a + d, b + c, b + d\}$, and $\max\{\max\{a, b\}, c\} = \max\{a, b, c\}$. ■

Lemma 2: The set $\mathcal{Z} = \{(x, u) : \bar{H}f(x, u, w) + \bar{G}u + \bar{E}w \leq \bar{h}, \forall w \in W\}$ with $\bar{H} \geq 0$ (cf. Assumption **A1**), can be written equivalently as $\mathcal{Z} = \{(x, u) : \tilde{H}x + \tilde{G}u \leq \tilde{h}\}$ with $\tilde{H} \geq 0$.

Proof: Since $\bar{H} \geq 0$ and $f(\cdot, w) \in (\mathcal{F}_{\text{mps}}^+)^n$ for each w , the function $x \mapsto \bar{H}f(x, u, w)$ is in $(\mathcal{F}_{\text{mps}}^+)^n$ for any (u, w) by Lemma 1. Recall that $\{z : \max_{j \in \underline{l}} \{\varphi_j(z)\} \leq \alpha\} = \{z : \varphi_j(z) \leq \alpha, \forall j \in \underline{l}\}$. Then, \mathcal{Z} has the equivalent representation $\mathcal{Z} = \{(x, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w \leq h, \forall w \in W\}$, where $\tilde{H} \geq 0$. Define $f_j^* := \max_{w \in W} \{\tilde{F}_j w\}$, where \tilde{F}_j denotes the j^{th} row of \tilde{F} . Then the result follows by letting $\tilde{h} := h - f^*$, where $f^* := (f_1^*, f_2^*, \dots)$. Note that f^* can be computed by solving a set of convex optimization problems (recall that W is a compact, convex set). ■

Lemma 3: Let $\mathcal{Z} = \{(x, r, t, u) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m : \bar{H}x + \bar{F}r + \bar{K}t + \bar{G}u \leq \bar{h}\}$ be given, where $\bar{H} \geq 0$ and $\bar{K} \leq 0$. The set $\mathcal{X} := \{(x, r, t) : \exists u \text{ s.t. } (x, r, t, u) \in \mathcal{Z}\}$ is a polyhedral set of the form $\mathcal{X} = \{(x, r, t) : \tilde{H}x + \tilde{F}r + \tilde{K}t \leq \tilde{h}\}$, where $\tilde{H} \geq 0$ and $\tilde{K} \leq 0$.

Proof: See Appendix B. ■

Proposition 1: Suppose Ω is a polyhedral set given by $\Omega = \{(x, \mathbf{r}) : \Gamma x + \Phi \mathbf{r} \leq \gamma\}$ with $\Gamma \geq 0$, and assume that H in (10a) satisfies $H \geq 0$. Then, the set X defined in (10b) is a polyhedron given by $X = \{(x, \mathbf{r}) : \hat{H}x + \hat{F}\mathbf{r} \leq \hat{h}\}$, where $\hat{H} \geq 0$.

Proof: The set Z is described as follows:

$$Z = \left\{ (x, \mathbf{r}, u) : \begin{bmatrix} H \\ \Gamma \end{bmatrix} f(x, u, w) + \begin{bmatrix} G \\ 0 \end{bmatrix} u + \begin{bmatrix} Fr \\ \Phi \mathbf{r} \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} w \leq \begin{bmatrix} h \\ \gamma \end{bmatrix}, \forall w \in W \right\}, \quad (12)$$

with $H, \Gamma \geq 0$. From Lemma 2 it follows that Z can be written as $Z = \{(x, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \leq \tilde{h}\}$ where $\tilde{H} \geq 0$. The result follows by applying a particular case of Lemma 3. ■

The reason for introducing Assumption **A1** is now obvious, since $H \geq 0$ is crucial in the proof of Proposition 1; it would not be possible to convert the expression for Z into a set of linear inequalities if some components of H were negative.

