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MPC of Implicit Switching Max-Plus-Linear Discrete Event Systems – Timing Aspects

Ton J.J. van den Boom and Bart De Schutter

***Abstract*— In this paper we consider the control of discrete event systems that can be modeled as implicit switching max-plus-linear systems. In switching max-plus-linear systems we can switch between different modes of operation. In each mode the discrete event system is described by an implicit max-plus-linear state space model with different system matrices for each mode. The switching allows us to change the structure of the system, to break synchronization and to change the order of events. We introduce implicit switching max-plus-linear systems, and explain how model predictive control (MPC) can be applied to them. Next, we discuss the timing aspects of MPC for this type of discrete event systems.**

I. Introduction

In general, models that describe the behavior of a discrete event system are nonlinear in conventional algebra. However, there is a class of discrete event systems — the max-plus-linear discrete event systems — that can be described by a model that is "linear" in the max-plus algebra [1], [5], which has maximization and addition as its basic operations. The max-plus-linear discrete event systems can be characterized as the class of discrete event systems in which only synchronization and no concurrency or choice occurs.

In the literature on control for max-plus linear discrete event systems [3], [13] usually explicit input-output models or explicit state space models are used. However, in this paper we will consider implicit models in which the current state appears both at the left-hand side and the right-hand side of the state update equation, as this will offer some advantages when considering time-varying models (due to new measurements that become available) and when considering timing aspects of MPC for max-plus linear discrete event systems.

In [15], [16] switching max-plus-linear systems were introduced. The discrete event system can switch between different modes of operation, such that in each mode the system is described by an implicit max-plus-linear state space model instead of by an explicit model as in [15], [16]. The switching changes the structure of the system, and so allows us to break synchronization and to change the order of events. We consider model predictive control (MPC) for the class of implicit switching max-plus linear systems. In contrast to conventional MPC where the sample counter is directly related to the clock/time, in discrete event systems the counter is an event counter and there is no direct relation between the counter and the current time. As a result at a given time t and for a given event step k it could be that some of the components of the state $x(k,t)$ of the system are already known (because they happened before time t) whereas other component are not yet known (as their (estimated or predicted) value is larger than t). In our previous papers on MPC for max-plus linear systems we have not considered in a detailed way the possible problems and intricacies caused by this timing issue. In this paper however we will in particular focus on the timing aspects of MPC for (implicit switching) max-plus linear systems.

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II. Max-plus algebra and implicit switching max-plus-linear systems

A. Max-plus algebra and max-plus-linear systems

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as $x \oplus y = \max(x,y)$ and $x \otimes y = x + y$ for numbers $x, y \in \mathbb{R}_\varepsilon$ [1], [5], and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n} (a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$. The matrix $\mathcal{E}$ is the max-plus-algebraic zero matrix with $(\mathcal{E})_{ij} = \varepsilon,\ \forall i, j$. The matrix E is the max-plus-algebraic identity matrix with $[E]_{ij} = 0$ for all $i = j$, and $(E)_{ij} = \varepsilon$ for all $i \neq j$.

In [1], [5] it has been shown that discrete event systems in which there is synchronization but no concurrency can be described by a model of the form

$$\begin{aligned} x(k) &= A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1) \oplus \ldots \oplus \\ &\quad A_m(k) \otimes x(k-m) \oplus B(k) \otimes u(k) \\ &= \Big(\bigoplus_{i=0}^{m} A_i(k) \otimes x(k-i) \Big) \oplus B(k) \otimes u(k) \end{aligned} \tag{1}$$

The index k is called the event counter.So k indicates the batch number, the operation cycle, etc. of the discrete event system. For discrete event systems the state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, and the input $u(k)$ contains the time instants at which the input events occur for the kth time. The max-plus-linear system (1) is called implicit, because on both sides of the equality sign the state $x(k)$ appear. Usually (1) is transformed into an explicit equation of the form

$$x(k) = A'_1(k) \otimes x(k-1) \oplus \cdots \oplus A'_m(k) \otimes x(k-m) \oplus B' \otimes u(k)$$

with $A'_i = A^* \otimes A_i$ and $B' = A^* \otimes B$ where $A^* = E \oplus A \oplus A^{\otimes^2} \oplus \cdots$ [1]. However, as we will argue later for some applications (in particular those involving on-line adaptive and/or predictive control) the implicit form is more suited.

