

Technical report 06-006

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I. Necoara, T.J.J. van den Boom, B. De Schutter, and H. Hellendoorn, "Stabilization of max-plus-linear systems using model predictive control: The unconstrained case," *Automatica*, vol. 44, no. 4, pp. 971–981, Apr. 2008.

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.51.19 (secretary)
fax: +31-15-278.66.79
URL: <http://www.dcsc.tudelft.nl>

*This report can also be downloaded via http://pub.deschutter.info/abs/06_006.html

Stabilization of max-plus-linear systems using model predictive control: The unconstrained case [★]

Ion Necoara ^a, Ton J.J. van den Boom ^a, Bart De Schutter ^a, Hans Hellendoorn ^a

^a*Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands*

Abstract

Max-plus-linear (MPL) systems are a class of event-driven nonlinear dynamic systems that can be described by models that are “linear” in the max-plus algebra. In this paper we derive a solution to a finite-horizon model predictive control (MPC) problem for MPL systems where the cost is designed to provide a trade-off between minimizing the due date error and a just-in-time production. In general, MPC can deal with complex input and states constraints. However, in this paper we assume that these are not present and it is only required that the input should be a nondecreasing sequence, i.e. we consider the “unconstrained” case. Despite the fact that the controlled system is nonlinear, by employing recent results in max-plus theory we are able to provide sufficient conditions such that the MPC controller is determined analytically and moreover the stability in terms of Lyapunov and in terms of boundedness of the closed-loop system is guaranteed a priori.

Key words: Discrete-event systems, max-plus-linear systems, model predictive control, Lyapunov stability, bounded buffer stability.

1 Introduction

Discrete-event systems (DES) are event-driven dynamical systems that often arise in the context of manufacturing systems, telecommunication networks, railway networks, parallel computing, etc. In the last decades there has been an increasing amount of research on DES that can be modeled as max-plus-linear (MPL) systems. Most of the earlier literature on this class of systems addresses performance analysis [1, 4, 5, 9, 11] rather than control. Although there are papers on optimal control for MPL systems (see e.g. [6, 8, 13, 16, 19, 20]), the literature on stabilizing controllers for MPL systems is relatively sparse. In [6, 13, 16] optimal controllers for MPL systems were derived. The main differences between our approach and the approaches of [6, 13, 16] are that these papers use an input-output model instead of a state-space model, that they do not use a receding horizon approach, that stability is not guaranteed a priori in those papers, except [13], and the optimal input sequence is sometimes decreasing (i.e. physically infeasible), unless this sequence is projected on the set of nondecreasing input sequences, similarly as we do in this paper. In [8, 19] stability is not guaranteed a priori, and in [20] stability is only defined in terms of boundedness. Lyapunov stability is not

discussed in [20], it is not proved that stability in terms of boundedness implies bounded buffer levels, and moreover the imposed conditions are somewhat restrictive (in particular irreducibility of the system matrix A is required and the initial states must satisfy a certain inequality). We obtain an explicit expression for the model predictive controller (in fact the current paper extends our results from the conference paper [20] regarding explicit model predictive control (MPC)). But in [8, 19] an optimization problem must be solved in order to obtain the MPC input at each event step.

Model predictive control (MPC) is an attractive feedback strategy for linear or nonlinear processes [12, 15]: it is an easy-to-tune method, it is applicable to multi-variable systems, it can handle constraints, and it is capable of tracking pre-scheduled reference signals. The essence of MPC is to determine a control profile that optimizes a cost criterion over a prediction window and then to apply this control profile until new process measurements become available. Feedback is incorporated by using these measurements to update the optimization problem for the next step.

This paper considers the problem of designing a stabilizing MPC controller for the class of MPL systems where the cost is chosen such that it provides a trade-off between minimizing the due date error and a just-in-time production. We define two notions of stability for MPL systems: Lyapunov stability and stability in terms of boundedness. Although in general the MPC framework allows us to deal with state and input constraints (see Remark 2), in this paper we con-

[★] This paper was not presented at any IFAC meeting. Corresponding author I. Necoara. Tel. +31-015-2787171. Fax +31-015-2787171.

Email address: ion.necoara@esat.kuleuven.be (Ion Necoara).

sider an unconstrained formulation of the MPC, where the only constraint that we take into account is that the input is required to be a monotone nondecreasing sequence (since, in general, the input represents consecutive time instants). The main advantage of this paper compared to most of the results on optimal control and MPC for MPL systems is the fact that we guarantee *a priori* stability of the closed-loop system both in the sense of Lyapunov and in terms of boundedness, that the resulting closed-loop signals are non-decreasing (i.e. physically feasible), and that the MPC law is computed explicitly.

One of the key results of this paper is to provide sufficient conditions such that one can employ results from max-plus algebra to compute an explicit MPC controller for MPL systems that guarantees a priori closed-loop stability in terms of Lyapunov and in terms of boundedness. The usual approach for proving stability of the MPC is to use a terminal cost and a terminal set such that the optimal cost becomes a Lyapunov function [15]. In this paper we do not follow this classical proof for stability, but rather taking advantage of the special MPL system properties, particularly monotonicity, we show that by a proper tuning of the design parameters stability can still be guaranteed and convergence can be achieved even in a finite number of event steps.

This section proceeds with an introduction to MPL systems and the formulation of the control problem that we are going to solve in this paper. In Section 2 we derive two controllers together with their main properties, in particular stability of the controlled systems. In Section 3, taking into account that the input should be nondecreasing, we design a stabilizing MPC controller which turns out to be also a just-in-time controller. Using results from max-plus algebra, in particular the monotonicity property of the max and plus operators, we derive sufficient conditions such that the resulting MPC controller lies in between the two controllers derived in Section 2 and it can be determined explicitly. Moreover, the closed-loop MPC is stable, and convergence of the closed-loop state trajectory is even achieved in a finite number of event steps. We conclude with an example in Section 4.

1.1 Max-Plus Algebra

Define $\varepsilon := -\infty$ and $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic (MPA) addition (\oplus) and multiplication (\otimes) are defined as [1, 7]: $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$, for $x, y \in \mathbb{R}_\varepsilon$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$ one can extend the definition as follows: $(A \oplus B)_{ij} := A_{ij} \oplus B_{ij}$, $(A \otimes C)_{ij} := \bigoplus_{k=1}^n A_{ik} \otimes C_{kj}$ for all i, j . The matrix \mathcal{E} denotes the MPA zero matrix of appropriate dimension: $\mathcal{E}_{ij} := \varepsilon$ for all i, j . The matrix E is the MPA identity matrix of appropriate dimension: $E_{ii} := 0$ for all i and $E_{ij} := \varepsilon$ for all i, j with $i \neq j$. For any matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ let A^* be defined, whenever it exists, by $A^* := \lim_{k \rightarrow \infty} E \oplus A \oplus \dots \oplus A^{\otimes k}$. For a positive integer n , we denote with $\underline{n} := \{1, 2, \dots, n\}$. A matrix $\Gamma \in \mathbb{R}_\varepsilon^{n \times m}$ is *row-finite* if for any row $i \in \underline{n}$, $\max_{j \in \underline{m}} \Gamma_{ij} > \varepsilon$; *column-finite* is similarly defined. For

$A \in \mathbb{R}_\varepsilon^{m \times n}$ and $\rho \in \mathbb{R}$ the notation $A - \rho$ denotes a new matrix in $\mathbb{R}_\varepsilon^{m \times n}$ defined as $(A - \rho)_{ij} = A_{ij} - \rho$ for all i, j . We denote with $x \oplus' y := \min\{x, y\}$ and $x \otimes' y := x + y$ (the operations \otimes and \otimes' differ only in that $(-\infty) \otimes (+\infty) := -\infty$, while $(-\infty) \otimes' (+\infty) := +\infty$). The matrix multiplication and addition for (\oplus', \otimes') are defined similarly as for (\oplus, \otimes) . It can be shown that the following relations hold for any matrices A, B and any vectors x, y of appropriate dimensions over \mathbb{R}_ε (see [3, Section 1]):

$$\begin{aligned} A \otimes' (B \otimes x) &\geq (A \otimes' B) \otimes x, \quad ((-A^T) \otimes' A) \otimes x \geq x & (1a) \\ x \leq y &\Rightarrow A \otimes x \leq A \otimes y \quad \text{and} \quad A \otimes' x \leq A \otimes' y, & (1b) \end{aligned}$$

where “ \leq ” denotes the partial order defined by the nonnegative orthant. The rules for the order of evaluation of the max-plus-algebraic operators are similar to those of conventional algebra. So max-plus-algebraic power has the highest priority, and \otimes has a higher priority than \oplus .

