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Stabilization of max-plus-linear systems using receding horizon control: The unconstrained case – Extended report*

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Abstract

Max-plus-linear (MPL) systems are a class of discrete event systems that can be described by models that are "linear" in the max-plus algebra. In this paper we focus on MPL systems such as they arise in the context of e.g. manufacturing systems, telecommunication networks, railway networks, and parallel computing. We derive a receding horizon control scheme for MPL systems that guarantees a priori stability (in the sense of boundedness of the normalized state) of the closed-loop system in the "unconstrained" case. We also discuss the main properties of the resulting receding horizon controllers.

1 Introduction

Discrete-event systems (DES) are event-driven dynamical systems (i.e. the state transitions are initiated by events, rather than a clock) and they often arise in the context of manufacturing systems, telecommunication networks, railway networks, parallel computing, etc. In the last couple of decades there has been an increase in the research on DES that can be modeled as max-plus-linear (MPL) systems [1, 4, 7]. Most of the earlier literature on this class of systems addresses performance analysis rather than control. There are two main directions in MPL DES control: one direction uses optimal control based on residuation theory [3, 8, 10, 11], and the other a receding horizon control (RHC) based approach [6]. Although there are several papers on optimal and RHC control for MPL DES, the literature on the stabilizing controller for this class of systems is relatively sparse. In fact, to the authors' best knowledge, the only papers explicitly dealing with stabilizing control of MPL DES are [8] and [13].

Receding horizon control (RHC), also known as model predictive control, is an attractive feedback strategy for linear or nonlinear processes subject to input and state constraints [9]. The essence of RHC is to determine a control profile that optimizes a cost criterion over a prediction window and then to apply this control profile until new process measurements become available. Feedback is incorporated by using these measurements to update the optimization problem for the next step.

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This paper considers the problem of designing a stabilizing receding horizon controller for the class of MPL DES. We consider a trade-off between tracking a reference state trajectory and just-in-time production for the so-called "unconstrained" case, in which only the constraint that the input (i.e., the sequence of feeding times) should be nondecreasing is taken into account. In this particular case we derive a stable RHC scheme for which the analytic solution exists. The main advantage of this paper compared to most of the results on RHC of MPL DES is the fact that we guarantee *a priori* stability of the closed-loop system. Moreover, the conditions that we will derive in this paper are less strict than those of [13] (where output tracking is considered). We also prove several properties of the RHC controllers, and we characterize a whole class of stabilizing controllers for MPL DES.

2 Max-plus algebra and MPL DES

2.1 Max-plus algebra

Define $\varepsilon := -\infty$ and $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic (MPA) addition ($\oplus$) and multiplication ($\otimes$) are defined as [1, 4]: $x \oplus y := \max\{x,y\}$, $x \otimes y := x + y$, for $x, y \in \mathbb{R}_\varepsilon$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ one can extend the definition as follows:

$$(A \oplus B)_{ij} := A_{ij} \oplus B_{ij},$$
$$(A \otimes C)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kj},$$

for all i, j. Define the matrix $\mathcal{E}_{m\times n}$ as the $m \times n$ MPA zero matrix: $(\mathcal{E}_{m\times n})_{ij} := \varepsilon$, for all i, j. The matrix E_n is the $n \times n$ MPA identity matrix: $(E_n)_{ii} := 0$, for all i and $(E_n)_{ij} := \varepsilon$, for all i, j with $i \neq j$. For any matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, let A^* be defined, whenever it exists, by

$$A^* := E_n \oplus A \oplus \cdots \oplus A^{\otimes^n} \oplus A^{\otimes^{n+1}} \oplus \cdots$$

For a positive integer n, we denote $\underline{n} := \{1, 2, \cdots, n\}$. Given $x \in \mathbb{R}_\varepsilon^n$ we define $\|x\|_\oplus := \max_{i\in\underline{n}} x_i$ and $\|x\|_\infty := \max_{i\in\underline{n}} |x_i|$. A matrix $\Gamma \in \mathbb{R}_\varepsilon^{n\times m}$ is *row-finite* if for any row $i \in \underline{n}$, $\max_{j\in\underline{m}} \Gamma_{ij} \neq \varepsilon$.

We denote with $x \oplus' y := \min\{x,y\}$ and $x \otimes' y := x + y$ (the operations $\otimes$ and $\otimes'$ differ only in that $(-\infty) \otimes (+\infty) := -\infty$, while $(-\infty) \otimes' (+\infty) := +\infty$). The matrix multiplication and addition for $(\oplus', \otimes')$ are defined similarly as for $(\oplus, \otimes)$. It can be shown that the following relations hold for any matrices A, B and vectors x, y of appropriate dimensions over $\mathbb{R}_\varepsilon$ (see also [2]):

$$A \otimes' (B \otimes x) \geq (A \otimes' B) \otimes x, \tag{1}$$
$$((-A^T) \otimes' A) \otimes x \geq x \tag{2}$$
$$x \leq y \Rightarrow\ A \otimes x \leq A \otimes y \text{ and } A \otimes' x \leq A \otimes' y. \tag{3}$$

Lemma 2.1 [1] *(i) The inequality $A \otimes x \leq b$ has a unique largest solution given by $x_{\rm opt} = (-A^T) \otimes' b = -(A^T \otimes (-b))$ (by the largest solution we mean that for all x satisfying $A \otimes x \leq b$ we have $x \leq x_{\rm opt}$).*
(ii) The equation $x = A \otimes x \oplus b$ has a solution $x = A^* \otimes b$. If $A_{ij} < 0$ for all i, j, then the solution is unique.

2.2 MPL systems

DES with only synchronization and no concurrency can be modeled by an MPA model of the form [1, 4][1]

$$x_{\rm sys}(k) = A_{\rm sys} \otimes x_{\rm sys}(k-1) \oplus B_{\rm sys} \otimes u_{\rm sys}(k), \tag{4}$$

where $x_{\rm sys}(k) \in \mathbb{R}_\varepsilon^n$ represents the state, $u_{\rm sys}(k) \in \mathbb{R}_\varepsilon^m$ is the input and where $A_{\rm sys} \in \mathbb{R}_\varepsilon^{n\times n}$, $B_{\rm sys} \in \mathbb{R}_\varepsilon^{n\times m}$ are the system matrices. In the context of DES k is an event counter while $u_{\rm sys}, x_{\rm sys}$ are dates (feeding times and processing times, respectively). A typical constraint that appears in the context of DES where the input represents times, is that the signal $u_{\rm sys}$ should be increasing: $u_{\rm sys}(k+1) - u_{\rm sys}(k) \geq 0$.

Let λ^* be the largest MPA eigenvalue of $A_{\rm sys}$ ($\lambda \in \mathbb{R}_\varepsilon$ is an MPA eigenvalue of $A_{\rm sys}$ if there exists an MPA eigenvector $v \in \mathbb{R}_\varepsilon^n$ with at least one finite entry such that $A_{\rm sys} \otimes v = \lambda \otimes v$). In the next section we will consider a reference signal that the state should track of the following form:

$$r_{\rm sys}(k) = x_{\rm sys,t} + k\rho. \tag{5}$$

Since through the term $B_{\rm sys} \otimes u_{\rm sys}$ it is only possible to create delays in the starting times of activities, we should choose $\rho \geq \lambda^*$. If $\lambda^* > \varepsilon$ (in practical applications we even have $\lambda^* \geq 0$), then there exists an MPA invertible matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ such that[2] the matrix $\bar{A} = P^{\otimes^{-1}} \otimes A_{\rm sys} \otimes P$ satisfies $\bar{A}_{ij} \leq \lambda^*$ for all $i, j \in \underline{n}$ [5].