B. Implicit switching max-plus-linear systems

In this paper we will consider *implicit* switching max-plus-linear systems, i.e. discrete event systems that can switch between different modes of operation. The switching allows us to change the structure of the system, to break synchronization and to change the order of events. In each different mode $\ell = 1,\ldots,n_{\mathrm{m}}$, the implicit switching max-plus-linear system is described by an implicit max-plus-linear state equation

$$\begin{aligned} x(k,t) &= A_0^{(\ell(k,t))}(k,t)\otimes x(k,t)\oplus A_1^{(\ell(k,t))}(k,t)\otimes x(k-1,t) \\ &\oplus\ldots\oplus A_m^{(\ell(k,t))}(k,t)\otimes x(k-m,t)\oplus B^{(\ell(k,t))}(k,t)\otimes u(k,t) \\ &= \Big(\bigoplus_{i=1}^{m} A_i^{(\ell(k,t))}(k,t)\otimes x(k-i,t)\Big)\oplus B^{(\ell(k,t))}(k,t)\otimes u(k,t) \end{aligned} \tag{2}$$

in which the matrices $A_i^{(\ell)}$, $i=1,\ldots,m$ and $B^{(\ell)}$ are the system matrices for the ℓ-th mode, and $x(k,t)$, $u(k,t)$ and $\ell(k,t)$ denote the state, input and mode signal, respectively, for the kth cycle as they are known, measured, estimated or predicted at time t. For implicit switching max-plus-linear systems we need time t explicitly as an argument to be able to switch within a cycle.

The motivation for introducing implicit switching max-plus-linear systems is the following. In [15], [16] we have introduced switching max-plus-linear systems of the form (2) with only $A_1^{(\ell)}$ present (so $A_0^{(\ell)}=\mathcal{E}$ and $A_i^{(\ell)}=\mathcal{E}$ for $i=2,\ldots,m$). The main disadvantage of this description is that we are not able to consider the intermediate switching explicitly in time, but we can only concentrate on the final result $\ell(k)$. For systems where the A-matrices are uncertain and may change in time (which is usually the case, when the entries of the A matrices correspond to production times or traveling times), we will find it is more appropriate to describe the different modes over both the event axis (k) and the time axis (t).

The moments of switching are determined by a switching mechanism. Consider the system (2) at a time t, for which a switching might take place. To distinguish between the signals before and after the switching we will denote t^- as the time just before switching, and t^+ as the time just after switching. Consequently, $x(k,t^-)$ and $u(k,t^-)$ denote the state and input just before the switching, and $x(k,t^+)$ and $u(k,t^+)$ denote the state and input just after the switching. Note that some of the events in $x(k,t^-)$ may already have taken place ($x_i(k,t^-)<t$) and others still have to taken place ($x_i(k,t^-)\geq t$). Let $N_x(k,t)$ be the set of indices that correspond to states that already have taken place (so $x_i(k,t^-)<t$ for $i\in N_x(k,t)$), and define a vector $x_{\mathrm{past}}(k,t^-)$, consisting of all entries of $x_i(k,t^-)$, $i\in N_x(k,t)$. In the same way, let $N_u(k,t)$ be the set of indices for which $u_j(k,t^-)<t$ for $j\in N_u(k,t)$)

We define the switching variable $z(k,t)$, which consists of the time variable t, the state $x(k,t)$, the mode $\ell(k,t)$, the input variable $u(k,t)$ and an (additional) control variable $v(k,t)$:

$$z(k,t)=\begin{bmatrix} t \\ x_{\mathrm{past}}(k,t^-) \\ \vdots \\ x_{\mathrm{past}}(k-m,t^-) \\ \ell(k,t^-) \\ \ell(k-1,t^-) \\ u(k,t^-) \\ v(k,t) \end{bmatrix} \in \mathbb{R}_{\varepsilon}^{n_z} \ . \tag{3}$$

We partition $\mathbb{R}_{\varepsilon}^{n_z}$ in n_{m} subsets $\mathscr{Z}^{(i)}$, $i=1,\ldots,n_{\mathrm{m}}$. The mode $\ell(k,t)$ is now obtained by determining in which set $z(k,t)$ is for event k and for time t. So if $z(k,t)\in\mathscr{Z}^{(i)}$, then $\ell(k,t)=i$.