Lemma 1 [1, 11] (i) *The inequality $A \otimes x \leq b$ has the largest¹ solution $x_{\text{opt}} = (-A^T) \otimes' b = -(A^T \otimes (-b))$.* (ii) *The equation $x = A \otimes x \oplus b$ has a solution $x = A^* \otimes b$ provided that A^* exists. If $A_{ij} < 0$ for all i, j , then the solution is unique.*

1.2 Max-Plus-Linear Systems

DES with only synchronization and no concurrency can be modeled by an MPA model of the following form [1, 7, 11]:

$$\begin{aligned} x_{\text{sys}}(k) &= A_{\text{sys}} \otimes x_{\text{sys}}(k-1) \oplus B_{\text{sys}} \otimes u_{\text{sys}}(k) & (2a) \\ y_{\text{sys}}(k) &= C_{\text{sys}} \otimes x_{\text{sys}}(k), & (2b) \end{aligned}$$

where $x_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^n$ represents the state, $u_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^m$ is the input, $y_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^p$ is the output and where $A_{\text{sys}} \in \mathbb{R}_\varepsilon^{n \times n}$, $B_{\text{sys}} \in \mathbb{R}_\varepsilon^{n \times m}$, $C_{\text{sys}} \in \mathbb{R}_\varepsilon^{p \times n}$ are the system matrices². In the context of DES k is an event counter while $u_{\text{sys}}, x_{\text{sys}}$ and y_{sys} are times (feeding times, processing times and finishing times, respectively). Note that the state denotes time, and thus it can be easily measured. Since the input represents time, a typical constraint that appears in the context of MPL systems is that the signal u_{sys} should be nondecreasing, i.e.

$$u_{\text{sys}}(k+1) - u_{\text{sys}}(k) \geq 0 \quad \forall k \geq 0. \quad (3)$$

Remark 2 In [8, 17] we have considered linear state and input constraints of the form $H_k x_{\text{sys}}(k) + G_k u_{\text{sys}}(k) + F_k r_{\text{sys}}(k) \leq h_k$, where $r_{\text{sys}}(k)$ is a reference signal as defined below. In this paper we consider the “unconstrained” case, i.e. only the constraints (3) are taken into account. \diamond

¹ By the largest solution we mean that for all x satisfying $A \otimes x \leq b$ we have $x_{\text{opt}} \geq x$.

² We may assume without loss of generality that B_{sys} is column-finite and C_{sys} is row-finite since otherwise the corresponding inputs and outputs can be eliminated from the description model.

Let λ_{\max} be the largest MPA eigenvalue of A_{sys} ($\lambda \in \mathbb{R}_{\varepsilon}$ is an MPA eigenvalue if there exists $v \in \mathbb{R}_{\varepsilon}^n$ with at least one finite entry such that $A_{\text{sys}} \otimes v = \lambda \otimes v$ [1]). We consider a reference signal (due dates) that the output should track:

$$r_{\text{sys}}(k) = y_{\text{sys},t} + k\rho, \quad (4)$$

where $y_{\text{sys},t} \in \mathbb{R}^p$ is a vector of offsets³. Note that we can consider a more general signal $r_{\text{sys}}(\cdot)$ such that there exists a finite K^r for which $r_{\text{sys}}(k) = y_{\text{sys},t} + k\rho$ for all $k \geq K^r$. The subsequent derivations remain the same.

Since through the term $B_{\text{sys}} \otimes u_{\text{sys}}$ it is only possible to create delays in the starting times of activities, we should choose the growth rate of the due dates such that is larger than the growth rate of the system, i.e. $\rho \geq \lambda_{\max}$. If $\lambda_{\max} > \varepsilon$ (in practical applications we ever have $\lambda_{\max} \geq 0$), then there exists an MPA invertible matrix $P \in \mathbb{R}_{\varepsilon}^{n \times n}$ such that the matrix $\bar{A} = P^{\otimes -1} \otimes A_{\text{sys}} \otimes P$ satisfies $\bar{A}_{ij} \leq \lambda_{\max}$ for all $i, j \in \underline{n}$ (see e.g. [14, Lemma 3]), where $P^{\otimes -1}$ denotes the inverse of the matrix P in the max-plus algebra, i.e. $P^{\otimes -1} \otimes P = P \otimes P^{\otimes -1} = E_n$ (see [7] for a method to compute $P^{\otimes -1}$).

In order to simplify the proofs in the sequel we make the following change of coordinates. First, we consider $\bar{x}(k) = P^{\otimes -1} \otimes x_{\text{sys}}(k)$. We denote with $\bar{B} = P^{\otimes -1} \otimes B_{\text{sys}}$, $\bar{C} = C_{\text{sys}} \otimes P$ and $\bar{y}(k) = y_{\text{sys}}(k)$, $\bar{u}(k) = u_{\text{sys}}(k)$. Rewriting (2a)–(2b) in the new coordinates, i.e. replacing $x_{\text{sys}}(k)$ with $P \otimes \bar{x}(k)$ we obtain the following equivalent system:

$$\bar{x}(k) = \bar{A} \otimes \bar{x}(k-1) \oplus \bar{B} \otimes \bar{u}(k), \quad \bar{y}(k) = \bar{C} \otimes \bar{x}(k).$$

We now consider the *normalized* system: $x(k) = \bar{x}(k) - \rho k$, $u(k) = \bar{u}(k) - \rho k$, $y(k) = \bar{y}(k) - \rho k$, $A = \bar{A} - \rho$ (recall that this means to subtract in the conventional algebra all entries of $\bar{x}, \bar{u}, \bar{y}$ and of \bar{A} by ρk and ρ , respectively) and $B = \bar{B}$, $C = \bar{C}$. Using the standard max-plus operations the normalized system can be written as:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \quad (5a)$$

$$y(k) = C \otimes x(k). \quad (5b)$$

As we will see in the sequel, the normalized system allows us to pose the notion of Lyapunov stability for MPL systems and it also simplifies the proofs. Therefore, in the sequel we consider only MPL systems in the form (5a)–(5b), where the matrix A satisfies $A_{ij} < 0$ for all i, j , provided that $\rho > \lambda_{\max}$ (since $A = \bar{A} - \rho$ and $\bar{A}_{ij} \leq \lambda_{\max}$ for all $i, j \in \underline{n}$, according to the definition of \bar{A}). For the normalized systems the variables x, u and y represent the delays (deviations) with respect to some nominal signals (see Appendix A for their definition) and thus we can consider the problem of obtaining asymptotic constant values of timers for this new system. Since we only make a change of coordinates and a

³ In practice, such a reference signal is often used as it corresponds to a regular and smooth due date signal with a constant output rate.

subtraction with ρk , it follows that if a control law μ is optimal for the normalized system, then $\mu + \rho k$ is optimal for the original system (see also the Appendix A). In the new coordinates, the constraint (3) becomes:

$$u(k+1) - u(k) \geq -\rho \quad \forall k \geq 0. \quad (6)$$

The MPL system (5a) is *controllable* if and only if (iff) each component of the state can be made arbitrarily large by applying an appropriate controller to the system initially at rest. It can be checked (see e.g. [10, Theorem 3.3]) that the system is controllable iff the matrix $\Gamma_n := [B \ A \otimes B \ \dots \ A^{\otimes n-1} \otimes B]$ is row-finite (this definition is equivalent to the one given in [1, 10], where the system is controllable if all states are connected to some input). Similarly, the system (5a)–(5b) is *observable* iff each state is connected to some output, i.e. the matrix $\Omega_n := [C^T \ (C \otimes A)^T \ \dots \ (C \otimes A^{\otimes n-1})^T]^T$ is column-finite (see e.g. [10, Theorem 3.10]). The following key assumption will be used throughout the paper:

Assumption A: We consider that $\rho > \lambda_{\max} \geq 0$ and the system is controllable and observable. \diamond

The conditions from Assumption A are quite weak and are usually met in applications. Note that ρ can be chosen arbitrarily close to λ_{\max} (see also the previous discussion). From Assumption A it follows that $A_{ij} < 0$ for all $i, j \in \underline{n}$. In the new coordinates the output should be regulated to the desired target $y_{\text{sys},t}$.