We make the following change of coordinates $\bar{x}(k) = P^{\otimes^{-1}} \otimes x_{\rm sys}(k)$. We denote with $\bar{B} = P^{\otimes^{-1}} \otimes B_{\rm sys}$ and $\bar{u}(k) = u_{\rm sys}(k)$. In the new coordinates the system (4) becomes:

$$\bar{x}(k) = \bar{A} \otimes \bar{x}(k-1) \oplus \bar{B} \otimes \bar{u}(k).$$

We now consider the normalized system with $x(k) = \bar{x}(k) - \rho k$, $u(k) = \bar{u}(k) - \rho k$, $A = \bar{A} - \rho$ (i.e. by subtracting in the conventional algebra all entries of $\bar{x}, \bar{u}$ and of $\bar{A}$ by ρk and ρ, respectively) and $B = \bar{B}$. The normalized system can be written as:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k). \tag{6}$$

The MPL system (6) is *controllable* if and only if (iff) each component of the state can be made arbitrarily large by applying an appropriate controller to the system initially at rest. It can be checked that the system is controllable iff the matrix $\Gamma_n := [B \;\; A \otimes B \cdots A^{\otimes^{n-1}} \otimes B]$ is row-finite (note that this definition is equivalent to the one given in [1], where the system is controllable if all states are connected to some input).

The following key assumption will be used throughout the paper:

Assumption A: We assume that $\rho > \lambda^* \geq 0$ and that the system is controllable.

The conditions of this assumption are quite weak and are usually met in applications (see also the previous discussion). Note that from Assumption **A** it follows that $A_{ij} < 0$, for all $i, j \in \underline{n}$. In the new coordinates the state should be regulated to the desired target $x_{\rm t} := P^{\otimes^{-1}} \otimes x_{\rm sys,t}$.

[1] In general there is also an output equation of the form $y(k) = C \otimes x(k)$, but in this paper we assume that all the states can be measured (i.e. $C = E_n$). Note however that the results of this paper can also be extended to take the output into account and to do output tracking instead of state tracking.

[2] Here, $P^{\otimes^{-1}}$ denotes the MPA inverse of P: $P^{\otimes^{-1}} \otimes P = P \otimes P^{\otimes^{-1}} = E_n$. In fact, P is the MPA diagonal matrix with on its diagonal the entries of the MPA eigenvector v^* corresponding to λ^*, and ε elsewhere, i.e. $P = \mathrm{diag}_\oplus(v^*)$. Moreover, $P^{\otimes^{-1}} = \mathrm{diag}_\oplus(-v^*)$.

Since $A_{ij} < 0$ for all $i, j \in \underline{n}$, we have $A^* = E_n \oplus A \oplus \cdots \oplus A^{\otimes^{n-1}}$ (see [1, Theorem 3.20]). Note that for any finite, constant input u there exists a state equilibrium x (i.e. $x = A \otimes x \oplus B \otimes u$), viz. $x = A^* \otimes B \otimes u$. Note that x is unique (according to Lemma 2.1 (ii)) and finite (since Γ_n is row-finite). We associate to x_t the largest[3] equilibrium pair (x_{el}, u_{el}) satisfying $x_{el} \leq x_t$. From the previous discussion it follows that (x_{el}, u_{el}) is unique, finite and given by:

$$u_{el} = (-(A^* \otimes B))^T \otimes' x_t, \, x_{el} = A^* \otimes B \otimes u_{el}. \tag{7}$$

3 Stabilizing MPL DES controllers for the unconstrained case

3.1 Stabilizing control for MPL DES

In this section we consider the normalized system (6), where the matrix A satisfies $A_{ij} < 0$, for all $i, j \in \underline{n}$ (according to Assumption **A**) and with the constraint that the original input signal (u_{sys}) should be nondecreasing, i.e.

$$u(k+1) - u(k) \geq -\rho, \;\; \forall k \geq 0. \tag{8}$$

Given a desired target $x_t \in \mathbb{R}^n$, let (x_{el}, u_{el}) be the largest equilibrium pair satisfying $x_{el} \leq x_t$ (cf. (7)). We define also an upper bound on x_t: $x_{ub} = A^* \otimes x_t \geq x_t$, $u_{ub} = (-B)^T \otimes' (A^* \otimes x_t)$. It is clear that these pairs are uniquely determined and finite. Note that $u_{el} \leq u_{ub}$ and whenever x_t is an equilibrium state (i.e. there exists a finite u_t such that $x_t = A \otimes x_t \oplus B \otimes u_t$) then $x_{el} = x_{ub} = x_t$ and consequently $u_{el} = u_{ub} = u_t$.

Definition 3.1 *Given a state feedback controller $\mu : \mathbb{R}^n_\varepsilon \to \mathbb{R}^m_\varepsilon$, then the closed-loop system $x(k) = A \otimes x(k-1) \oplus B \otimes \mu(x(k-1))$ is stable iff the state remains bounded, i.e. for every $\delta > 0$ there exists a real-valued function $\theta(\delta) > 0$ such that $\|x(0) - x_{el}\|_\infty \leq \delta$ implies $\|x(k) - x_{el}\|_\infty \leq \theta(\delta)$ for all $k \geq 0$.*

Now we formulate the control problem that we will solve in the sequel:

Problem 1: Design a state feedback controller $\mu : \mathbb{R}^n_\varepsilon \to \mathbb{R}^m_\varepsilon$ for the MPL system (6) such that the closed-loop system is stable.

3.2 Stabilizing state feedback controller

Assume we are at event step k. Given the previous[4] state $x(k-1)$ and input $u(k-1)$, we define two controllers: a feedback controller

$$u^f(k) := (-B^T) \otimes' (A \otimes x(k-1) \oplus B \otimes (u(k-1) - \rho) \oplus x_t) \tag{9}$$

and a "constant" controller:

$$u^c(k) := u_{el} \oplus (u(k-1) - \rho). \tag{10}$$

Later on, we will show that under some conditions the RHC controller lies between these two controllers. Let us now study the (stabilizing) properties of these two controllers. Note that $u^f(k)$ satisfies the constraint (8). Indeed, from (3) it follows that $u^f(k) \geq (-B^T) \otimes' (B \otimes (u(k-1) - \rho))$ and from (1) and (2) we conclude that $u^f(k) \geq u(k-1) - \rho$. Using similar arguments we can prove that $u^f(k) \geq u_{el}$, for all $k \geq 1$. Similarly, $u^c(k)$ satisfies the constraint (8) and $u^c(k) \geq u_{el}$, for all $k \geq 1$.

[3]By the largest we mean that any other feasible equilibrium pair (x, u) satisfies $x \leq x_{el}, u \leq u_{el}$.

[4]Timing aspects and the interplay between event steps and time steps are discussed in [12].

We study now the stabilizing properties of these two controllers. With the controller (9), the closed-loop normalized system (6) becomes

$$x^{\mathrm{f}}(k) = A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes u^{\mathrm{f}}(k), \tag{11}$$

where the initial conditions $x^{\mathrm{f}}(0) = x(0)$ and $u^{\mathrm{f}}(0) = u(0)$ are given. Note that $u^{\mathrm{c}}(k) = u_{\mathrm{el}} \oplus (u(0) - \rho k)$ and the corresponding closed-loop system, for $x^{\mathrm{c}}(0) = x(0)$ and $u^{\mathrm{c}}(0) = u(0)$ is given by:

$$x^{\mathrm{c}}(k) = A \otimes x^{\mathrm{c}}(k-1) \oplus B \otimes u^{\mathrm{c}}(k). \tag{12}$$

First let us note that:

$$\begin{cases} x^{\mathrm{f}}(k) \leq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus x_{\mathrm{t}} \\ x^{\mathrm{f}}(k) \geq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus B \otimes u_{\mathrm{el}}. \end{cases} \tag{13}$$