Note that due to causality it is important to guarantee that events that already have taken place at switching time t should not be changed anymore. We therefore introduce the following causality conditions:

$$x_i(k,t^+)=x_i(k,t^-),\ \text{for all } i\in N_x(k,t) \tag{4}$$

$$u_j(k,t^+)=u_j(k,t^-),\ \text{for all } j\in N_u(k,t) \tag{5}$$

In practice this means that mode transitions that will change $x_{\mathrm{past}}(k,t^-)$ will not be allowed.

Remark 1: Let $\mathscr{G}(A_i^{(\ell)})=(\mathscr{N}(A_i^{(\ell)}),\mathscr{D}(A_i^{(\ell)}))$ be the precedence graph (or communication graph) associated with matrix $A_i^{(\ell)}$, where $\mathscr{N}(A_i^{(\ell)})$ denote the set of nodes and $\mathscr{D}(A_i^{(\ell)})$ the set of arcs of the graph [11]. For systems of the form (2) we will find that the nodes will not change and so $\mathscr{N}(A_i^1)=\ldots=\mathscr{N}(A_i^{n_{\mathrm{m}}})$ for all i. In that case switching means changing the arcs of the graph, i.e. adding or removing arcs or changing the weights of the arcs. Now let us restrict ourselves to systems in which the weights of the arcs (=entries of $A_i^{(\ell)}$) are either ε or non-negative (for physical systems this is usually the case). Further let $\mathscr{N}_{\mathrm{past}}(k,t)$ be the set of nodes that correspond to the past state $x_{\mathrm{past}}(k,t^-)$. In that case condition (4) means that the incoming arcs at nodes $\mathscr{N}_{\mathrm{past}}(k,t)$ are not allowed to change (i.e. all existing incoming arcs should stay and keep the same weight and new incoming arcs should not be added).

Remark 2: Due to causality, the mode switching is only determined by the past values of the states and input signals. In some cases the switching may be dependent on a predicted value of the state. In that case this predicted state can always be computed using the values of $x_{\mathrm{past}}(k-j,t^-)$, $j=1,\ldots,m$, together with the values of $A_i^{(\ell(k-j,t))}(k-j,t)$ and the future mode and input sequence.

III. The model predictive control problem

Consider the switching max-plus-linear model (2)–(3). We have two possible input signals, $v(k,t)$ and $u(k,t)$. The input signal $u(k,t)$ correspond to time a specific input event occurs, the input signal $v(k,t)$ is an additional (usually integer valued) signal that gives some additional control of the switching mechanism. For more background on the choices of the input signals we refer to [15], [16]. Just as in conventional Model Predictive Control (MPC) [12] we define the input sequences $\tilde{u}(k,t) = [\ u^T(k,t), \ldots, u^T(k+N_\mathrm{p}-1,t)\ ]^T$ and $\tilde{v}(k,t) = [\ v^T(k,t), \ldots, v^T(k+N_\mathrm{p}-1,t)\ ]^T$ where N_p is the prediction horizon. Let $\mathscr{V}(k,t)$ and $\mathscr{U}(k,t)$ be the sets of feasible future control sequences $\tilde{v}(k,t)$ and $\tilde{u}(k,t)$, respectively.