Since $A_{ij} < 0$ for all $i, j \in \underline{n}$, we have $A^* = E_n \oplus A \oplus \dots \oplus A^{\otimes n-1}$ [1, Theorem 3.20]. Note that for any finite vector u there exists a state equilibrium x (i.e. $x = A \otimes x \oplus B \otimes u$), given by $x = A^* \otimes B \otimes u$. Note that x is unique (see Lemma 1 (ii)) and finite (since Γ_n is row-finite). We associate to $y_{\text{sys},t}$ the largest⁴ equilibrium pair (x_e, u_e) satisfying $C \otimes x_e \leq y_{\text{sys},t}$. From the previous discussion and taking into account that Ω_n and B are column-finite and C is row-finite, it follows that (x_e, u_e) is unique, finite, and given by:

$$u_e = -(C \otimes A^* \otimes B)^T \otimes' y_{\text{sys},t}, \quad x_e = A^* \otimes B \otimes u_e. \quad (7)$$

Throughout the paper $\|\cdot\|$ represents the ∞ -norm (i.e. $\|x\| := \max_{i \in \underline{n}} |x_i|$). We consider a state feedback law (e.g. an MPC law) $\mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the closed-loop system:

$$x(k) = A \otimes x(k-1) \oplus B \otimes \mu(x(k-1)), \quad y(k) = C \otimes x(k). \quad (8)$$

Definition 3 (i) *The closed-loop system (8) is Lyapunov stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x(0) - x_e\| \leq \delta$ implies $\|x(k) - x_e\| \leq \varepsilon$ for all $k \geq 0$.*

(ii) *The closed-loop system (8) is stable in terms of boundedness if for any $\delta > 0$ there exists a $\theta > 0$ such that $\|x(0) - x_e\| \leq \delta$ implies $\|x(k) - x_e\| \leq \theta$ for all $k \geq 0$.*

(iii) *The closed-loop system (8) is finitely convergent if there*

⁴ By the largest we mean that any other feasible equilibrium pair (x, u) satisfies $x \leq x_e, u \leq u_e$.

exists a finite k_0 such that $x(k) = x_e$ for all $k \geq k_0$.
The closed-loop system is finitely Lyapunov stable if it is Lyapunov stable and finitely convergent.

In [1, 13, 18] stability for DES is defined in terms of boundedness of the buffer levels. In Appendix A we will prove that stability in terms of boundedness for the normalized system (Definition 3 (ii)) implies boundedness of the buffer levels for the original system (2a)–(2b). In this paper however, in addition to stability in terms of boundedness we also prove Lyapunov stability. We formulate now the control problem that we solve in the sequel:

Problem definition: Design a state feedback law $\mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for the MPL system (5a)–(5b) such that the closed-loop system is finitely Lyapunov stable and/or stable in terms of boundedness and the constraint (6) is satisfied. \diamond

2 Stabilizing controllers

2.1 Feedback and ultimately constant slope controller

We consider the normalized system (5a)–(5b), where $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$, $C \in \mathbb{R}_\varepsilon^{p \times n}$ and the matrix A satisfies $A_{ij} < 0$ for all $i, j \in \underline{n}$ (according to Assumption A), subject to the constraint (6). We define two types of control input signals:

- one that corresponds to a feedback controller

$$u^f(k) := (-B^T) \otimes' (A \otimes x(k-1) \oplus B \otimes (u(k-1) - \rho) \oplus x_e), \quad (9)$$

- an ultimately constant slope (UCS) control signal

$$u^c(k) := u_e \oplus (u(k-1) - \rho), \quad (10)$$

with $x(k-1)$ and $u(k-1)$ the previous state and input.

For the original system the UCS controller results in a control input that has a constant slope ρ for k large enough since $u_{\text{sys}}^c(k) = (u_e + \rho k) \oplus u_{\text{sys}}^c(k-1)$. Later on, we will show that under some conditions the MPC controller lies in between these two controllers. However, we do not need to compute them explicitly in order to obtain the MPC controller.

With the feedback controller (9) the normalized system (5a) becomes

$$x^f(k) = A \otimes x^f(k-1) \oplus B \otimes u^f(k) \quad (11a)$$

$$u^f(k) = (-B^T) \otimes' (A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e), \quad (11b)$$

for initial conditions $x^f(0) = x(0)$ and $u^f(0) = u(0)$. With the UCS controller (10) the normalized system becomes

$$x^c(k) = A \otimes x^c(k-1) \oplus B \otimes u^c(k) \quad (12a)$$

$$u^c(k) = u_e \oplus (u^c(k-1) - \rho), \quad (12b)$$

where the initial conditions $x^c(0) = x(0)$ and $u^c(0) = u(0)$ are given. Under these initial conditions by iterating backwards (12b) and using Assumption A (in particular that $\rho > 0$) it follows that $u^c(k) = u_e \oplus (u(0) - \rho k)$.

First let us note that the feedback controller u^f defined in (11b) satisfies the constraint (6). Indeed, from (11b) it follows that $u^f(k) \geq (-B^T) \otimes' (B \otimes (u^f(k-1) - \rho))$ and from (11a) we conclude that $u^f(k) \geq u^f(k-1) - \rho$. Using similar arguments we can prove that $u^f(k) \geq u_e$ for all $k \geq 1$. Clearly the UCS controller u^c defined in (12b) satisfies the constraint (6) and $u^c(k) \geq u_e$ for all $k \geq 1$. Furthermore,

$$\begin{cases} x^f(k) \leq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e \\ x^f(k) \geq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus B \otimes u_e \end{cases} \quad (13)$$

Indeed, from Lemma 1 (i) and (11b) it follows that $B \otimes u^f(k) \leq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e$ and thus $x^f(k) \leq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e$. The second inequality is straightforward (recall that $u^f(k) \geq u^f(k-1) - \rho$ and $u^f(k) \geq u_e$ and using the monotonicity property (1b) it follows that $x^f(k) \geq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus B \otimes u_e$). The next inequality is also useful:

$$B \otimes (u^f(k-1) - \rho) = (B \otimes u^f(k-1)) - \rho \leq x^f(k-1) - \rho, \quad (14)$$

since $x^f(k-1) \geq B \otimes u^f(k-1)$.

Lemma 4 *The following inequalities hold:*

$$u^f(k) \geq u^c(k) \text{ and } x^f(k) \geq x^c(k) \quad \forall k \geq 0. \quad (15)$$

PROOF. We prove the lemma by induction. For $k=0$ we have $u^f(0) = u^c(0) = u(0)$ and $x^f(0) = x^c(0) = x(0)$. Let us assume that the inequalities of the lemma are valid for a given $k-1$. Now we prove that they also hold for k . Since u^f satisfies the constraint (6) and using our induction hypothesis we obtain $u^f(k) \geq u^f(k-1) - \rho \geq u^c(k-1) - \rho$. Moreover, $u^f(k) \geq u_e$. We conclude that $u^f(k) \geq (u^c(k-1) - \rho) \oplus u_e = u^c(k)$. Using again the induction hypothesis and the monotonicity property (1b) it follows that: $x^f(k) \geq A \otimes x^c(k-1) \oplus B \otimes u^f(k) \geq A \otimes x^c(k-1) \oplus B \otimes u^c(k) = x^c(k)$. \diamond

2.2 Stability of the feedback and UCS controller

The stabilizing properties of the two controllers discussed before are summarized in the next theorem.

Theorem 5 *The following statements hold:*

(i) *For any initial conditions $x^f(0) = x(0) \in \mathbb{R}^n$ and $u^f(0) = u(0) \in \mathbb{R}^m$ there exists a finite K^f such that $x^f(k) = x_e$ for all $k \geq K^f$.*

(ii) *For any initial conditions $x^c(0) = x(0) \in \mathbb{R}^n$ and $u^c(0) = u(0) \in \mathbb{R}^m$ there exists a finite K^c such that $x^c(k) = x_e$ for*

all $k \geq K^c$.

(iii) The controlled systems (11a) and (12a) are finitely Lyapunov stable. In particular, they are also stable in terms of boundedness.