B. Properties of $\mathbb{P}_{\max}(x, \mathbf{r}, u)$

Proposition 2: Suppose $\ell(\cdot)$ satisfies Assumption **A2**. If, in addition $V(\cdot, \mathbf{r}) \in \mathcal{F}_{\text{mps}}^+$ for any fixed \mathbf{r} and $V(\cdot) \in \mathcal{F}_{\text{mps}}$, then $J(\cdot, \mathbf{r}, u) \in \mathcal{F}_{\text{mps}}^+$ for any fixed (\mathbf{r}, u) and $J(\cdot) \in \mathcal{F}_{\text{mps}}$.

Proof: Since $\ell(\cdot), V(\cdot) \in \mathcal{F}_{\text{mps}}$ and $\ell(\cdot, u, r, w), V(\cdot, \mathbf{r}) \in \mathcal{F}_{\text{mps}}^+$ (according to Assumption **A2**), it follows from Lemma 1 that we can write $\ell(f(x, u, w), u, r, w) + V(f(x, u, w), \mathbf{r}) = \max_{j \in \underline{l}} \{\alpha_j^T x + \beta_j^T w + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}$, where $\alpha_j \geq 0$ for all $j \in \underline{l}$, so that

$$J(x, \mathbf{r}, u) = \max_{w \in W} \{ \max_{j \in \underline{l}} \{ \alpha_j^T x + \beta_j^T w + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j \} \} = \max_{j \in \underline{l}} \{ \alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \},$$

where $\theta_j := \tilde{\theta}_j + \max_{w \in W} \{ \beta_j^T w \}$ for all $j \in \underline{l}$. Note that $\{\theta_j\}_{j \in \underline{l}}$ can be computed by solving a sequence of convex optimization problems. \blacksquare

Note that if Assumption **A2** would not hold, then one cannot guarantee that the cost function will be a max expression of affine terms in $(x, \mathbf{r}, u, \mathbf{w})$ such that the vectors that multiply the state (i.e. α_j) are non-negative and thus $J(\cdot)$ would not be a max expression with the vectors that multiply the state being non-negative, a property which will be crucial in the next section.

C. Properties of $\mathbb{P}_{\min}(x, \mathbf{r})$

Lemma 4: Suppose Ω is a polyhedral set given by $\Omega = \{(x, \mathbf{r}) : \Gamma x + \Phi \mathbf{r} \leq \gamma\}$ with $\Gamma \geq 0$, and assume that H in (10a) satisfies $H \geq 0$. Suppose $Z \neq \emptyset$ and $J(\cdot) \in \mathcal{F}_{\text{mps}}$. Then, there exists a $(\bar{x}, \bar{\mathbf{r}}) \in X$ such that $V^0(\bar{x}, \bar{\mathbf{r}})$ is finite if and only if $V^0(x, \mathbf{r})$ is finite for all $(x, \mathbf{r}) \in X$.

Proof: From the proof of Proposition 1 it follows that Z is a non-empty polyhedron: $Z = \{(x, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \leq \tilde{h}\}$ with $\tilde{H} \geq 0$. Since $J(\cdot) \in \mathcal{F}_{\text{mps}}$, we can write $J(x, \mathbf{r}, u) = \max_{j \in \underline{l}} \{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$. The prototype minimization problem $\mathbb{P}_{\min}(x, \mathbf{r})$ becomes:

$$\begin{aligned} V^0(x, \mathbf{r}) &= \min_u \{ \max_{j \in \underline{l}} \{ \alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \} : (x, \mathbf{r}, u) \in Z \} \\ &= \min_{(\nu, u)} \{ \nu : \alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \leq \nu, \forall j \in \underline{l}, \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \leq \tilde{h} \}, \end{aligned} \quad (13)$$

i.e. we have obtained a feasible linear program (LP) for any fixed $(x, \mathbf{r}) \in X$. But the feasible set of the dual of (13) does not depend on x or \mathbf{r} . Assume that $V^0(\bar{x}, \bar{\mathbf{r}})$ is finite for some $(\bar{x}, \bar{\mathbf{r}}) \in X$. From strong duality for linear programs [10] it follows that the dual problem of (13) is feasible, and independent of x and \mathbf{r} . Using again strong duality, we conclude that

$V^0(x, \mathbf{r})$ is finite if $(x, \mathbf{r}) \in X$ and $V^0(x, \mathbf{r}) = +\infty$ if $(x, \mathbf{r}) \notin X$. The reverse implication is obvious. ■