We now aim at computing the optimal $\tilde{u}(k,t)$ and $\tilde{v}(k,t)$ that minimize a cost criterion $J(k,t)$, possibly subject to linear constraints on the inputs and the states. The cost criterion reflects the input and output cost functions (J_in and J_out, respectively) in the event period $[k, k+N_\mathrm{p}-1]$:

$$J(k,t) = J_\mathrm{out}(k,t) + \lambda J_\mathrm{in}(k,t) \quad , \tag{6}$$

where λ is a weighting parameter. The output cost function is usually chosen as

$$J_\mathrm{out}(k,t) = \sum_{j=0}^{N_\mathrm{p}-1} \| e(k+j,t) \| \quad ,$$

where $\|\cdot\|$ is an appropriate norm (usually the two-norm, the one-norm or the infinity-norm), and e is the due date error $e_i(k,t) = \max(x_i(k,t) - r_i(k,t), 0)$, where $r(k,t)$ is the desired due date of the state. The input cost function consists of two parts, $J_\mathrm{in} = J_{\mathrm{in},u} + J_{\mathrm{in},v}$. The first part $J_{\mathrm{in},u}$ depends on $\tilde{u}(k,t)$ and is usually chosen as

$$J_{\mathrm{in},u}(k,t) = -\sum_{j=0}^{N_\mathrm{p}-1} \| u(k+j,t) \| \quad ,$$

(see also [7]). The second part $J_{\mathrm{in},v}$ is a function of $\tilde{v}(k,t)$. For different applications, $J_{\mathrm{in},v}$ will have different appearances. If the input signal $v(k,t)$ has no influence on the switching (i.e. the partition $\mathbb{R}_\varepsilon^{n_z}$ in n_m subsets $\mathscr{Z}^{(i)}$ does not depend on $v(k,t)$), then we will choose $J_{\mathrm{in},v} = 0$. If $v(k,t)$ has an influence on the switching, then $J_{\mathrm{in},v}$ is usually chosen as

$$J_{\mathrm{in},v}(k,t) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_v} \| w_i v_i(k+j,t) \| ,$$

where w_i are weighting constants, that penalizes an increase of the variable $v_i(k+j,t)$.

Since the input signal $u(k,t)$ corresponds to consecutive event occurrence times, we have the additional condition for $j = 0, \ldots, N_\mathrm{p}-1$:

$$\Delta u(k+j,t) = u(k+j,t) - u(k+j-1,t) \geq 0 \,.$$

Furthermore, in order to reduce the number of decision variables and the corresponding computational complexity we introduce a control horizon constraint on the signals $\Delta u(k,t)$ and $v(k,t)$, which means that these signals should be constant from the point $k+N_\mathrm{c}-1$ on, so

$$\Delta u(k+j,t) = \Delta u(k+N_\mathrm{c}-1,t) \,,$$
$$v(k+j,t) = v(k+N_\mathrm{c}-1,t) \,,$$

for $j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1$. Now the MPC control problem for event step k and time t can be defined as:

$$\min_{\{\tilde{u}(k,t)\in\mathscr{U},\tilde{v}(k,t)\in\mathscr{V}(k,t)\}} J(k,t) \tag{7}$$

subject to

$$x(k+j,t) = \Big(\bigoplus_{i=0}^{m} A_i^{(\ell(k+j,t))}(k+j,t) \otimes x(k+j-i,t)\Big)$$
$$\oplus B^{(\ell(k+j,t))}(k+j,t) \otimes u(k+j,t) \tag{8}$$
$$z(k+j,t) \in \mathscr{Z}^{(\ell(k+j,t))} \tag{9}$$
$$\Delta u(k+j,t) \geq 0 \tag{10}$$
$$\Delta v(k+l,t) = 0 \tag{11}$$
$$\Delta u(k+l,t) - \Delta u(k+N_\mathrm{c}-1,t) = 0 \tag{12}$$
$$A_\mathrm{c}(k,t)\tilde{u}(k,t) + B_\mathrm{c}(k,t)\tilde{x}(k,t) \leq c_\mathrm{c}(k,t) \tag{13}$$

for $j = 0, \ldots, N_\mathrm{p}-1, \ l = N_\mathrm{c}, \ldots, N_\mathrm{p}-1$

where (13) may represent additional linear constraints on the inputs and the states.