PROOF. (i) From (13) and (14) it follows that

$$x^f(k) \leq A \otimes x^f(k-1) \oplus (x^f(k-1) - \rho) \oplus x_e. \quad (16)$$

Iterating this formula backwards we obtain $x^f(k) \leq \bigoplus_{t=0}^k (A^{\otimes k-t} \otimes (x^f(0) - t\rho)) \oplus x_e$. Since $A_{ij} < 0$ for all $i, j \in \underline{n}$, it follows that [11, Section 2.3]

$$A^{\otimes k} \otimes x^f(0) \rightarrow \mathcal{E} \text{ as } k \rightarrow \infty. \quad (17)$$

We denote with $z_0 = x^f(0)$ and iteratively $z_k = \bigoplus_{t=0}^k (A^{\otimes k-t} \otimes (x^f(0) - t\rho)) = \max\{A^{\otimes k} \otimes x^f(0), z_{k-1} - \rho\}$. From (17) and since $\rho > 0$ it follows that

$$z_k \rightarrow \mathcal{E} \text{ as } k \rightarrow \infty. \quad (18)$$

Therefore, there exists a finite integer K_0^f such that $\bigoplus_{t=0}^k (A^{\otimes k-t} \otimes (x^f(0) - t\rho)) \leq x_e$ for any $k \geq K_0^f$. In conclusion, $x^f(k) \leq x_e$ for any $k \geq K_0^f$.

On the other hand, from (13) it follows that $x^f(k) \geq A \otimes x^f(k-1) \oplus B \otimes u_e$. Iterating this formula and using the definition of $x_e = \bigoplus_{t=1}^n (A^{\otimes n-t} \otimes B \otimes u_e)$ it follows that $x^f(k) \geq A^{\otimes k} \otimes x^f(0) \oplus x_e$ for all $k \geq n$.

From the previous discussion it follows that there exists $K^f \geq \max\{K_0^f, n\}$ such that $x^f(k) = x_e$ for all $k \geq K^f$.

(ii) Since $\rho > 0$ and $u^c(k) = u_e \oplus (u(0) - \rho k)$ it follows that $u^c(k) = u_e$ for k large enough. Note that $x^c(k) = A^{\otimes k} \otimes x^c(0) \oplus (\bigoplus_{t=1}^k (A^{\otimes k-t} \otimes B \otimes (u(0) - t\rho))) \oplus (\bigoplus_{t=1}^k A^{\otimes k-t} \otimes B \otimes u_e)$. From (17) we have $A^{\otimes k} \otimes x^c(0) \rightarrow \mathcal{E}$ as $k \rightarrow \infty$. Using similar arguments as in (18) it follows that $\bigoplus_{t=1}^k (A^{\otimes k-t} \otimes B \otimes (u(0) - t\rho)) \rightarrow \mathcal{E}$ as $k \rightarrow \infty$. Since $x_e = \bigoplus_{t=1}^n (A^{\otimes n-t} \otimes B \otimes u_e)$, there exists a $K^c \geq n$ such that $x^c(k) = x_e$ for all $k \geq K^c$.

(iii) From (i) and (ii) we conclude that we have finite convergence for the controlled systems (11a) and (12a). Let us now prove Lyapunov stability of both controlled systems. Let $\varepsilon > 0$ and consider $\|x(0) - x_e\| \leq \varepsilon$ (i.e. $\delta = \varepsilon$). From Lemma 4 it follows that for all $k \geq 0$: $\max\{\|x^f(k) - x_e\|, \|x^c(k) - x_e\|\} \leq \max_{i \in \underline{n}} \{(x^f(k) - x_e)_i, (x_e - x^c(k))_i\}$.

Note that $a^T \otimes x - a^T \otimes y \leq \max_{i \in \underline{n}} \{x_i - y_i\}$ for any $a \in \mathbb{R}_\varepsilon^n$, $a \neq \mathcal{E}$, and $x, y \in \mathbb{R}^n$ [11, Lemma 3.10]. From the last inequality and (16) it follows that $\max_{i \in \underline{n}} \{(x^f(k) - x_e)_i\} \leq \max_{i \in \underline{n}} \{((A \otimes x^f(k-1) \oplus (x^f(k-1) - \rho) \oplus x_e) - x_e)_i\} = \max_{i \in \underline{n}} \{((A \otimes x^f(k-1) \oplus (x^f(k-1) - \rho) \oplus x_e) - (A \otimes x_e \oplus$

$(x_e - \rho) \oplus x_e)_i\} \leq \max_{i \in \underline{n}} \{(x^f(k-1) - x_e)_i, 0\}$. Iterating this inequality backwards we obtain $\max_{i \in \underline{n}} \{(x^f(k) - x_e)_i\} \leq \max_{i \in \underline{n}} \{(x^f(0) - x_e)_i, 0\} \leq \varepsilon$ for all $k \geq 0$. Moreover, $x_e = A^{\otimes k} \otimes x_e \oplus (\bigoplus_{t=1}^k (A^{\otimes k-t} \otimes B \otimes u_e))$ for all $k \geq 1$. From second part of the proof we see that $x^c(k) \geq A^{\otimes k} \otimes x^c(0) \oplus (\bigoplus_{t=1}^k (A^{\otimes k-t} \otimes B \otimes u_e))$. It follows that $\max_{i \in \underline{n}} \{(x_e - x^c(k))_i\} \leq \max_{i \in \underline{n}} \{(x_e - x^c(0))_i, 0\} \leq \varepsilon$ for all $k \geq 0$. So $\max\{\|x^f(k) - x_e\|, \|x^c(k) - x_e\|\} \leq \varepsilon$ for all $k \geq 0$, i.e. the controlled systems (11a) and (12a) are finitely Lyapunov stable. Using the same arguments as above it follows that the controlled systems (11a) and (12a) are also stable in terms of boundedness (we may take $\theta = \delta$). \diamond

An immediate consequence of Theorem 5 is the following:

Corollary 6 For any input signal $u(\cdot)$ fulfilling the constraint (6) and such that $u^c(k) \leq u(k) \leq u^f(k)$ for all k , the corresponding trajectory satisfies $x^c(k) \leq x(k) \leq x^f(k)$ for all k . Moreover, there exists a finite K such that $x(k) = x_e$ for all $k \geq K$. Consequently the controlled system obtained by applying this input signal is finitely Lyapunov stable and stable in terms of boundedness. \diamond

Note that once we have finite convergence the output of the controlled system satisfies $\max\{y(k) - y_{\text{sys},t}, 0\} = 0$ or equivalently $\max\{y_{\text{sys}}(k) - r_{\text{sys}}(k), 0\} = 0$ for all $k \geq K$ (where $K \leq K^f$ according to Theorem 5), which implies $y_{\text{sys}}(k) \leq r_{\text{sys}}(k)$ (i.e. we obtain a just-in-time production).

3 Stabilizing MPC controller

Given the state and the input at event step $k-1$, the following cost function is introduced:

$$J(x(k-1), \tilde{u}(k)) = \sum_{j=0}^{N-1} \sum_{i=1}^n \max\{x_i(k+j|k-1) - (x_e)_i, 0\} - \beta \sum_{j=0}^{N-1} \sum_{i=1}^m u_i(k+j|k-1), \quad (19)$$

where N is the prediction horizon, $\beta > 0$ is the weight, and $x(k+j|k-1)$ is the system state at event step $k+j$ as predicted at event step $k-1$, based on the MPL difference equation (5a), the state $x(k-1)$ and the future input sequence

$$\tilde{u}(k) = [u^T(k|k-1) \cdots u^T(k+N-1|k-1)]^T.$$

The first term in the cost expresses the tardiness (i.e. penalizes every delay with respect to the desired due date target x_e), while the second term maximizes the feeding times (we want to feed raw material as late as possible). So, the cost function is designed to obtain a just-in-time controller. We define the following optimization problem:

$$J^*(x(k-1)) = \min_{\tilde{u}(k)} J(x(k-1), \tilde{u}(k)) \text{ s.t.} \quad (20)$$

$$\begin{cases} x(k+j|k-1) = A \otimes x(k+j-1|k-1) \oplus \\ \quad B \otimes u(k+j|k-1) \\ u(k+j|k-1) - u(k+j-1|k-1) \geq -\rho \quad \forall j \end{cases} \quad (21)$$

where $x(k-1|k-1) = x(k-1)$, $u(k-1|k-1) = u(k-1)$. Let $\tilde{u}^\sharp(k)$ be the optimal solution of (20)–(21). Using the receding horizon principle at event counter k we apply only the input $u^{\text{MPC},N}(k) = \tilde{u}^\sharp(k|k-1)$. Note that $\tilde{u}^\sharp(k)$ depends on $x(k-1)$ and consequently $u^{\text{MPC},N}(k)$ depends on $x(k-1)$. In this way we can define an implicit MPC state feedback law. The evolution of the closed-loop system obtained from applying the MPC law is denoted with