Indeed, from Lemma 2.1 we have $B \otimes u^{\mathrm{f}}(k) \leq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus x_{\mathrm{t}}$ and thus $x^{\mathrm{f}}(k) \leq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus x_{\mathrm{t}}$. The second inequality is straightforward (recall that $u^{\mathrm{f}}(k) \geq u^{\mathrm{f}}(k-1) - \rho$ and $u^{\mathrm{f}}(k) \geq u_{\mathrm{el}}$ and using the monotonicity property (3) it follows that $x^{\mathrm{f}}(k) \geq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus B \otimes u_{\mathrm{el}}$. The following inequality is also useful: since $x^{\mathrm{f}}(k-1) \geq B \otimes u^{\mathrm{f}}(k-1)$ it follows that

$$B \otimes (u^{\mathrm{f}}(k-1) - \rho) = (B \otimes u^{\mathrm{f}}(k-1)) - \rho \leq x^{\mathrm{f}}(k-1) - \rho \tag{14}$$

We have

Lemma 3.2 *The following inequalities are satisfied:*

$$u^{\mathrm{f}}(k) \geq u^{\mathrm{c}}(k) \ \textit{and} \ x^{\mathrm{f}}(k) \geq x^{\mathrm{c}}(k), \ \forall k \geq 0. \tag{15}$$

Proof: We prove the lemma by induction. For $k = 0$ we have $u^{\mathrm{f}}(0) = u^{\mathrm{c}}(0) = u(0)$ and $x^{\mathrm{f}}(0) = x^{\mathrm{c}}(0) = x(0)$. Let us assume that the inequalities of the lemma are valid for a given k. Now we prove that they also hold for $k+1$. Since $u^{\mathrm{f}}(\cdot)$ satisfies the constraint (8) and using our induction hypothesis we obtain

$$u^{\mathrm{f}}(k+1) \geq u^{\mathrm{f}}(k) - \rho \geq u^{\mathrm{c}}(k) - \rho$$

Moreover, $u^{\mathrm{f}}(k+1) \geq u_{\mathrm{el}}$. We conclude that $u^{\mathrm{f}}(k+1) \geq (u^{\mathrm{c}}(k) - \rho) \oplus u_{\mathrm{el}} = u^{\mathrm{c}}(k+1)$. Using again the induction hypothesis and the monotonicity property (3) it follows that:

$$\begin{aligned} x^{\mathrm{f}}(k+1) &\geq A \otimes x^{\mathrm{c}}(k) \oplus B \otimes u^{\mathrm{f}}(k+1) \geq \\ &A \otimes x^{\mathrm{c}}(k) \oplus B \otimes u^{\mathrm{c}}(k+1) = x^{\mathrm{c}}(k+1). \end{aligned}$$

This concludes the proof. □

Proposition 3.3 *The feedback controller* (9) *is the largest controller that guarantees the fastest single step decrease while fulfilling the constraint* (8).

Proof: We prove this proposition by contradiction. Given x and u let us assume that there exists a feasible $\bar{u}$ with the property

$$\bar{u} \geq u^{\mathrm{f}} \text{ and } \bar{u} \neq u^{\mathrm{f}} \tag{16}$$

such that $\bar{x} = A \otimes x \oplus B \otimes \bar{u} \leq x^{\mathrm{f}} = A \otimes x \oplus B \otimes u^{\mathrm{f}}$.
Since $\bar{u} \geq u^{\mathrm{f}}$, using the monotonicity property of max operator (3) it follows that $\bar{x} \geq x^{\mathrm{f}}$. In conclusion, $\bar{x} = x^{\mathrm{f}}$, i.e. we cannot have a larger decrease than x^{f}.
From (13) we have $B \otimes \bar{u} \leq \bar{x} = x^{\mathrm{f}} \leq A \otimes x \oplus B \otimes (u - \rho) \oplus x_{\mathrm{t}}$. Hence, $\bar{u} \leq (-B^T) \otimes' (A \otimes x \oplus B \otimes (u - \rho) \oplus x_{\mathrm{t}}) = u^{\mathrm{f}}$. We have obtained a contradiction with (16). □

The stabilizing properties of the two state feedback controllers are summarized in the next theorem:

Theorem 3.4 *The following statements hold:*

(i) For any initial condition $x^{\mathrm{f}}(0) = x(0)$ *and* $u^{\mathrm{f}}(0) = u(0)$ *there exists a finite* K^{f} *such that* $x^{\mathrm{f}}(k) \leq x_{\mathrm{ub}}$ *and* $u_{\mathrm{el}} \leq u^{\mathrm{f}}(k+1) \leq u_{\mathrm{ub}}$*, for all* $k \geq K^{\mathrm{f}}$.

(ii) For any initial condition $x^{\mathrm{c}}(0) = x(0)$ *and* $u^{\mathrm{c}}(0) = u(0)$ *there exists a finite* K^{c} *such that* $x^{\mathrm{c}}(k) = x_{\mathrm{el}}$ *and* $u^{\mathrm{c}}(k) = u_{\mathrm{el}}$*, for all* $k \geq K^{\mathrm{c}}$.

(iii) The closed-loop systems (11) *and* (12) *are stable.*

Proof: (i) From (13) and (14) it follows that: $x^{\mathrm{f}}(k) \leq A \otimes x^{\mathrm{f}}(k-1) \oplus B \otimes (u^{\mathrm{f}}(k-1) - \rho) \oplus x_{\mathrm{t}} \leq A \otimes x^{\mathrm{f}}(k-1) \oplus (x^{\mathrm{f}}(k-1) - \rho) \oplus x_{\mathrm{ub}}$. By induction it is straightforward to prove that:

$$x^{\mathrm{f}}(k) \leq \bigoplus_{t=0}^{k} (A^{\otimes^{k-t}} \otimes (x^{\mathrm{f}}(0) - t\rho)) \oplus x_{\mathrm{ub}}. \tag{17}$$

Recall that $A_{ij} < 0$ for all $i, j \in \underline{n}$. Then, it is well-known that [1]:

$$A^{\otimes^k} \otimes x^{\mathrm{f}}(0) \to \boldsymbol{\mathcal{E}}_{n\times 1} \text{ as } k \to \infty. \tag{18}$$

We denote with $z_0 = x^{\mathrm{f}}(0)$ and iteratively $z_k = \bigoplus_{t=0}^{k} (A^{\otimes^{k-t}} \otimes x^{\mathrm{f}}(0) - t\rho) = \max\{A^{\otimes^k} \otimes x^{\mathrm{f}}(0), z_{k-1} - \rho\}$. From (18) and $\rho > 0$ it follows that

$$z_k \to \boldsymbol{\mathcal{E}}_{n\times 1} \text{ as } k \to \infty. \tag{19}$$

Therefore, there exists a finite integer K^{f} such that $\bigoplus_{t=0}^{k} (A^{\otimes^{k-t}} \otimes (x^{\mathrm{f}}(0) - t\rho)) \leq x_{\mathrm{ub}}$ for any $k \geq K^{\mathrm{f}}$. In conclusion, $x^{\mathrm{f}}(k) \leq x_{\mathrm{ub}}$ for any $k \geq K^{\mathrm{f}}$.

Now consider k satisfying $k \geq K^{\mathrm{f}}$. Therefore, $x^{\mathrm{f}}(k) \leq x_{\mathrm{ub}}$. We obtain $A \otimes x^{\mathrm{f}}(k) \leq A \otimes x_{\mathrm{ub}} \leq x_{\mathrm{ub}}$. Similarly, from (14) we have $B \otimes (u^{\mathrm{f}}(k) - \rho) \leq x^{\mathrm{f}}(k) - \rho \leq x_{\mathrm{ub}}$. Using now (3) we obtain:

$$u^{\mathrm{f}}(k+1) \leq (-B^T) \otimes' x_{\mathrm{ub}} = u_{\mathrm{ub}}.$$

By induction, using the same procedure it follows that $u^{\mathrm{f}}(K^{\mathrm{f}} + l) \leq u_{\mathrm{ub}}$, for all $l \geq 1$. On the other hand $u^{\mathrm{f}}(k) \geq u_{\mathrm{el}}$ for all $k \geq 1$. We conclude that $u_{\mathrm{el}} \leq u^{\mathrm{f}}(K^{\mathrm{f}} + l) \leq u_{\mathrm{ub}}$, for all $l \geq 1$.