MPC uses a receding horizon principle. This means that after computation of the optimal future control sequences $\tilde{u}(k,t)$ and $\tilde{v}(k,t)$, only the first control samples $u(k,t)$ and $v(k,t)$ will be implemented, subsequently the horizon is shifted one sample, and the optimization is restarted with new information of the measurements.

In principle we have all elements to solve the receding horizon control problem (7)–(13). In general we will have an optimal control problem with both real parameters and integer parameters. As was already discussed in [15], [16], the optimization problem can often be recast in a form for which reliable algorithms are available[1]

Timing

MPC for (switching) MPL systems is different from conventional MPC in the sense that the event counter k is not directly related to a specific time [16]. So far we have assumed that $x(k-j,t)$, $j = 0, \ldots, m$ are available when we want optimize over the future control sequence at time t. However, only the components of $x_\mathrm{past}(k-j,t)$ are available at time instant t. Therefore, we will now present a method to address the timing issues of the controller.

We consider the case of full state information[2]. Let $[x_\mathrm{true}(k-j,t)]_i$, $j = 0, \ldots, m$ be the measured (true) occurrence time of the $(k-j)$th occurrence of internal event $i \in N(k-j,t)$ at time t, and let $[x_\mathrm{est}(k-j,t)]_i$, $j = 0, \ldots, m$ be an estimation of the $(k-j)$th occurrence time of internal event $i \notin N(k-j,t)$ at time t. The estimation can be done using the following procedure: Let $p(t)$ be the smallest integer such that $[x_\mathrm{true}(k-p(t))]_i < t$ for all $i = 1, \ldots, n$.

[1]In some particular cases, the problem can be recast as a Extended Linear Complementary Problem (ELCP) that can be solved efficiently [7], [8]. If the optimization is over a binary valued vector $v(k)$ we obtain an integer optimization problem (without any real valued variables), which can be solved using genetic algorithms [6], tabu search [10], or a branch-and-bound method [4]. In some particular cases the problem can be recast as a Mixed Integer Linear Programming (MILP) or a Mixed Integer Quadratic Programming (MIQP) [2], [9].

[2]Since the components of x correspond to event times, they are in general easy to measure. Also note that measurements of occurrence times of events are in general not as susceptible to noise and measurement errors as measurements of continuous-time signals involving variables such as temperature, speed, pressure, etc.

Hence, $[x_{\text{true}}(k-p(t))]_i$ is completely known at time t. If we define $x_{\text{est}}(k-p(t),t) = x_{\text{true}}(k-p(t))$, we can reconstruct the unknown state components using the recursion

$$x_{\text{est}}(k-j,t) = \Big(\bigoplus_{i=0}^{m} A_i^{(\ell(k-j,t))}(k,t) \otimes x_{\text{est}}(k-i-j,t)\Big)$$
$$\oplus B^{(\ell(k-j,t))}(k,t) \otimes u(k-j,t)$$
$$z(k-j,t) \in \mathscr{Z}^{(\ell(k-j,t))}$$

for $j = 0,\ldots,p(t)-1$, where for the components of $u(k-j,t)$ that are less than t we take the actually applied input times, and for the other components we take the computed estimated values. The value of the state $x(k,t)$, $j = 0,\ldots,p(t)-1$ that can be used to compute the MPC controller at time t is given by $x(k-j,t)$ with components $[x(k-j,t)]_i$ for $i = 1,\ldots,n$ such that

$$[x(k-j,t)]_i = \begin{cases} [x_{\text{true}}(k-j,t)]_i & \text{if } i \in N(k-j,t) \\ [x_{\text{est}}(k-j,t)]_i & \text{if } i \notin N(k-j,t) \end{cases}$$

Finally, before the implementation of the controller can be done, one has to determine at what time instants a new optimization should be done. In principle, the appropriate input sequences $\tilde{u}(k,t)$ and $\tilde{v}(k,t)$ should be recomputed as soon as a new measurement of state $[x_{\text{true}}(k-j,t)]_i$ comes available. If the measured $[x_{\text{true}}(k-j,t)]_i$ is equal to the estimated $[x_{\text{est}}(k-j,t)]_i$, an optimization is superfluous and the already computed input sequences will be optimal.