$$x^{\text{MPC},N}(k) = A \otimes x^{\text{MPC},N}(k-1) \oplus B \otimes u^{\text{MPC},N}(k), \quad (22)$$

where $x^{\text{MPC},N}(0) = x(0)$, $u^{\text{MPC},N}(0) = u(0)$ are given initial conditions. Let us define the matrices

$$\tilde{D} = \begin{bmatrix} B & \mathcal{E} & \cdots & \mathcal{E} \\ A \otimes B & B & \cdots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ A^{\otimes N-1} \otimes B & A^{\otimes N-2} \otimes B & \cdots & B \end{bmatrix}, \tilde{C} = \begin{bmatrix} A \\ A^{\otimes 2} \\ \vdots \\ A^{\otimes N} \end{bmatrix}$$

and the vectors $\bar{u}(k) = [u^T(k-1) - \rho \cdots u^T(k-1) - N\rho]^T$, $\bar{u}_e = [u_e^T \cdots u_e^T]^T$, $\bar{x}_e = [x_e^T \cdots x_e^T]^T$ and $\bar{x}(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \bar{u}(k) \oplus \bar{x}_e$, or in vector notation $\bar{x}(k) = [\bar{x}^T(k|k-1) \cdots \bar{x}^T(k+N-1|k-1)]^T$.

The next lemma shows that the MPC controller $u^{\text{MPC},N}$ is bounded from below by the UCS controller u^c .

Lemma 7 $u^c(k) \leq u^{\text{MPC},N}(k)$, $x^c(k) \leq x^{\text{MPC},N}(k) \quad \forall k \geq 0$

PROOF. First, let us show that $\tilde{u}^\sharp(k) \geq \bar{u}_e$. The states corresponding to $\tilde{u}^\sharp(k)$ are given by: $\tilde{x}^\sharp(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^\sharp(k)$, or in vector notation $\tilde{x}^\sharp(k) = [(\tilde{x}^\sharp(k|k-1))^T \cdots (\tilde{x}^\sharp(k+N-1|k-1))^T]^T$. Let us assume that $\tilde{u}^\sharp(k) \not\geq \bar{u}_e$. Define $\tilde{u}^{\text{feas}}(k) = \tilde{u}^\sharp(k) \oplus \bar{u}_e$ and $\tilde{x}^{\text{feas}}(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^{\text{feas}}(k)$. Note that $\tilde{u}^{\text{feas}}(k)$ is a feasible solution of the problem (20)–(21), i.e. it satisfies the constraints (21). Since $A^{\otimes j} \otimes B \otimes u_e \leq x_e$ for all j , $\tilde{x}^{\text{feas}}(k) = \tilde{x}^\sharp(k) \oplus \tilde{D} \otimes \bar{u}_e \leq \tilde{x}^\sharp(k) \oplus \bar{x}_e$. It follows that

$$\begin{aligned} J(x(k-1), \tilde{u}^{\text{feas}}(k)) &\leq \sum_{j=0}^{N-1} \sum_{i=1}^n \max\{x_i^\sharp(k+j|k-1) - \\ &(x_e)_i, 0\} - \beta \sum_{j=0}^{N-1} \sum_{i=1}^m u_i^{\text{feas}}(k+j|k-1) < \\ &\sum_{j=0}^{N-1} \sum_{i=1}^n \max\{x_i^\sharp(k+j|k-1) - (x_e)_i, 0\} - \\ &\beta \sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\sharp(k+j|k-1) = J(x(k-1), \tilde{u}^\sharp(k)), \end{aligned}$$

and thus we get contradiction with the optimality of $\tilde{u}^\sharp(k)$.

Now we go on with the proof of the lemma using induction. For $k=0$ we have $u^c(0) = u^{\text{MPC},N}(0) = u(0)$ and $x^c(0) = x^{\text{MPC},N}(0) = x(0)$. We assume that $u^c(k-1) \leq u^{\text{MPC},N}(k-1)$ and $x^c(k-1) \leq x^{\text{MPC},N}(k-1)$ and we prove that these inequalities also hold for k . From the induction hypothesis we have $u^{\text{MPC},N}(k) \geq u^{\text{MPC},N}(k-1) - \rho \geq u^c(k-1) - \rho$. Moreover, $u^{\text{MPC},N}(k) = \tilde{u}^\sharp(k|k-1) \geq u_e$. So $u^{\text{MPC},N}(k) \geq (u^c(k-1) - \rho) \oplus u_e = u^c(k)$. From the monotonicity property (1b) it follows that $x^c(k) = A \otimes x^c(k-1) \oplus B \otimes u^c(k) \leq A \otimes x^{\text{MPC},N}(k-1) \oplus B \otimes u^{\text{MPC},N}(k) = x^{\text{MPC},N}(k)$. \diamond

We will show in the sequel that when the parameters N and β are chosen properly the MPC controller is bounded from above as well and therefore it will stabilize the system (5a)–(5b). In fact, the next proposition shows us that by a proper tuning of the design parameters the MPC controller can be interpreted as a just-in-time controller.

Proposition 8 Assume $\beta < \frac{1}{mN}$ and consider the maximization problem

$$\tilde{u}^\sharp(k) = \arg \max_{\tilde{u}(k)} \sum_{j=0}^{N-1} \sum_{i=1}^m u_i(k+j|k-1) \quad (23)$$

$$\text{s.t.} \begin{cases} \tilde{D} \otimes \tilde{u}(k) \leq \bar{x}(k) \\ u(k+j|k-1) - u(k+j-1|k-1) \geq -\rho \quad \forall j. \end{cases} \quad (24)$$

Then, $\tilde{u}^\sharp(k)$ is also the optimal solution of (20)–(21).

PROOF. Note that we do not have to impose also the constraint $u(k|k-1) - u(k-1|k-1) \geq -\rho$ in (24). This inequality is automatically satisfied, since $\bar{u}(k)$ is a feasible solution for (20)–(21) and consequently a feasible solution for the optimization problem (23)–(24) and thus $\bar{u}(k) \leq \tilde{u}^\sharp(k)$.

We will prove this lemma by contradiction. Define $\tilde{x}^c(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \bar{u}(k)$ then $\bar{x}(k) = \tilde{x}^c(k) \oplus \bar{x}_e$. First let us consider an $\tilde{u}^\flat(k)$ that satisfies (24) but for which $\sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\flat(k+j|k-1) < \sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\sharp(k+j|k-1)$. Define $\tilde{x}^\flat(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^\flat(k)$. Then, for each $i \in \underline{n}$ and $j \in \{0, 1, \dots, N-1\}$ it follows that $\max\{x_i^\flat(k+j|k-1), (x_e)_i\} = \max\{x_i^c(k+j|k-1), \tilde{D}_{jn+i} \otimes \tilde{u}^\flat, (x_e)_i\} = \max\{\bar{x}_i(k+j|k-1), \tilde{D}_{jn+i} \otimes \tilde{u}^\flat\} = \bar{x}_i(k+j|k-1)$, where \tilde{D}_{jn+i} denotes the $(jn+i)$ -th row of \tilde{D} . It follows that $J(x(k-1), \tilde{u}^\flat(k)) = \sum_{j=0}^{N-1} \sum_{i=1}^n (\bar{x}_i(k+j|k-1) - (x_e)_i) - \beta \sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\flat(k+j) > \sum_{j=0}^{N-1} \sum_{i=1}^n (\bar{x}_i(k+j|k-1) - (x_e)_i) - \beta \sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\sharp(k+j) = J(x(k-1), \tilde{u}^\sharp(k))$, and thus $\tilde{u}^\flat(k)$ cannot be the optimal solution of (20)–(21).