Proposition 3: Suppose Ω is a polyhedral set given by $\Omega = \{(x, \mathbf{r}) : \Gamma x + \Phi \mathbf{r} \leq \gamma\}$ with $\Gamma \geq 0$, and assume that H in (10a) satisfies $H \geq 0$. Suppose also that $Z \neq \emptyset$, $J(\cdot) \in \mathcal{F}_{\text{mps}}$, and $V^0(\cdot)$ is proper. Then, the value function $V^0(\cdot)$ is in \mathcal{F}_{mps} and has domain X , where X is a polyhedral set. The (set-valued) control law $\kappa(x, \mathbf{r})$ is a polyhedron for a given $(x, \mathbf{r}) \in X$. Moreover, it is always possible to select a continuous and PWA control law $\nu(\cdot)$ such that $\nu(x, \mathbf{r}) \in \kappa(x, \mathbf{r})$ for all $(x, \mathbf{r}) \in X$.

Proof: It follows from (13) that $\mathbb{P}_{\min}(x, \mathbf{r})$ is a parametric LP of the type $\min_z \{c^T z : \bar{H}\phi + \bar{G}z \leq \bar{h}\}$, where the vector of parameters is ϕ and the optimization variable is z (in our case, from (13) we conclude that $\phi = [x^T \ \mathbf{r}^T]^T$ and $z = [\nu \ u^T]^T$). The properties stated above then follow from the properties of a parametric LP (see [14]–[16]). In particular, since the value function of a parametric LP is a convex PWA function [14] and using the equivalence between convex PWA functions and max-plus-scaling functions [10], it follows that $V^0(\cdot) \in \mathcal{F}_{\text{mps}}$. ■

Theorem 1: Suppose that the same assumptions as in Proposition 3 hold. If, in addition, $J(\cdot, \mathbf{r}, u) \in \mathcal{F}_{\text{mps}}^+$ for any (\mathbf{r}, u) , then the value function $V^0(\cdot, \mathbf{r}) \in \mathcal{F}_{\text{mps}}^+$ for each fixed \mathbf{r} .

Proof: See Appendix B. ■

D. Main result

Based on the invariance properties of the two prototype problems \mathbb{P}_{\max} and \mathbb{P}_{\min} , we can now derive the properties of $V_s^0(\cdot)$, $\kappa_s(\cdot)$ and X_s for all $s \in \underline{N}$. The main result of this paper follows by applying Propositions 1–3 and Theorem 1 to the DP equations (8):

Theorem 2: Suppose that Assumptions **A1** and **A2** (see Section I-B) hold, Z_s is non-empty and $V_s^0(\cdot)$ is proper for all $s \in \underline{N}$. Then, $V_s^0(\cdot)$ is a max-plus-scaling function having the non-empty polyhedral domain X_s . Furthermore, there exists a continuous PWA function $\mu_{N-s+1}^0(\cdot)$ such that $\mu_{N-s+1}^0(x, \mathbf{r}) \in \kappa_s(x, \mathbf{r})$ for all $(x, \mathbf{r}) \in X_s$.

Since the proofs of all the above results are constructive, the sequences $\{V_s^0(\cdot), \kappa_s(\cdot), X_s\}_{s=1}^N$ and $\{\mu_s^0(\cdot)\}_{s=1}^N$ can be computed iteratively, without gridding, by noting the following:

- Given X_{s-1} , compute X_s by first computing Z_s , as in the proof of Proposition 1, followed by a projection operation,
- Given $V_{s-1}^0(\cdot)$, a max-plus-scaling expression of $J_s(\cdot)$ can be computed by referring to the proof of Proposition 2,

TABLE I
NUMBER OF REGIONS n_R AS A FUNCTION OF THE PREDICTION HORIZON N .

N	1	2	3	4	5	6	7	8	9	10
n_R	2	7	7	10	13	15	19	23	25	25

- Given $J_s(\cdot)$ and Z_s , compute $V_s^0(\cdot)$, $\kappa_s(\cdot)$ and $\mu_{N-s+1}^0(\cdot)$ as in the proof of Proposition 3 or Theorem 1, by using parametric LP algorithms [14]–[16].