IV. Example: A railway network

In this example we consider the railroad network of Figure 1, which is a refined version of the network, presented in [14]. There are 4 stations in this railroad network (A, B, C and D) that are connected by 6 single tracks (1/7, 2/4, 3, 5, 6/8, 9). There are three trains available. The first train follows the route $D \to A \to B \to D$, the second train follows the route $A \to B \to C \to A$, and the third train follows the route $D \to A \to C \to D$. We assume that there exists a periodic timetable that schedules the earliest departure times of the trains. The period of the timetable is $T = 60$ minutes. So if a departure of a train from station B is scheduled at 5.30 a.m., then there is also scheduled a departure of a train from station B at 6.30 a.m., 7.30 a.m., and so on.

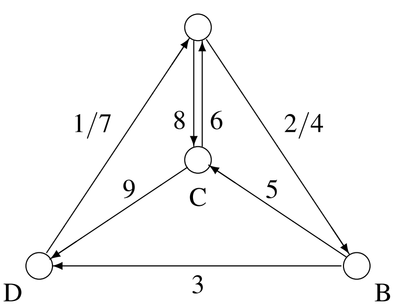

Fig. 1. The railroad network.

Each track of the railway network has a number and a train allocated to it. For the sake of simplicity we will say "(virtual) train j" to denote the (physical) train on a specific track. The number of tracks in the network is equal to 6, the number of physical trains in the network is equal to 3, and the number of virtual trains in the network is equal to 9. (We say virtual to denote that some of the virtual trains are actually the same physical train). Let $d_j(k,t)$, $j = 1,\ldots,9$ be the time instant at which train j departs from its departure station in the kth period, and let $a_j(k,t)$, $j = 1,\ldots,9$ be the time instant at which train j arrives at its arrival station in the kth period. Let $r_j(k)$ be the departure time for this train according to the time schedule, and let $\tau_j(k,t)$ be the transportation time for this train j.

TABLE I

The nominal transportation times and the departure times

train	from to	transportation time	departure arrival	same train as	give connection to	follow	wait for arrival of
1	D-A	12	00-12	3^-	9^-	7^-	
2	A-B	12	15-27	1	6^-	4^-	
3	B-D	20	30-50	2			
4	A-B	12	19-31	6^-	7^-	2	
5	B-C	10	34-44	4			
6	C-A	25	47-12	5			8
7	D-A	12	04-16	9^-		1	
8	A-C	25	19-44	7			6^-
9	C-D	10	47-57	8	5		

Note: 3^- denotes train 3 in the previous cycle

Table I summarizes the information in connection with the nominal transportation times and the departure times. All the times are measured in minutes.

The *continuity constraints* are that the trains on tracks 1, 2 and 3 are physically the same train, and the same holds for the trains on tracks 4, 5 and 6 and for the trains on tracks 7, 8 and 9. *Connection constraints* are introduced to allow the passengers to change trains. In this network, train 1 has to wait for train 9 in the previous cycle with minimum connection time $c^{\min} = 3$. In the same way, train 2 waits for train 6 in the previous cycle, train 4 wait for train 7, and train 9 waits for train 5. The minimum stopping time of train j at station j to allow passenger to get off or on the train is fixed at $s^{\min} = 1$. *Follow constraints* are introduced to guarantee sufficient separation time between two trains on the same track (moving in the same direction). In this network, train 4 is scheduled behind train 2 (train 4 follows train 2) with a minimum separation time $f^{\min} = 4$. In the same way, train 2 follows train 4 in the previous cycle, train 7 follows train 1, and train 1 follows train 7 in the previous cycle. Finally, a *wait constraint* is introduced to guarantee that two trains (moving in opposite direction) are not on the same track at the same time: Train 6 is scheduled behind train 8 (train 4 waits for train 2) with a minimum separation time $w^{\min} = 1$. In the same way train 8 waits for train 6 in the previous cycle.