Next let us consider $\tilde{u}^\dagger(k)$ that satisfies (21) but not the inequality $\tilde{D}\tilde{u} \leq \bar{x}(k)$. Define $\delta = \max_{i \in \underline{n}, j \in \{0, \dots, N-1\}} \{(\tilde{D}_{jn+i} \otimes \tilde{u}^\dagger(k)) - \bar{x}_i(k+j|k-1)\} > 0$. Then, there exist i_0, j_0 such that

$x_{i_0}^\dagger(k+j_0|k-1) = \tilde{D}_{j_0 n+i_0} \otimes \tilde{u}^\dagger(k) = \bar{x}_{i_0}(k+j_0|k-1) + \delta$ and thus $\sum_{j=0}^{N-1} \sum_{i=1}^n \max\{x_i^\dagger(k+j|k-1) - (x_e)_i, 0\} \geq \sum_{j=0}^{N-1} \sum_{i=1}^n \max\{\bar{x}_i(k+j|k-1) - (x_e)_i, 0\} + \delta$. Note that $\tilde{u}^\dagger(k) = (\tilde{u}^\dagger(k) - \delta) \oplus \bar{u}(k)$ fulfills (24) and the corresponding cost satisfies:

$$\begin{aligned} J(x(k-1), \tilde{u}^\dagger(k)) &\leq \sum_{j=0}^{N-1} \sum_{i=1}^n \max\{x_i^\dagger(k+j|k-1) - (x_e)_i, \\ &0\} - \delta - \beta \left(\sum_{j=0}^{N-1} \sum_{i=1}^m u_i^\dagger(k+j|k-1) - Nm\delta \right) = \\ J(x(k-1), \tilde{u}^\dagger(k)) + (\beta Nm - 1)\delta &< J(x(k-1), \tilde{u}^\dagger(k)), \end{aligned}$$

and thus $\tilde{u}^\dagger(k)$ cannot be the optimal solution of (20)–(21). This proves that $\tilde{u}^\#(k)$ is also the optimal solution of the original optimization problem (20)–(21). \diamond

From the proof of Proposition 8 it follows that the optimal control sequence $\tilde{u}^\#(k)$ is a just-in-time control sequence over the prediction window $[k, k+N-1]$, since we search for the latest input dates $\tilde{u}(k)$ such that the state dates occur at times as close as possible to the desired ones or at the latest before the desired ones.

Let us define $\tilde{u}^{*N}(k) := (-\tilde{D}^T) \otimes \bar{x}(k)$ or in vector notation it can be written as $\tilde{u}^{*N}(k) = [(u^{*N}(k|k-1))^T \dots (u^{*N}(k+N-1|k-1))^T]^T$. The following lemma provides an explicit solution to the optimization problem (23)–(24).

Lemma 9 *The optimization problem (23)–(24) has an unique solution given by:*

$$\begin{aligned} u^\#(k+N-1|k-1) &= u^{*N}(k+N-1|k-1) \\ u^\#(k+j|k-1) &= \min\{u^{*N}(k+j|k-1), \\ &u^\#(k+j+1|k-1) + \rho\}, \end{aligned} \quad (25)$$

for $j = N-2, \dots, 0$.

PROOF. The feasibility conditions (24) for $u^\#(k+N-1|k-1)$ are given by: $B \otimes u^\#(k+N-1|k-1) \leq \bar{x}(k+N-1|k-1)$, and from Lemma 1 (i) it is clear that the largest $u^\#(k+N-1|k-1)$ is given by $u^\#(k+N-1|k-1) = u^{*N}(k+N-1|k-1) = (-B^T) \otimes \bar{x}(k+N-1|k-1)$.

From the feasibility conditions (24), $u^\#(k+N-2|k-1)$ has to satisfy:

$$\begin{aligned} A \otimes B \otimes u^\#(k+N-2|k-1) &\leq \bar{x}(k+N-1|k-1), \\ B \otimes u^\#(k+N-2|k-1) &\leq \bar{x}(k+N-2|k-1), \\ u^\#(k+N-2|k-1) &\leq u^\#(k+N-1|k-1) + \rho, \end{aligned}$$

and thus the largest $u^\#(k+N-2|k-1)$ is given by $u^\#(k+N-2|k-1) = \min\{u^{*N}(k+N-2|k-1), u^\#(k+N-1|k-1) +$

$\rho\}$. Using the same reasoning we obtain: $u^\#(k+j|k-1) \leq u^{*N}(k+j|k-1)$, $u^\#(k+j|k-1) \leq u^\#(k+j+1|k-1) + \rho$ for all j , i.e. the formula (25). \diamond

Lemma 10 *Any feasible solution $\tilde{u}_{\text{feas}}(k)$ of (23)–(24) satisfies $\tilde{u}_{\text{feas}}(k) \leq \tilde{u}^\#(k)$.*

PROOF. From Lemma 1 and $\tilde{D} \otimes \tilde{u}_{\text{feas}}(k) \leq \bar{x}(k)$ it follows that $\tilde{u}_{\text{feas}}(k) \leq \tilde{u}^{*N}(k)$ and thus $u_{\text{feas}}(k+N-1|k-1) \leq u^{*N}(k+N-1|k-1) = u^\#(k+N-1|k-1)$. Note that $u_{\text{feas}}(k+N-2|k-1)$ satisfies

$$\begin{aligned} u_{\text{feas}}(k+N-2|k-1) &\leq u^{*N}(k+N-2|k-1) \\ u_{\text{feas}}(k+N-2|k-1) &\leq u_{\text{feas}}(k+N-1|k-1) + \rho \leq \\ u^\#(k+N-1|k-1) + \rho. \end{aligned}$$

In conclusion $u_{\text{feas}}(k+N-2|k-1) \leq \min\{u^{*N}(k+N-2|k-1), u^\#(k+N-1|k-1) + \rho\} = u^\#(k+N-2|k-1)$. Applying this reasoning backwards, it follows that $u_{\text{feas}}(k+j|k-1) \leq u^\#(k+j|k-1)$ for all $j = N-1, \dots, 0$, i.e. $\tilde{u}_{\text{feas}}(k) \leq \tilde{u}^\#(k)$. \diamond

Since $\bar{x}(k|k-1) = A \otimes x(k-1) \oplus B \otimes (u(k-1) - \rho) \oplus x_e$ and $B \otimes u^\#(k|k-1) \leq \bar{x}(k|k-1)$, it follows that

$$u^{\text{MPC},N}(k) \leq (-B^T) \otimes (A \otimes x(k-1) \oplus B \otimes (u(k-1) - \rho) \oplus x_e), \quad (26)$$

for any previous state $x(k-1)$ and input $u(k-1)$. The next theorem characterizes the stabilizing properties of the MPC controller. Contrary to the classical MPC where stability is proved using the optimal cost as a Lyapunov function (see e.g. [2, 12, 15]), here the proof is based on the particular properties of max-plus algebra, especially the monotonicity property (1b). We will show that the MPC policy lies in between an infinite horizon policy and a feedback policy.

Theorem 11 *Suppose that $\beta < \frac{1}{mN}$, then (i) The following inequalities hold*

$$\begin{aligned} u^c(k) &\leq u^{\text{MPC},N}(k) \leq u^f(k), \\ x^c(k) &\leq x^{\text{MPC},N}(k) \leq x^f(k) \quad \forall k \geq 0. \end{aligned} \quad (27)$$

In particular, the closed-loop system (22) is finitely Lyapunov stable and stable in terms of boundedness.

(ii) *If $N = 1$, then $u^{\text{MPC},1}(k) = u^f(k)$. For two prediction horizons $N_1 < N_2$ the following inequalities hold*

$$\begin{aligned} u^{\text{MPC},N_1}(k) &\geq u^{\text{MPC},N_2}(k), \\ x^{\text{MPC},N_1}(k) &\geq x^{\text{MPC},N_2}(k) \quad \forall k \geq 0. \end{aligned} \quad (28)$$

PROOF. (i) The left-hand side of inequalities (27) follows from Lemma 7.

The right-hand side of inequalities (27) is proved using induction. For $k=0$ we have $u^{\text{MPC},N}(0) = u^f(0) = u(0)$ and $x^{\text{MPC},N}(0) = x^f(0) = x(0)$. Let us assume that $u^{\text{MPC},N}(k-1) \leq u^f(k-1)$ and $x^{\text{MPC},N}(k-1) \leq x^f(k-1)$ are valid and we prove that they also hold for k . From (26) and our induction hypothesis we have:

$$B \otimes u^{\text{MPC},N}(k) \leq A \otimes x^{\text{MPC},N}(k-1) \oplus B \otimes (u^{\text{MPC},N}(k-1) - \rho) \oplus x_e \leq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e.$$

On the other hand, $u^f(k)$ is the largest solution of

$$B \otimes u^f(k) \leq A \otimes x^f(k-1) \oplus B \otimes (u^f(k-1) - \rho) \oplus x_e.$$

From Lemma 1 (i) it follows that $u^{\text{MPC},N}(k) \leq u^f(k)$. Then, $x^{\text{MPC},N}(k) = A \otimes x^{\text{MPC},N}(k-1) \oplus B \otimes u^{\text{MPC},N}(k) \leq A \otimes x^f(k-1) \oplus B \otimes u^f(k) = x^f(k+1)$.