III. EXAMPLE

We consider the following example:

$$x(k) = \begin{bmatrix} -w_1(k) + w_2(k) + 2 & \varepsilon \\ -w_1(k) - w_2(k) + 5 & w_1(k) - 2 \end{bmatrix} \otimes x(k-1) \oplus \begin{bmatrix} -w_1(k) + 3 \\ -w_2(k) + 2 \end{bmatrix} \otimes u(k), \quad y(k) = [0 \ \varepsilon] \otimes x(k)$$

We assume a bounded disturbance: $W = \{[w_1 \ w_2]^T : 2 \leq w_1 \leq 3, 1 \leq w_2 \leq 2, w_1 + w_2 \leq 4\}$.

We consider $N = 10$, the due date signal is $\mathbf{r} = [5 \ 7 \ 9.5 \ 11.8 \ 14 \ 16.7 \ 19.4 \ 21.6 \ 23.8 \ 26]^T$, and the initial conditions are $x(0) = [6 \ 8]^T$, $u(0) = 7$. The system is subject to input-output constraints: $x_2(k) - u(k) \leq 2$, $x_1(N) + x_2(N) \leq 2r_N$, $u(k+1) - u(k) \geq 0$, $-6 + r_k \leq u(k) \leq 6 + r_k$.

We use the stage cost defined in (2) with $\gamma = 0.1$, and a random sequence of disturbances:

$$\mathbf{w} = \begin{bmatrix} 2.4 & 2.1 & 2 & 2.8 & 2.1 & 2.1 & 2.2 & 2 & 2.4 & 2.2 \\ 1 & 1.4 & 1.3 & 1 & 1.2 & 1.6 & 1.1 & 1.7 & 1.4 & 1.6 \end{bmatrix}^T. \quad \text{Since } \mathbf{r} \text{ is known in advance, in}$$

this example we do not consider it as a parameter in the parametric LPs. Algorithms for solving parametric LPs can be found in [14]–[16].

The left plots in Figure 1 show the output y and the signal $y - r$ for the feedback optimal control approach presented in this paper, the open-loop optimal control approach as in [4], and the residuation approach of [3]. As expected, the feedback optimal control approach gives a better tracking than the open-loop and residuation approach (note that $V_N^0(x, \mathbf{r}) = -13.8 \ll V_N^{0, \text{open-loop}}(x, \mathbf{r}) = -3.02$). The right plots in Figure 1 show the input u and the 'normalized' signal $u - 2k$ (note that 2 is the constant slope of the residuation input signal) for the three approaches. Note that the optimal open-loop input sequence coincides with the residuation input sequence, except for the last sample. Table I displays the number of regions n_R of the parametric LP as a function of the prediction horizon N . Note that the number of regions increases with the prediction horizon.