(ii) Since $\rho > 0$, $u^{\mathrm{c}}(k) = u_{\mathrm{el}}$ for k large enough. Note that

$$x^{\mathrm{c}}(k) = A^{\otimes^k} \otimes x^{\mathrm{c}}(0) \oplus (\bigoplus_{t=1}^{k} A^{\otimes^{k-t}} \otimes B \otimes (u^{\mathrm{c}}(0) - t\rho)) \oplus (\bigoplus_{t=1}^{k} A^{\otimes^{k-t}} \otimes B \otimes u_{\mathrm{el}}).$$

From (18) we have $A^{\otimes^k} \otimes x^{\mathrm{c}}(0) \to \boldsymbol{\mathcal{E}}_{n\times 1}$ as $k \to \infty$. So, $\bigoplus_{t=1}^{k} A^{\otimes^{k-t}} \otimes B \otimes (u^{\mathrm{c}}(0) - t\rho) \to \boldsymbol{\mathcal{E}}_{n\times 1}$ as $k \to \infty$ (this can be proved in a similar way as (19)). Since $x_{\mathrm{el}} = \bigoplus_{t=1}^{n} A^{\otimes^{n-t}} \otimes B \otimes u_{\mathrm{el}}$, it follows that there exists a $K^{\mathrm{c}} \geq n$ such that $x^{\mathrm{c}}(k) = x_{\mathrm{el}}$ and $u^{\mathrm{c}}(k) = u_{\mathrm{el}}$, for all $k \geq K^{\mathrm{c}}$.

(iii) Let us now prove stability of the closed-loop systems (11) and (12). Let $\delta > 0$ and consider $\|x(0) - x_{\mathrm{el}}\|_\infty \leq \delta$.

From $u^{\mathrm{c}}(k) \geq u_{\mathrm{el}}$ it follows that $x^{\mathrm{c}}(k) \geq x_{\mathrm{el}}$ for all $k \geq n$.

Since the system is controllable (by Assumption **A**), for any $1 \leq k \leq n-1$ and for any index $i \in \underline{n}$, one of the two following conditions are satisfied:

$$x_i^{\mathrm{c}}(k) \geq B_{ij} + (u_{\mathrm{el}})_j, \;\text{ with } B_{ij} \neq \varepsilon \tag{20}$$

$$\exists\, p \in \underline{n} \text{ s. t. } x_i^{\mathrm{c}}(k) \geq (A^{\otimes^p})_{lj} + x_j^{\mathrm{c}}(k-p), \;\text{ with} (A^{\otimes^p})_{lj} \neq \varepsilon. \tag{21}$$

Note that $x_j^{\mathrm{c}}(k-p)$ is either equal to $x_j^{\mathrm{c}}(0)$ or satisfies (20).
Hence, for *any* $k \geq 0$ and for *any* index $i \in \underline{n}$ we have $(x_{\mathrm{el}} - x^{\mathrm{c}}(k))_i \leq \theta_1(\delta) := \max\big\{0, (x_{\mathrm{el}})_{i_1} - B_{i_1 j} - (u_{\mathrm{el}})_j, (x_{\mathrm{el}})_{i_2} - (A^{\otimes^p})_{li_1} - x_{i_1}(0), (x_{\mathrm{el}})_{i_3} - (A^{\otimes^p})_{li_1} - B_{i_1 j} - (u_{\mathrm{el}})_j\big\}$ for some indices i_1, i_2, i_3, j.
So from $x^{\mathrm{c}}(k) \leq x^{\mathrm{f}}(k) \leq z_k \oplus x_{\mathrm{ub}}$ it follows that:

$$\begin{aligned}
\|x^{\mathrm{f}}(k) - x_{\mathrm{el}}\|_\infty &= \max_{i \in \underline{n}} \{(x^{\mathrm{f}}(k) - x_{\mathrm{el}})_i, (x_{\mathrm{el}} - x^{\mathrm{f}}(k))_i\} \\
&\leq \max_{i \in \underline{n}} \{((z_k \oplus x_{\mathrm{ub}}) - x_{\mathrm{el}})_i, (x_{\mathrm{el}} - x^{\mathrm{c}}(k))_i\} \\
&\leq \max_{i \in \underline{n}} \{(z_k - x_{\mathrm{el}})_i, (x_{\mathrm{ub}} - x_{\mathrm{el}})_i, \theta_1(\delta)\} \\
&\leq \max_{i \in \underline{n}} \{(z_k - x_{\mathrm{el}})_i, \theta_2(\delta)\} \\
&\leq \max_{i,j} \{(A^{\otimes^j} \otimes x(0) - (k-j)\rho - x_{\mathrm{el}})_i, \theta_2(\delta)\} \\
&\leq \max_{i,j} \{(A^{\otimes^j} \otimes x(0) - (k-j)\rho - A^{\otimes^j} \otimes x_{\mathrm{el}})_i, \theta_2(\delta)\} \\
&\leq \max_{i,j} \{(A^{\otimes^j} \otimes x(0) - A^{\otimes^j} \otimes x_{\mathrm{el}})_i, \theta_2(\delta)\} \\
&\leq \max_{i} \{(x(0) - x_{\mathrm{el}})_i, \theta_2(\delta)\} \leq \theta(\delta)
\end{aligned}$$

with $\theta_2(\delta) = \max\{\max_{i \in \underline{n}} (x_{\mathrm{ub}} - x_{\mathrm{el}})_i, \theta_2(\delta)\}$ and $\theta(\delta) = \max\{\delta, \theta_1(\delta)\}$, and where for the last transition we have used that fact that from standard properties of the max operator[5] it follows that: $a^T \otimes x - a^T \otimes y \leq \|x - y\|_\oplus$, for any $a \in \mathbb{R}_\varepsilon^n$ and $x, y \in \mathbb{R}^n$. □

An immediate consequence of Theorem 3.4 is:

Proposition 3.5 *For any input signal $u(\cdot)$ fulfilling the constraint* (8) *and $u^{\mathrm{c}}(k) \leq u(k) \leq u^{\mathrm{f}}(k)$, the corresponding trajectory satisfies $x^{\mathrm{c}}(k) \leq x(k) \leq x^{\mathrm{f}}(k)$, for all k and consequently $u(\cdot)$ is stabilizing. Moreover, there exists a finite K such that $x_{\mathrm{el}} \leq x(k) \leq x_{\mathrm{ub}}$, for all $k \geq K$.*

3.3 Stabilizing receding horizon controller

Given the state and the input at event step $k-1$, the following cost function is introduced:

$$J(x(k-1), \tilde{u}(k)) = \sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{x_i(k+j|k-1) - (x_{\mathrm{t}})_i, 0\} - \beta \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i(k+j|k-1).$$

where N is the prediction horizon, $x(k+j|k-1)$ is the system state at event step $k+j$ as predicted at event step $k-1$, based on the MPL difference equation (6), the state $x(k-1)$ and the future input sequence

$$\tilde{u}(k) = [u^T(k|k-1) \cdots u^T(k+N-1|k-1)]^T.$$

[5] Recall that by definition $\varepsilon - \varepsilon = \varepsilon$.