The stability properties of the MPC controller follow from Corollary 6.

(ii) For $N=1$ from the feasibility condition (24) it is clear that $\tilde{u}^\#(k) = u^\#(k) = u^f(k)$ (according to (9)).

For two prediction horizons $N_1 < N_2$, we denote with $\tilde{D}_{(N_1)}$ the matrix \tilde{D} from (24) corresponding to the prediction horizon $N = N_1$. Similarly, we define $\tilde{D}_{(N_2)}$. Note that

$$\tilde{D}_{(N_2)} = \begin{bmatrix} \tilde{D}_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix} \text{ (where } * \text{ stands for matrix blocks of}$$

appropriate dimensions but that are not relevant for this proof). We denote with $\bar{x}_{(N_1)}(k)$ the vector $\bar{x}(k)$ from (24) corresponding to $N = N_1$ and $\tilde{u}_{(N_1)}^\#(k)$ the optimal solution of (23)–(24) corresponding to N_1 . Similarly, we define $\bar{x}_{(N_2)}(k)$ and $\tilde{u}_{(N_2)}^\#(k)$.

We prove the inequalities (28) by induction. For $k=0$ the statement is true: $u^{\text{MPC},N_1}(0) = u^{\text{MPC},N_2}(0) = u(0)$ and $x^{\text{MPC},N_1}(0) = x^{\text{MPC},N_2}(0) = x(0)$. Let us assume that $u^{\text{MPC},N_1}(k-1) \geq u^{\text{MPC},N_2}(k-1)$ and $x^{\text{MPC},N_1}(k-1) \geq x^{\text{MPC},N_2}(k-1)$. Define $\tilde{u}_{(N_2)}^\#(k : k+N_1-1)$ the sub-vector of $\tilde{u}_{(N_2)}^\#(k)$ containing the first mN_1 components. We have:

$$\begin{aligned} \tilde{D}_{(N_2)} \otimes \tilde{u}_{(N_2)}^\#(k) &= \begin{bmatrix} \tilde{D}_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix} \otimes \begin{bmatrix} \tilde{u}_{(N_2)}^\#(k : k+N_1-1) \\ * \end{bmatrix} \\ &\leq \bar{x}_{(N_2)}(k) = \begin{bmatrix} \bar{x}_{(N_2)}(k : k+N_1-1) \\ * \end{bmatrix} \leq \begin{bmatrix} \bar{x}_{(N_1)}(k) \\ * \end{bmatrix}. \end{aligned}$$

It follows that $\tilde{D}_{(N_1)} \otimes \tilde{u}_{(N_2)}^\#(k : k+N_1-1) \leq \bar{x}_{(N_1)}(k)$, i.e. $\tilde{u}_{(N_2)}^\#(k : k+N_1-1)$ is feasible for (23)–(24) with prediction horizon $N = N_1$. From Lemma 10 we obtain that $\tilde{u}_{(N_2)}^\#(k : k+N_1-1) \leq \tilde{u}_{(N_1)}^\#(k)$. Therefore, $u^{\text{MPC},N_1}(k) = u_{(N_1)}^\#(k|k-$

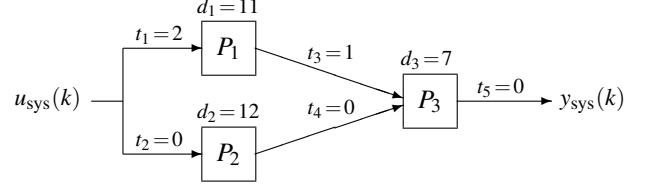


Figure 1. A production system.

$1) \geq u_{(N_2)}^\#(k|k-1) = u^{\text{MPC},N_2}(k)$. Similarly, $x^{\text{MPC},N_1}(k) \geq x^{\text{MPC},N_2}(k)$. \diamond

4 Example

We consider the example from [8], which represents a production system with three processing units (see Figure 1) that has the following state space description:

$$\begin{aligned} x_{\text{sys}}(k) &= \begin{bmatrix} 11 & \varepsilon & \varepsilon \\ \varepsilon & 12 & \varepsilon \\ 23 & 24 & 7 \end{bmatrix} \otimes x_{\text{sys}}(k-1) \oplus \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \otimes u_{\text{sys}}(k) \\ y_{\text{sys}}(k) &= [\varepsilon \ \varepsilon \ 7] \otimes x_{\text{sys}}(k). \end{aligned}$$

In this example the largest MPA eigenvalue (the growth rate of the system) is $\lambda_{\max} = 12$ and $P = \text{diag}_{\oplus}([0 \ 0 \ 12])$ ⁵. We consider the following due dates $r_{\text{sys}}(k) = 25 + 14.4k$, i.e. the offset vector is $y_{\text{sys},t} = 25$ and $\rho = 14.4$. We choose the prediction horizon $N = 5$ and the weight $\beta = 0.18$. Note that $\beta < \frac{1}{N}$, i.e. the condition from Theorem 11 is satisfied. The initial conditions are chosen to be $x_{\text{sys}}(0) = [37 \ 29 \ 26]^T$ and $u_{\text{sys}}(0) = 35$. The normalized system has the form: $y(k) = [\varepsilon \ \varepsilon \ 19] \otimes x(k)$ and

$$x(k) = \begin{bmatrix} -3.4 & \varepsilon & \varepsilon \\ \varepsilon & -2.4 & \varepsilon \\ -3.4 & -2.4 & -7.4 \end{bmatrix} \otimes x(k-1) \oplus \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \otimes u(k).$$

The equilibrium pair is given by $x_e = [6 \ 4 \ 6]^T$ and $u_e = 4$. The controllers $u^c, u^f, u^{\text{MPC},5}$ corresponding to the normalized system are depicted in Figure 2 and the corresponding state trajectories in Figure 3. In this particular case we see that $u^c(k) \leq u^f(k) = u^{\text{MPC},5}(k)$ but $x^c(k) = x^f(k) = x^{\text{MPC},5}$ for all k . Note that we have finitely Lyapunov stability since convergence is achieved in a finite number of event steps.

5 Conclusions

In this paper we have discussed the problem of guaranteeing a priori stability of an MPL system using the MPC framework. We have provided sufficient conditions that guarantee

⁵ A matrix $D = \text{diag}_{\oplus}(v)$, where $v \in \mathbb{R}^n$, is defined as follows: $D_{ii} = v_i$ for all i and $D_{ij} = \varepsilon$ for all $i \neq j$.

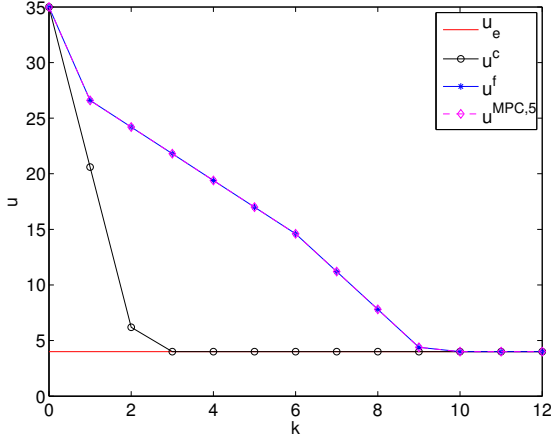


Figure 2. The controllers $u^c, u^f, u^{MPC,5}$.