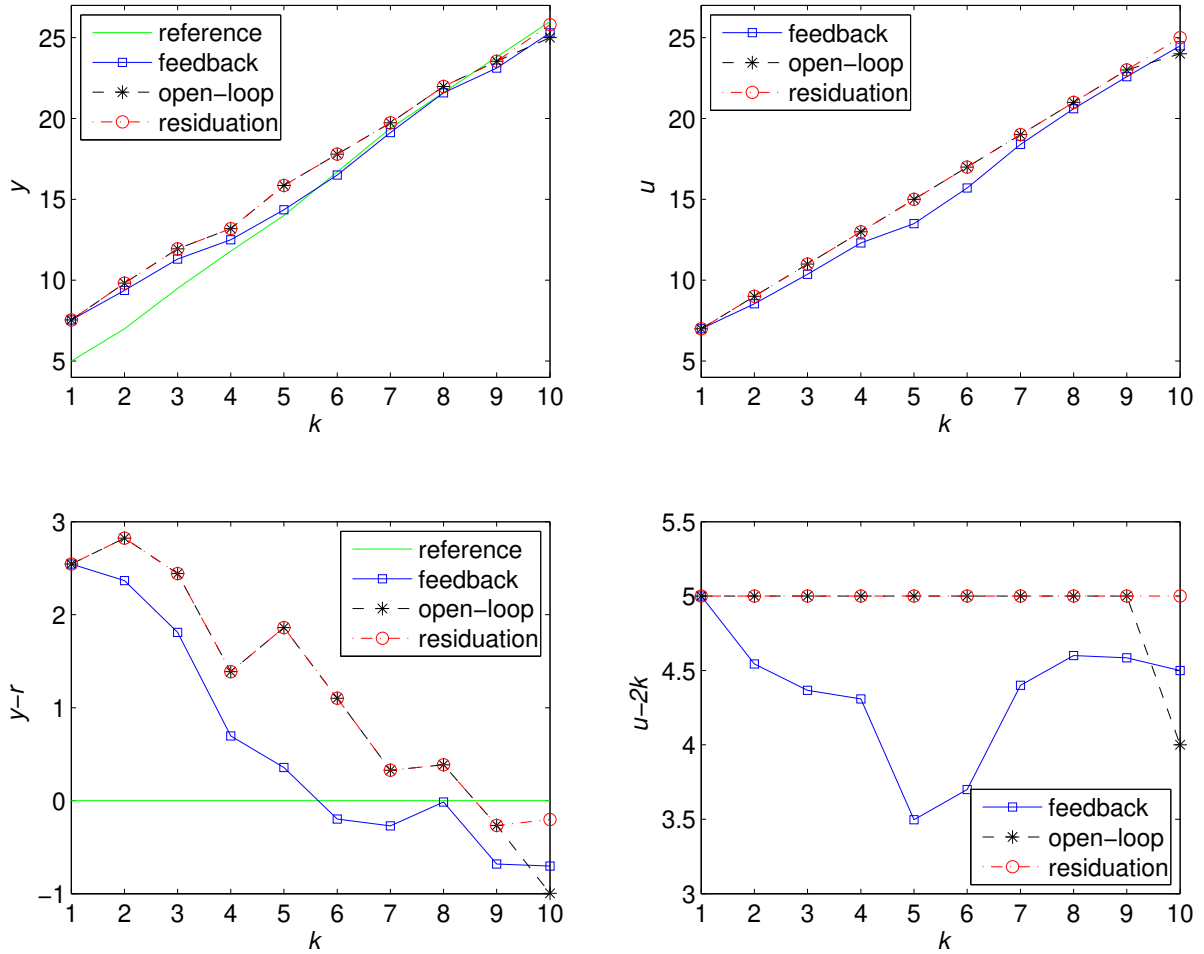


Fig. 1. Left: y and $y - r$ for the feedback approach (full-squares), the open-loop approach of [4] (dashed-stars), and the residuation approach of [3] (dash-dotted-circles); the reference signal r is indicated by the full line. Right: u and $u - 2k$.

IV. CONCLUSIONS

We have shown that we can compute an optimal control policy over a prediction horizon of N steps by solving N parametric LP problems. The key assumptions that allow us to guarantee convexity of the partial return functions and their domains at each dynamic programming iteration, were that the stage cost be a max-plus-non-negative-scaling expression in the state *and* that the matrices associated with the state constraints all have non-negative entries.

Future research topics include: in-depth investigation of the timing issues, extension of the results to min-plus-linear systems, and relaxations for robust optimal MPL control problems.

APPENDIX A

The $m \times n$ MPA zero matrix $\mathcal{E}_{m \times n}$ and the $n \times n$ MPA identity matrix E_n are defined as follows: $[\mathcal{E}_{m \times n}]_{ij} := \varepsilon$ for all i, j , and $[E_n]_{ii} := 0$ for all i and $[E_n]_{ij} := \varepsilon$ for all $i \neq j$. The constraint $u(k+1) - u(k) \geq 0$ does not satisfy Assumption **A1**. However, by remodeling the system, this constraint can be removed. Indeed, introducing a new state vector $\tilde{x} = [x^T, \tilde{u}^T]^T$ and the dynamics:

$$\tilde{x}(k) = \begin{bmatrix} A(w(k)) & B(w(k)) \\ \mathcal{E}_{m \times n} & E_m \end{bmatrix} \otimes \tilde{x}(k-1) \oplus \begin{bmatrix} B(w(k)) \\ E_m \end{bmatrix} \otimes u(k), \quad y(k) = [C(w(k)) \ \mathcal{E}_{p \times m}] \otimes \tilde{x}(k)$$

with the constraint $\tilde{u}(k) \leq u(k)$, it is clear that both systems have the same behavior. Note that the previous constraint can be written equivalently as $[0 \ I_m] \tilde{x}(k) - I_m u(k) \leq 0$. Moreover, if the original system is subject to constraints of the form $Hx(k) + Gu(k) + Fr(k) + Ew(k) \leq h(k)$, then they can be written equivalently as $[H \ 0] \tilde{x}(k) + Gu(k) + Fr(k) + Ew(k) \leq h(k)$. So, Assumption **A1** still holds after performing this remodeling since $[H \ 0] \geq 0$ if $H \geq 0$ and $[0 \ I_m] \geq 0$.

APPENDIX B

Proof of Lemma 3: $\mathcal{X} = \text{Proj}_{n+p+q} \mathcal{Z}$ is clearly a polyhedron. Note that it is sufficient to consider the case $m = 1$; the general case can be proved in a similar fashion, using induction. We derive the properties of \tilde{H} , \tilde{F} and \tilde{K} using Fourier-Motzkin elimination. Let $\bar{H}_i, \bar{F}_i, \bar{K}_i$ and \bar{G}_i be the i^{th} row of respectively the matrices \bar{H} , \bar{F} , \bar{K} and the vector \bar{G} . Define $I_+ := \{i \in \underline{q} : \bar{G}_i > 0\}$, $I_- := \{i \in \underline{q} : \bar{G}_i < 0\}$ and $I_0 := \{i \in \underline{q} : \bar{G}_i = 0\}$. We have three cases:

- 1) $i \in I_0 \Rightarrow \bar{H}_i x + \bar{F}_i r + \bar{K}_i t \leq h_i$ and $\bar{H}_i \geq 0, \bar{K}_i \leq 0$;
- 2) $j \in I_+ \Rightarrow u \leq -\frac{1}{\bar{G}_j} \bar{H}_j x - \frac{1}{\bar{G}_j} \bar{F}_j r - \frac{1}{\bar{G}_j} \bar{K}_j t + \frac{\bar{h}_j}{\bar{G}_j}$ and $\frac{1}{\bar{G}_j} \bar{H}_j \geq 0, \frac{1}{\bar{G}_j} \bar{K}_j \leq 0$;
- 3) $l \in I_- \Rightarrow u \geq -\frac{1}{\bar{G}_l} \bar{H}_l x - \frac{1}{\bar{G}_l} \bar{F}_l r - \frac{1}{\bar{G}_l} \bar{K}_l t + \frac{\bar{h}_l}{\bar{G}_l}$ and $-\frac{1}{\bar{G}_l} \bar{H}_l \geq 0, -\frac{1}{\bar{G}_l} \bar{K}_l \leq 0$.

It is then easy to combine the above and show that the set \mathcal{X} is described by the following inequalities: $\bar{H}_i x + \bar{F}_i r + \bar{K}_i t \leq \bar{h}_i \ \forall i \in I_0$; $(-\frac{1}{\bar{G}_l} \bar{H}_l + \frac{1}{\bar{G}_j} \bar{H}_j) x + (-\frac{1}{\bar{G}_l} \bar{F}_l + \frac{1}{\bar{G}_j} \bar{F}_j) r + (-\frac{1}{\bar{G}_l} \bar{K}_l + \frac{1}{\bar{G}_j} \bar{K}_j) t \leq -\frac{\bar{h}_l}{\bar{G}_l} + \frac{\bar{h}_j}{\bar{G}_j} \ \forall j \in I_+, l \in I_-$. The result follows, since the rows of \tilde{H} are composed of the vectors $\bar{H}_i \geq 0$ and $-\frac{1}{\bar{G}_l} \bar{H}_l + \frac{1}{\bar{G}_j} \bar{H}_j \geq 0$ for all $i \in I_0, j \in I_+, l \in I_-$, while the rows of \tilde{K} are composed of the vectors $\bar{K}_i \leq 0$ and $-\frac{1}{\bar{G}_l} \bar{K}_l + \frac{1}{\bar{G}_j} \bar{K}_j \leq 0$ for all $i \in I_0, j \in I_+, l \in I_-$. ■