In the context of DES the first term of J expresses the tardiness (i.e. the delay with respect to the desired due date target x_t), while the second term maximizes the feeding times. We define the following receding horizon control (RHC) based optimization problem:

$$J^*(x(k-1)) = \min_{\tilde{u}(k)} J(x(k-1), \tilde{u}(k)) \tag{22}$$

$$\text{s.t.} \begin{cases} x(k+j|k-1) = A \otimes x(k+j-1|k-1) \oplus B \otimes u(k+j|k-1) \\ u(k+j|k-1) - u(k+j-1|k-1) \geq -\rho \end{cases} \quad \forall j \in \{0, \cdots, N-1\}. \tag{23}$$

where $x(k-1|k-1) = x(k-1), u(k-1|k-1) = u(k-1)$. Let $\tilde{u}^\natural(k)$ be the optimal solution of the optimization problem (22)–(23). Using the receding horizon principle at event counter k we apply the input $u^{\mathrm{RHC,N}}(k) = u^\natural(k|k-1)$. The evolution of the closed-loop system obtained from applying the receding horizon controller is denoted with

$$x^{\mathrm{RHC,N}}(k) = A \otimes x^{\mathrm{RHC,N}}(k-1) \oplus B \otimes u^{\mathrm{RHC,N}}(k),$$

with given initial conditions $x^{\mathrm{RHC,N}}(0) = x(0)$, $u^{\mathrm{RHC,N}}(k) = u(0)$.

Let us define the matrices

$$\tilde{D} = \begin{bmatrix} B & \varepsilon & \cdots & \varepsilon \\ A \otimes B & B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ A^{\otimes^{N-1}} \otimes B & A^{\otimes^{N-2}} \otimes B & \cdots & B \end{bmatrix}, \tilde{C} = \begin{bmatrix} A \\ A^{\otimes^2} \\ \vdots \\ A^{\otimes^N} \end{bmatrix}$$

and the vectors $\bar{u}(k) = [u^T(k-1) - \rho \cdots u^T(k-1) - N\rho]^T$, $\bar{u}_\mathrm{el} = [u_\mathrm{el}^T \cdots u_\mathrm{el}^T]^T$, $\bar{x}_\mathrm{t} = [x_\mathrm{t}^T \cdots x_\mathrm{t}^T]^T$ and $\bar{x}(k) = [\bar{x}^T(k|k-1) \cdots \bar{x}^T(k+N-1|k-1)]^T = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \bar{u}(k) \oplus \bar{x}_\mathrm{t}$.

Now we give some properties of the receding horizon controller. The next lemma shows that the receding horizon controller $u^{\mathrm{MPC,N}}(\cdot)$ is bounded from below by $u^\mathrm{c}(\cdot)$:

Lemma 3.6 $u^\mathrm{c}(k) \leq u^{\mathrm{RHC,N}}(k)$, $x^\mathrm{c}(k) \leq x^{\mathrm{RHC,N}}(k)$, $\forall k \geq 0$.

Proof: First, let us show that $\tilde{u}^\natural(k) \geq \bar{u}_\mathrm{el}$. The states corresponding to $\tilde{u}^\natural(k)$ are given by $\tilde{x}^\natural(k) = [(x^\natural(k|k-1))^T \cdots (x(k+N-1|k-1))^T]^T = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^\natural(k)$. Let us assume that $\tilde{u}^\natural(k) \ngeq \bar{u}_\mathrm{el}$. Define $\tilde{u}^{\mathrm{feas}}(k) = \tilde{u}^\natural(k) \oplus \bar{u}_\mathrm{el}$ and $\tilde{x}^{\mathrm{feas}}(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^{\mathrm{feas}}(k)$. Note that $\tilde{u}^{\mathrm{feas}}(k)$ is feasible for the optimization problem (22)–(23). Since $A^{\otimes^j} \otimes B \otimes u_\mathrm{el} \leq x_\mathrm{t}$ for all j, $\tilde{x}^{\mathrm{feas}}(k) = \tilde{x}^\natural(k) \oplus \tilde{D} \otimes \bar{u}_\mathrm{el} \leq \tilde{x}^\natural(k) \oplus \bar{x}_\mathrm{t}$. It follows that:

$$\begin{aligned} J(x(k-1), \tilde{u}^{\mathrm{feas}}(k)) &\leq \sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{x_i^\natural(k+j|k-1) - (x_\mathrm{t})_i, 0\} - \beta \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\mathrm{feas}}(k+j|k-1) \\ &< \sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{x_i^\natural(k+j|k-1) - (x_\mathrm{t})_i, 0\} - \\ &\qquad \beta \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^\natural(k+j|k-1) = J(x(k-1), \tilde{u}^\natural(k)) \end{aligned}$$

and thus we get contradiction with the optimality of $\tilde{u}^\natural(k)$.

Now we go on with the proof of the lemma using induction. For $k=0$, we have $u^\mathrm{c}(0) = u^{\mathrm{RHC,N}}(0) = u(0)$ and $x^\mathrm{c}(0) = x^{\mathrm{RHC,N}}(0) = x(0)$. We assume that $u^\mathrm{c}(k-1) \leq u^{\mathrm{RHC,N}}(k-1)$ and

$x^{\mathrm{c}}(k-1) \leq x^{\mathrm{RHC,N}}(k-1)$ and we prove that these inequalities also hold for k. From the induction hypothesis we have $u^{\mathrm{RHC,N}}(k) \geq u^{\mathrm{RHC,N}}(k-1) - \rho \geq u^{\mathrm{c}}(k-1) - \rho$. Moreover, $u^{\mathrm{RHC,N}}(k) = u^{\natural}(k|k-1) \geq u_{\mathrm{el}}$. It follows that $u^{\mathrm{RHC,N}}(k) \geq (u^{\mathrm{c}}(k-1) - \rho) \oplus u_{\mathrm{el}} = u^{\mathrm{c}}(k)$. From (3) it follows that $x^{\mathrm{c}}(k) = A \otimes x^{\mathrm{c}}(k-1) \oplus B \otimes u^{\mathrm{c}}(k) \leq A \otimes x^{\mathrm{RHC,N}}(k-1) \oplus B \otimes u^{\mathrm{RHC,N}}(k) = x^{\mathrm{RHC,N}}(k)$. $\square$

Proposition 3.7 *Assume $\beta < \frac{1}{mN}$ and consider the maximization problem*

$$\tilde{u}^{\sharp}(k) = \arg\max_{\tilde{u}(k)} \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i(k+j|k-1) \tag{24}$$

$$\text{s.t.} \begin{cases} \tilde{D} \otimes \tilde{u}(k) \leq \bar{x}(k) \\ u(k+j|k-1) - u(k+j-1|k-1) \geq -\rho \end{cases} \quad \forall j \in \{1, \cdots, N-1\}. \tag{25}$$

Then $\tilde{u}^{\sharp}(k)$ is also the optimal solution of the optimization problem (22)–(23).

Proof: First let us note that we do not have to impose also the constraint $u(k|k-1) - u(k-1|k-1) \geq -\rho$ in (25). This inequality is automatically satisfied, since $\bar{u}(k)$ is a feasible solution for (22)–(23) and consequently a feasible solution for the optimization problem (24)–(25) and thus $\bar{u}(k) \leq \tilde{u}^{\sharp}(k)$.