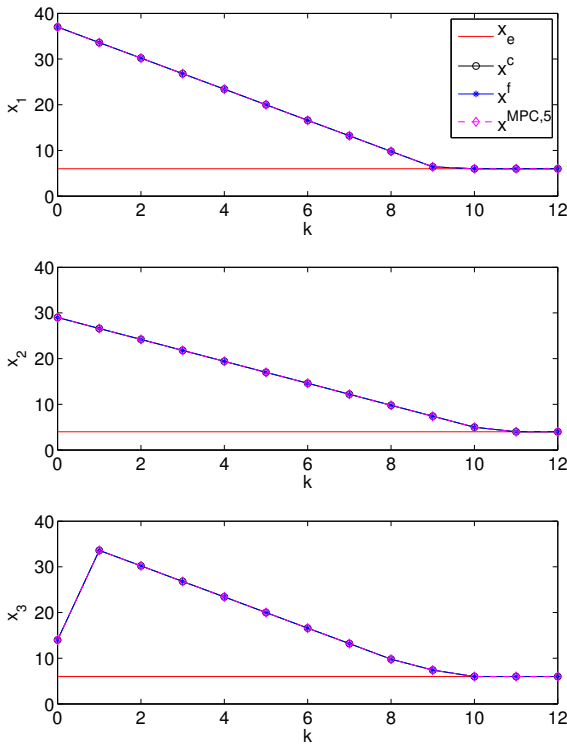


Figure 3. State trajectory using different controllers: $x^c, x^f, x^{MPC,5}$.

that the MPC controller based on a finite-horizon optimal control problem guarantees a priori stability of the closed-loop systems and the resulting control input sequence is nondecreasing. Contrary to the classical case where stability of the MPC is proved using a terminal cost/terminal set approach, we have shown that by a proper tuning of the design parameters stability can be achieved even in finite number of event steps and an analytic solution can be obtained for the MPC controller although the system is nonlinear. The key assumptions that allow one to guarantee stability of the closed-loop system were that the growth rate of the due dates be larger than the growth rate of the system and the cost

function be designed to provide a just-in-time controller.

A Appendix

In this appendix we show that stability in terms of boundedness for the normalized system implies boundedness of the buffer levels for the original system.

Remark 12 *Since we assume that B_{sys} and C_{sys} are column-finite and row-finite, respectively, then by their multiplication in the max-plus algebra with an MPA invertible matrix these properties are preserved. It follows that the matrices B and C are also column-finite and row-finite, respectively.* \diamond

For a row-finite matrix $A \in \mathbb{R}_E^{m \times n}$ the following property holds [11, Lemma 3.10]:

$$\|A \otimes x - A \otimes y\| \leq \|x - y\| \quad \forall x, y \in \mathbb{R}^n. \quad (\text{A.1})$$

It is clear that if $u(k)$ and $x(k)$ are the input and state trajectories for the nominal system (5a), then $x_{\text{sys}}(k) = P \otimes x(k) + \rho k$ is the state trajectory for the original system (2a) corresponding to the input trajectory $u_{\text{sys}}(k) = u(k) + \rho k$. Let us define the nominal trajectories $x_{\text{sys}}^e(k) := P \otimes x_e + \rho k$ and $u_{\text{sys}}^e(k) := u_e + \rho k$. Moreover, we denote the state trajectories for the original system (2a) obtained by applying the controllers $u_{\text{sys}}^f(k) = u^f(k) + \rho k$ and $u_{\text{sys}}^c(k) = u^c(k) + \rho k$ with $x_{\text{sys}}^f(k)$ and $x_{\text{sys}}^c(k)$, respectively. From the previous discussion it follows that $x_{\text{sys}}^f(k) = P \otimes x^f(k) + \rho k$ and $x_{\text{sys}}^c(k) = P \otimes x^c(k) + \rho k$.

First let us prove that the feedback controller u^f and the UCS controller u^c are bounded. Recall that in Section 2.1 we have shown that $u^f(k) \geq u^c(k) \geq u_e$ for all k , i.e. these two controllers are bounded from below. It remains to prove that the feedback controller u^f is also bounded from above. Let us assume that u^f is not bounded from above, i.e. there exists $j_0 \in \underline{m}$ such that the signal $\{u_{j_0}^f(k)\}_{k \geq 0}$ diverges towards $+\infty$. Since the matrix B is row-finite (according to Remark 12) it follows that there exists an $i_0 \in \underline{n}$ such that $B_{i_0 j_0} \in \mathbb{R}$ and thus $x_{i_0}^f(k) = \max\{*, B_{i_0 j_0} + u_{j_0}^f(k)\}$ is unbounded (where $*$ stands for scalar expressions that are not relevant for this proof), which is a contradiction with the result from Theorem 5 (iii). It follows that

$$\|u_{\text{sys}}^f(k) - u_{\text{sys}}^e(k)\|, \|u_{\text{sys}}^c(k) - u_{\text{sys}}^e(k)\| \text{ is bounded } \forall k \geq 0.$$

Since P and C are row-finite, using (A.1) the following inequalities can be deduced:

$$\begin{aligned} \|x_{\text{sys}}(k) - x_{\text{sys}}^e(k)\| &\leq \|x(k) - x_e\| \\ \|y_{\text{sys}}(k) - r_{\text{sys}}(k)\| &\leq \|x(k) - x_e\| + \|(C \otimes x_e) - y_{\text{sys}, t}\|. \end{aligned}$$

Using the last two inequalities we conclude that if the normalized system is stable in terms of boundedness (as defined

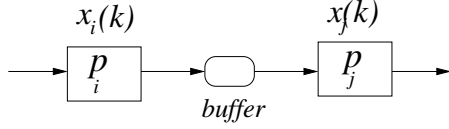


Figure A.1. Buffer between two processing units.

in Definition 3 (ii)), then the original closed-loop system obtained by applying the feedback law $\mu(x(k-1)) + \rho k$ has the property that

$$\|x_{\text{sys}}(k) - x_{\text{sys}}^e(k)\| \text{ is bounded } \forall k \geq 0, \quad (\text{A.2})$$

and the corresponding output satisfies

$$\|y_{\text{sys}}(k) - r_{\text{sys}}(k)\| \text{ is bounded } \forall k \geq 0. \quad (\text{A.3})$$

Now let us prove that if $\|u_{\text{sys}}(k) - u_{\text{sys}}^e(k)\|$, $\|x_{\text{sys}}(k) - x_{\text{sys}}^e(k)\|$ and $\|y_{\text{sys}}(k) - r_{\text{sys}}(k)\|$ are bounded for all $k \geq 0$, then the buffer levels are also bounded. It is sufficient to prove boundedness of the internal buffers since for the input or output buffers the proof is similar. Let us consider the buffer between the internal states i and j (see Figure A.1), then the buffer level of this buffer at time t is given by:

$$\mathcal{L}(t) = \sum_{k=0}^{\infty} I_{\{t \geq (x_{\text{sys}})_i(k) + p_i\}} - \sum_{k=0}^{\infty} I_{\{t \leq (x_{\text{sys}})_j(k)\}},$$

where p_i is a fixed positive scalar (the processing time of the i^{th} processing unit) and I_S is the indicator function defined as $I_S = 1$, if S is true and $I_S = 0$, if S is false. Since $\|x_{\text{sys}}(k) - x_{\text{sys}}^e(k)\|$ is bounded it follows that there exist some finite scalars m_i, M_i, m_j and M_j such that

$$\begin{aligned} m_i + \rho k &\leq (x_{\text{sys}})_i(k) \leq M_i + \rho k \\ m_j + \rho k &\leq (x_{\text{sys}})_j(k) \leq M_j + \rho k \end{aligned}$$

for all $k \geq 0$. If k satisfies $t \geq (x_{\text{sys}})_i(k) + p_i$, then $t \geq (x_{\text{sys}})_i(k) + p_i \geq p_i + m_i + \rho k$. Let us define⁶ $k_m(t) = \lfloor (t - m_i - p_i) / \rho \rfloor$. Then, the following inequality holds: $\sum_{k=0}^{\infty} I_{\{t \geq (x_{\text{sys}})_i(k) + p_i\}} \leq k_m(t)$. Similarly, if k satisfies $t \leq (x_{\text{sys}})_j(k)$, then $t \leq M_j + \rho k$. Let us define $k_M(t) = \lceil (t - M_j) / \rho \rceil$. The following inequality also holds: $\sum_{k=0}^{\infty} I_{\{t \leq (x_{\text{sys}})_j(k)\}} \geq k_M(t)$. It follows that

$$\begin{aligned} \mathcal{L}(t) &\leq k_m(t) - k_M(t) \leq (t - m_i - p_i) / \rho + 1 - \\ &\quad ((t - M_j) / \rho - 1) = 2 + (M_j - m_i - p_i) / \rho, \end{aligned}$$

which is finite (since $\rho > 0$) for each time t . Therefore, the buffer level of the buffer between processing units i and j is finite at any time t .

⁶ $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer not less than x .

Acknowledgements

Research supported by the STW projects ‘‘Model predictive control for hybrid systems’’ and ‘‘Multi-Agent Control of Large-Scale Hybrid Systems’’, and by the European 6th Framework Network of Excellence HYCON.

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