Proof of Theorem 1: Using Proposition 1 it follows that $Z = \{(x, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \leq \tilde{h}\}$, with $\tilde{H} \geq 0$. The function $J(\cdot)$ can be written as: $J(x, \mathbf{r}, u) = \max_{j \in \underline{l}} \{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$, where $\alpha_j \geq 0$ for all j . From Proposition 3 and the fact that $V^0(\cdot)$ is proper, it follows that $V^0(\cdot) \in \mathcal{F}_{\text{mps}}$ and its domain is X . The epigraph of $V^0(\cdot)$ is given by: $\text{epi } V^0 := \{(x, \mathbf{r}, t) : V^0(x, \mathbf{r}) \leq t, x \in X\} = \{(x, \mathbf{r}, t) : \exists u \text{ s.t. } (x, \mathbf{r}, u) \in Z, J(x, \mathbf{r}, u) \leq t\}$. So $\text{epi } V^0 = \{(x, \mathbf{r}, t) : \exists u \text{ s.t. } \bar{H}x + \bar{F}\mathbf{r} + \bar{K}t + \bar{G}u \leq \bar{h}\}$, where $\bar{H} = [\tilde{H}^T \ \alpha_1^T \ \dots \ \alpha_l^T]^T \geq 0$ and $\bar{K} = [0, -1, \dots, -1]^T \leq 0$. From Lemma 3 we obtain that the epigraph of $V^0(\cdot)$ is a polyhedron given by $\text{epi } V^0 = \{(x, \mathbf{r}, t) : \hat{H}x + \hat{F}\mathbf{r} + \hat{K}t \leq \hat{h}\}$, where $\hat{H} \geq 0, \hat{K} \leq 0$. Let l be the number of inequalities describing $\text{epi } V^0$. We arrange the indices $j \in \underline{l}$ s. t. $\hat{K}_j < 0$ for $j = 1, \dots, v$ but $\hat{K}_j = 0$ for $j = v + 1, v + 2, \dots, l$ (possibly $v = 0$, i.e. $\hat{K}_j = 0$ for all j). Here, D_j denotes the j^{th} row of a matrix D . Taking $a_j = -\hat{H}_j / \hat{K}_j, b_j = -\hat{F}_j / \hat{K}_j$ and $c_j = -\hat{h}_j / \hat{K}_j$ for $j = 1, 2, \dots, v$, we obtain

$$\text{epi } V^0 = \{(x, \mathbf{r}, t) : a_j x + b_j \mathbf{r} - c_j \leq t, \forall j \in \underline{v}; \hat{H}_j x + \hat{F}_j \mathbf{r} \leq \hat{h}_j \text{ for } j = v + 1, \dots, l\} \quad (14)$$

But $V^0(\cdot)$ is proper, therefore $v > 0$. Since $V^0(\cdot) \in \mathcal{F}_{\text{mps}}$, (14) gives us a representation of $V^0(\cdot)$ as $V^0(x, \mathbf{r}) = \max_{j \in \underline{v}} \{a_j x + b_j \mathbf{r} - c_j\}$, where $a_j = -\hat{H}_j / \hat{K}_j \geq 0$, for all $j \in \underline{v}$, i.e. $V^0(\cdot, \mathbf{r}) \in \mathcal{F}_{\text{mps}}^+$ for any fixed $\mathbf{r} \in \mathbb{R}^p$. Moreover, the domain of $V^0(\cdot)$ is $\{(x, \mathbf{r}) : \hat{H}_j x + \hat{F}_j \mathbf{r} \leq \hat{h}_j \text{ for } j = v + 1, \dots, l\}$ and coincides with X according to Proposition 3. ■

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