We will prove the lemma by contradiction. Define $\bar{x}^{\mathrm{c}}(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \bar{u}(k)$ then $\bar{x}(k) = \bar{x}^{\mathrm{c}}(k) \oplus \bar{x}_{\mathrm{t}}$. First let us consider an $\tilde{u}^{\flat}(k)$ that satisfies (25) but for which

$$\sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\flat}(k+j|k-1) < \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\sharp}(k+j|k-1).$$

Define $\tilde{x}^{\flat}(k) = \tilde{C} \otimes x(k-1) \oplus \tilde{D} \otimes \tilde{u}^{\flat}(k)$. Then, for each $i \in \underline{n}$ and $j \in \{0, 1, \cdots, N-1\}$ it follows that

$$\begin{aligned} \max\{x_i^{\flat}(k+j|k-1), (x_{\mathrm{t}})_i\} &= \max\{x_i^{\mathrm{c}}(k+j|k-1), \tilde{D}_{jn+i} \otimes \tilde{u}^{\flat}, (x_{\mathrm{t}})_i\} \\ &= \max\{\bar{x}_i(k+j|k-1), \tilde{D}_{jn+i} \otimes \tilde{u}^{\flat}\} \\ &= \bar{x}_i(k+j|k-1), \end{aligned}$$

where $\tilde{D}_{jn+i}$ denotes the $(jn+i)$-th row of $\tilde{D}$. In conclusion, we obtain that

$$\begin{aligned} J(x(k-1), \tilde{u}^{\flat}(k)) &= \sum_{j=0}^{N-1} \sum_{i=1}^{n} (\bar{x}_i(k+j|k-1) - (x_{\mathrm{t}})_i) - \beta \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\flat}(k+j) \\ &> \sum_{j=0}^{N-1} \sum_{i=1}^{n} (\bar{x}_i(k+j|k-1) - (x_{\mathrm{t}})_i) - \beta \sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\sharp}(k+j) = J(x(k-1), \tilde{u}^{\sharp}(k)) \end{aligned}$$

and thus $\tilde{u}^{\flat}(k)$ cannot be the optimizer of (22)–(23).

Next let us consider $\tilde{u}^{\dagger}(k)$ that satisfies (23) but does not satisfy the inequality $\tilde{D}\tilde{u} \leq \bar{x}(k)$. Define

$$\delta = \max_{i \in \underline{n}, j \in \{0, \cdots, N-1\}} \{\tilde{D}_{jn+i} \otimes \tilde{u}^{\dagger}(k) - \bar{x}_i(k+j|k-1)\} > 0.$$

Then, there exist i_0, j_0 such that

$$x_{i_0}^{\dagger}(k+j_0|k-1) = \tilde{D}_{j_0 n+i_0} \otimes \tilde{u}^{\dagger}(k) = \bar{x}_{i_0}(k+j_0|k-1) + \delta$$

and thus

$$\sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{x_i^{\dagger}(k+j|k-1) - (x_{\mathrm{t}})_i, 0\} \geq \sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{\bar{x}_i(k+j|k-1) - (x_{\mathrm{t}})_i, 0\} + \delta$$

Note that $\tilde{u}^{\ddagger}(k) = (\tilde{u}^{\dagger}(k) - \delta) \oplus \bar{u}(k)$ fulfills (25) and the corresponding cost satisfies:

$$\begin{aligned}
J(x(k-1), \tilde{u}^{\ddagger}(k)) \leq & \sum_{j=0}^{N-1} \sum_{i=1}^{n} \max\{x_i^{\dagger}(k+j|k-1) - (x_{\mathrm{t}})_i, 0\} - \delta - \\
& \beta(\sum_{j=0}^{N-1} \sum_{i=1}^{m} u_i^{\dagger}(k+j|k-1) - Nm\delta) \\
= & J(x(k-1), \tilde{u}^{\dagger}(k)) + (\beta Nm - 1)\delta \\
< & J(x(k-1), \tilde{u}^{\dagger}(k))
\end{aligned}$$

and thus $\tilde{u}^{\dagger}(k)$ cannot be the optimizer of (22)–(23). This proves that $\tilde{u}^{\sharp}(k)$ is also the optimizer of the original optimization problem (22)–(23). $\square$

Define $\tilde{u}^{*N}(k) := (-\tilde{D}^T) \otimes' \bar{x}(k)$. The following proposition provides an analytic solution to the optimization problem (24)–(25).

Proposition 3.8 *The optimization problem* (24)–(25) *has an unique solution given by:*

$$\begin{cases} u^{\sharp}(k+N-1|k-1) = u^{*N}(k+N-1|k-1) \\ u^{\sharp}(k+j|k-1) = \min\{u^{*N}(k+j|k-1), \\ u^{\sharp}(k+j+1|k-1) + \rho\}, \end{cases} \tag{26}$$

for $j = N-2, \cdots, 0$.

Proof: The feasibility conditions (25) for $u^{\sharp}(k+N-1|k-1)$ are given by:

$$B \otimes u^{\sharp}(k+N-1|k-1) \leq \bar{x}(k+N-1|k-1)$$

and from Lemma 2.1 is clear that the largest $u^{\sharp}(k+N-1|k-1)$ is given by $u^{\sharp}(k+N-1|k-1) = u^{*N}(k+N-1|k-1) = (-B^T) \otimes' \bar{x}(k+N-1|k-1)$.

From the feasibility conditions (25), $u^{\sharp}(k+N-2|k-1)$ has to satisfy:

$$\begin{aligned}
& A \otimes B \otimes u^{\sharp}(k+N-2|k-1) \leq \bar{x}(k+N-1|k-1) \\
& B \otimes u^{\sharp}(k+N-2|k-1) \leq \bar{x}(k+N-2|k-1) \\
& u^{\sharp}(k+N-2|k-1) \leq u^{\sharp}(k+N-1|k-1) + \rho
\end{aligned}$$

and thus the largest $u^{\sharp}(k+N-2|k-1)$ is given by $u^{\sharp}(k+N-2|k-1) = \min\{u^{*N}(k+N-2|k-1), u^{\sharp}(k+N-1|k-1) + \rho\}$. Using the same reasoning we obtain:

$$\begin{aligned}
& u^{\sharp}(k+j|k-1) \leq u^{*N}(k+j|k-1), \\
& u^{\sharp}(k+j|k-1) \leq u^{\sharp}(k+j+1|k-1) + \rho,
\end{aligned}$$

for all j, i.e. (26). $\square$

We prove now a lemma that will be useful in the sequel:

Lemma 3.9 *Any feasible solution* $\tilde{u}_{\mathrm{feas}}(k)$ *of* (24)–(25) *satisfies* $\tilde{u}_{\mathrm{feas}}(k) \leq \tilde{u}^{\sharp}(k)$.

Proof : From Lemma 2.1 and $\tilde{D} \otimes \tilde{u}_{\text{feas}}(k) \leq \bar{x}(k)$ it follows that $\tilde{u}_{\text{feas}}(k) \leq \tilde{u}^{*N}(k)$ and thus $u_{\text{feas}}(k+N-1|k-1) \leq u^{*N}(k+N-1|k-1) = u^{\sharp}(k+N-1|k-1)$. Note that $u_{\text{feas}}(k+N-2|k-1)$ satisfies

$$u_{\text{feas}}(k+N-2|k-1) \leq u^{*N}(k+N-2|k-1),$$
$$u_{\text{feas}}(k+N-2|k-1) \leq u_{\text{feas}}(k+N-1|k-1)+\rho \leq u^{\sharp}(k+N-1|k-1)+\rho.$$

In conclusion $u_{\text{feas}}(k+N-2|k-1) \leq \min\{u^{*N}(k+N-2|k-1), u^{\sharp}(k+N-1|k-1)+\rho\} = u^{\sharp}(k+N-2|k-1)$. Applying this reasoning backwards, we obtain $u_{\text{feas}}(k+j|k-1) \leq u^{\sharp}(k+j|k-1)$ for all $j = N-1, \cdots, 0$, i.e. $\tilde{u}_{\text{feas}}(k) \leq \tilde{u}^{\sharp}(k)$. □

The next theorem characterizes the stabilizing properties of the receding horizon controller:

Theorem 3.10 *Given a prediction horizon N such that $\beta < \frac{1}{mN}$, then*
(i) The following inequalities hold

$$\begin{cases} u^{\text{c}}(k) \leq u^{\text{RHC,N}}(k) \leq u^{\text{f}}(k) \\ x^{\text{c}}(k) \leq x^{\text{RHC,N}}(k) \leq x^{\text{f}}(k) \end{cases} \tag{27}$$

and thus the receding horizon controller stabilizes the system (6).
(ii) If $N = 1$ then $u^{\text{RHC,1}}(k) = u^{\text{f}}(k)$. For two prediction horizons $N_1 < N_2$ we have

$$\begin{cases} u^{\text{RHC,N}_1}(k) \geq u^{\text{RHC,N}_2}(k) \\ x^{\text{RHC,N}_1}(k) \geq x^{\text{RHC,N}_2}(k) \end{cases} \tag{28}$$

Proof : (i) The left-hand side of inequalities (27) follows from Lemma 3.6.

The right-hand side of inequalities (27) is proved using induction. For $k = 0$ we have $u^{\text{RHC,N}}(0) = u^{\text{f}}(0) = u(0)$ and $x^{\text{RHC,N}}(0) = x^{\text{f}}(0) = x(0)$. Let us assume that $u^{\text{RHC,N}}(k-1) \leq u^{\text{f}}(k-1)$ and $x^{\text{RHC,N}}(k-1) \leq x^{\text{f}}(k-1)$ are valid and we prove that they also hold for k. Since $x(k|k-1) = A \otimes x(k-1) \oplus B \otimes (u(k-1)-\rho) \oplus x_{\text{t}}$ and $B \otimes u^{\sharp}(k|k-1) \leq \bar{x}(k|k-1)$, we have

$$u^{\text{RHC,N}}(k) \leq (-B^T) \otimes' (A \otimes x(k-1) \oplus B \otimes (u(k-1)-\rho) \oplus x_{\text{t}}) \tag{29}$$

From (29) and our induction hypothesis we have:

$$\begin{aligned} B \otimes u^{\text{RHC,N}}(k) &\leq A \otimes x^{\text{RHC,N}}(k-1) \oplus B \otimes (u^{\text{RHC,N}}(k-1)-\rho) \oplus x_{\text{t}} \\ &\leq A \otimes x^{\text{f}}(k-1) \oplus B \otimes (u^{\text{f}}(k-1)-\rho) \oplus x_{\text{t}} \end{aligned}$$

On the other hand, $u^{\text{f}}(k)$ is the largest solution of the inequality

$$B \otimes u^{\text{f}}(k) \leq A \otimes x^{\text{f}}(k-1) \oplus B \otimes (u^{\text{f}}(k-1)-\rho) \oplus x_{\text{t}}$$

From Lemma 2.1 it follows that $u^{\text{RHC,N}}(k) \leq u^{\text{f}}(k)$. Then,

$$x^{\text{RHC,N}}(k) = A \otimes x^{\text{RHC,N}}(k-1) \oplus B \otimes u^{\text{RHC,N}}(k) \leq A \otimes x^{\text{f}}(k-1) \oplus B \otimes u^{\text{f}}(k) = x^{\text{f}}(k+1).$$

The stabilizing property of the receding horizon controller follows from Proposition 3.5.

(ii) For $N = 1$ from the feasibility condition (25) it is clear that $\tilde{u}^{\sharp}(k) = u^{\sharp}(k) = u^{\text{f}}(k)$ (see (9)). For two prediction horizons $N_1 < N_2$, we denote with $\tilde{D}_{(N_1)}$ the matrix $\tilde{D}$ from (25) corresponding to the prediction horizon $N = N_1$. Similarly, we define $\tilde{D}_{(N_2)}$. Note that $\tilde{D}_{(N_2)} = \begin{bmatrix} \tilde{D}_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix}$ (where $*$ stands for appropriate matrix blocks). We denote with $\bar{x}_{(N_1)}(k)$ the vector $\bar{x}(k)$ from (25) corresponding to $N = N_1$ and $\tilde{u}^{\sharp}_{(N_1)}(k)$ the optimal solution of (24)–(25) corresponding to N_1. Similarly, we define $\bar{x}_{(N_2)}(k)$ and $\tilde{u}^{\sharp}_{(N_2)}(k)$.

We prove the inequalities (28) by induction. For $k = 0$ the statement is true: $u^{\mathrm{RHC,N_1}}(0) = u^{\mathrm{RHC,N_2}}(0) = u(0)$ and $x^{\mathrm{RHC,N_1}}(0) = x^{\mathrm{RHC,N_2}}(0) = x(0)$. Let us now assume that $u^{\mathrm{RHC,N_1}}(k-1) \geq u^{\mathrm{RHC,N_2}}(k-1)$ and $x^{\mathrm{RHC,N_1}}(k-1) \geq x^{\mathrm{RHC,N_2}}(k-1)$. Define $\tilde{u}^{\sharp}_{(N_2)}(k : k+N_1-1)$ the sub-vector of $\tilde{u}^{\sharp}_{(N_2)}(k)$ containing the first mN_1 components. We have:

$$
\begin{aligned}
\tilde{D}_{(N_2)} \otimes \tilde{u}^{\sharp}_{(N_2)}(k) &= \begin{bmatrix} \tilde{D}_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix} \otimes \begin{bmatrix} \tilde{u}^{\sharp}_{(N_2)}(k : k+N_1-1) \\ * \end{bmatrix} \\
&\leq \bar{x}_{(N_2)}(k) = \begin{bmatrix} \bar{x}_{(N_2)}(k : k+N_1-1) \\ * \end{bmatrix} \\
&\leq \begin{bmatrix} \bar{x}_{(N_1)}(k) \\ * \end{bmatrix}
\end{aligned}
$$

It follows that $\tilde{D}_{(N_1)} \otimes \tilde{u}^{\sharp}_{(N_2)}(k : k+N_1-1) \leq \bar{x}_{(N_1)}(k)$, i.e $\tilde{u}^{\sharp}_{(N_2)}(k : k+N_1-1)$ is feasible for (24)–(25) with prediction horizon $N = N_1$. From Lemma 3.9 we obtain that $\tilde{u}^{\sharp}_{(N_2)}(k : k+N_1-1) \leq \tilde{u}^{\sharp}_{(N_1)}(k)$. Therefore, $u^{\mathrm{RHC,N_1}}(k) = u^{\sharp}_{(N_1)}(k|k-1) \geq u^{\sharp}_{(N_2)}(k|k-1) = u^{\mathrm{RHC,N_2}}(k)$. Similarly, we have $x^{\mathrm{RHC,N_1}}(k) \geq x^{\mathrm{RHC,N_2}}(k)$. □

4 Example

We consider the following system:

$$
x_{\mathrm{sys}}(k) = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes x_{\mathrm{sys}}(k-1) \oplus \begin{bmatrix} 2 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u_{\mathrm{sys}}(k) \tag{30}
$$

For this example the system matrix A_{sys} has a (unique) MPA eigenvalue $\lambda^* = 2.5$, and a corresponding MPA eigenvector $v^* = [0\ 1.5\ 2\ 1.5]^T$. We consider the due date signal $r_{\mathrm{sys}}(k) = [17\ 15\ 1\ 10]^T + 4.5k$ (so $\rho = 4.5$), and the initial condition $x_{\mathrm{sys}}(0) = [20\ \ 31.5\ \ 42\ \ 51.5]^T$ and $u_{\mathrm{sys}}(0) = 20$.

First we construct the normalized system corresponding to (30). We have

$$
P = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1.5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1.5 \end{bmatrix}, A = \begin{bmatrix} \varepsilon & -2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -2 \\ -2 & \varepsilon & \varepsilon & \varepsilon \end{bmatrix},
$$

$$
B = \begin{bmatrix} 2 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}, x(0) = \begin{bmatrix} 20 \\ 30 \\ 40 \\ 50 \end{bmatrix}, u(0) = 20, x_{\mathrm{t}} = \begin{bmatrix} 17 \\ 13.5 \\ -1 \\ 8.5 \end{bmatrix}.
$$

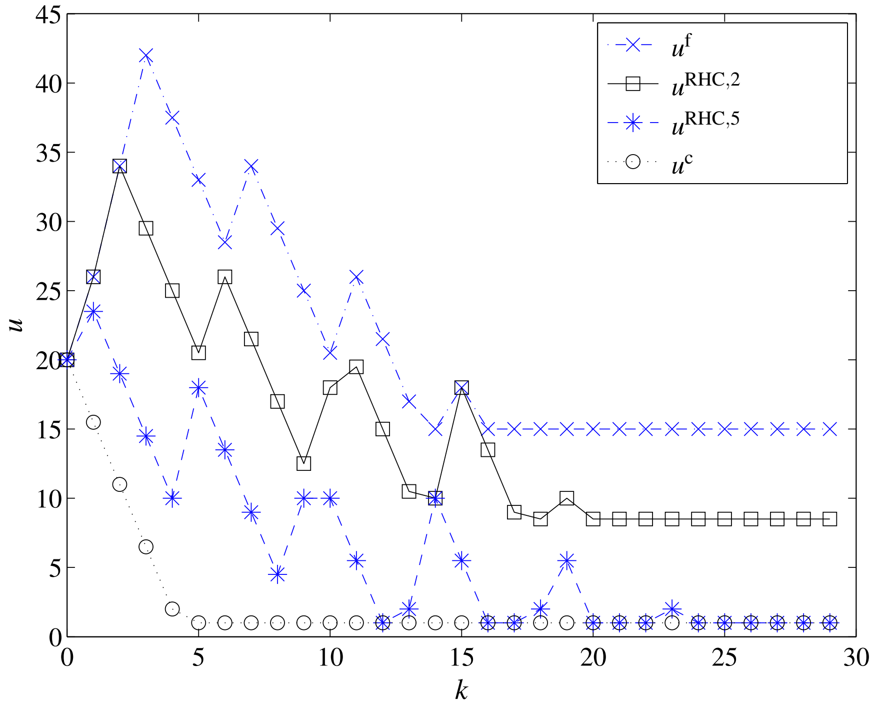

Figure 1: The state feedback, "constant" and RHC control signals for the normalized system.

Furthermore, we obtain $x_{\mathrm{el}} = [3\ -3\ -1\ 1], u_{\mathrm{el}} = 1$. It is easy to verify that the system and reference signal defined above satisfy Assumption **A**.

Now we design stabilizing state feedback and receding horizon controllers for this system. For the RHC controllers we consider the prediction horizons $N = 2$ and $N = 5$, and a weight $\beta = 0.1$ that satisfies the conditions of Proposition 3.7 and Theorem 3.10. In Figures 1 and 2 we have plotted respectively the control signals and the state trajectories for the closed-loop controlled normalized system. Clearly, all controllers are stabilizing. Moreover, the "constant" controller and the RHC controller with $N = 5$ also make all states less than the target states. This is not always the case for the state feedback controller and for the RHC controller with $N = 2$ (so in the latter case the prediction horizon is clearly selected too small). Also note that $u^{\mathrm{c}}(k) \leq u^{\mathrm{RHC,N}}(k) \leq u^{\mathrm{f}}(k)$ for all k and for $N = 2,5$. Furthermore, $u^{\mathrm{RHC,5}}(k) \leq u^{\mathrm{RHC,2}}(k)$ and $x^{\mathrm{RHC,5}}(k) \leq x^{\mathrm{RHC,2}}(k)$ for all k (cf. Theorem 3.10 (ii)).

5 Conclusions and future research

In this paper we have discussed the problem of stabilization of max-plus-linear (MPL) discrete-event systems. We have defined a stabilizing "constant" controller and a stabilizing state feedback controller that could be considered as a lower and upper bound respectively for the receding horizon control (RHC) controllers. For the RHC controllers we have considered a trade-off between minimizing the tardiness and maximizing the input times. Using only the constraint that expresses that the input signal should be nondecreasing and provided the trade-off weight is small enough, we have derived an analytic expression for the RHC controller and proved that stability can be achieved in finite time. We have also discussed also the main properties of the stabilizing state feedback, "constant", and RHC

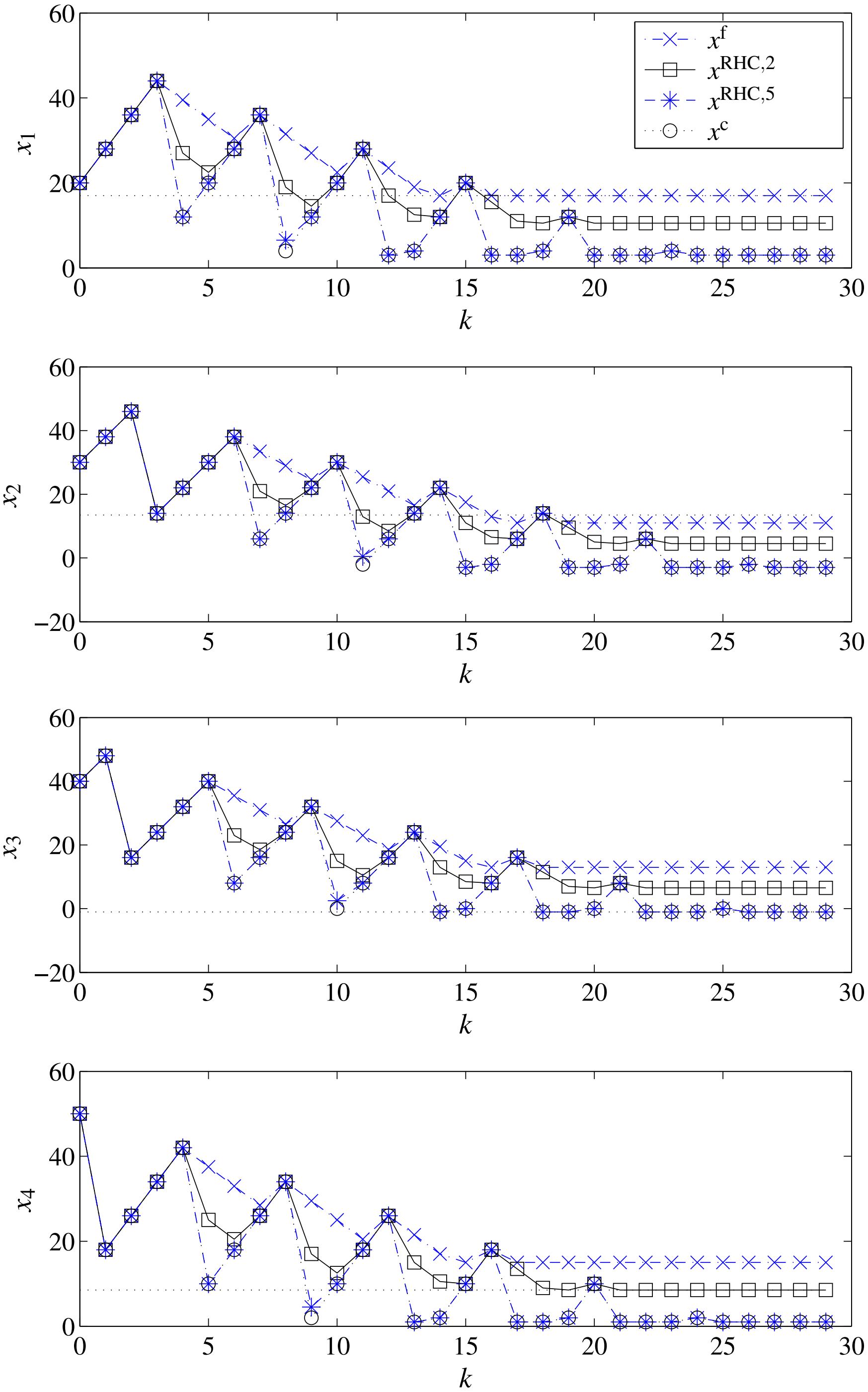

Figure 2: The evolution of the states for the state feedback, "constant" and RHC controllers for the normalized system. The dotted lines correspond to the target state x_{t}.

controllers.

Topics for future research include: extension to the "constrained" case where other constraints on the inputs and states are present, and extension to the case where disturbances are present.

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