Each train departs as soon as all the connections are guaranteed (except or a connection when it is broken), the passengers have gotten the opportunity to change over and the earliest departure time indicated in the timetable has passed.

Now we write down the equations that describe the evolution of the $a_j(k,t)$'s and $d_j(k,t)$'s. First we consider the train on track 1 and we determine $d_1(k,t)$, the time instant at which this train departs from station A for the kth time. The train has to wait at least until the train has arrived in station A for the $(k-1)$th time[3] and the passengers have got the time to get out of the train so we have $d_1(k,t) \geq a_3(k-1,t)+1$. Furthermore, the train on track 1 has to wait for the passengers of the train on track 9 in the $(k-1)$th cycle, which arrives in station B at time instant $a_9(k-1,t)$. The passengers have $c^{\min}=3$ minutes to change trains. Further the train on track 1 has to follow the train on track 7 in the previous cycle with a minimum separation time $f^{\min}=4$. According to the timetablethe train on track 1 can only depart after time instant $00+k60$. Hence, we have

$$\begin{aligned} d_1(k,t) &= \max(a_3(k-1,t)+s^{\min}, d_7(k-1,t)+f^{\min}, \\ &\qquad a_9(k-1,t)+c^{\min}, r_1(k)) \\ &= \max(a_3(k-1,t)+1, d_7(k-1,t)+4, \\ &\qquad a_9(k-1,t)+3, k60\,) \end{aligned}$$

for $k=1,2,\ldots$ with $a_3(0)=a_9(0)=\varepsilon$. The nominal arrival time of train 1 is now equal to the departure time plus the transportation time $(d_1(k,t)+\tau_1(k,t))$. However, train 1 can never take over train 7 from the previous and so $a_1(k,t) \geq a_7(k-1,t)+f^{\min}$. So the final arrival time becomes:

$$\begin{aligned} a_1(k,t) &= \max(\,d_1(k,t)+\tau_1(k,t), a_7(k-1,t)+f^{\min}\,) \\ &= \max(d_1(k,t)+\tau_1(k,t), a_7(k-1,t)+4) \end{aligned}$$

Using a similar reasoning, we find that the other departure and arrival times are given by

$$\begin{aligned} d_2(k,t) &= \max(a_1(k,t)+1, a_6(k-1,t)+3, \\ &\qquad d_4(k-1,t)+4, 15+k60) \\ d_3(k,t) &= \max(a_2(k,t)+1, 30+k60) \\ d_4(k,t) &= \max(a_6(k-1,t)+1, a_7(k-1,t)+3, \\ &\qquad d_2(k,t)+4, 19+k60) \\ d_5(k,t) &= \max(a_4(k,t)+1, 34+k60) \\ d_6(k,t) &= \max(a_5(k,t)+1, a_8(k,t)+1, 47+k60) \\ d_7(k,t) &= \max(a_9(k-1,t)+1, d_1(k,t)+4, 4+k60) \\ d_8(k,t) &= \max(a_7(k,t)+1, a_6(k-1,t)+1, 19+k60) \\ d_9(k,t) &= \max(a_8(k,t)+1, a_5(k,t)+3, 47+k60) \end{aligned}$$

[3]Under nominal operations the kth train on track 1 (e.g., the one that departs from station D at 10.00 a.m.) proceeds to the $(k-1)$th train on track 3 (which has departed from station B at 9.30 a.m.) and *not* to the kth train on track 3 (which will depart from station B at 10.30 a.m.).

$$\begin{aligned} a_2(k,t) &= \max(d_2(k,t)+\tau_2(k,t), a_4(k-1,t)+4) \\ a_3(k,t) &= d_3(k,t)+\tau_3(k,t) \\ a_4(k,t) &= \max(d_4(k,t)+\tau_4(k,t), a_2(k,t)+4) \\ a_5(k,t) &= d_5(k,t)+\tau_5(k,t) \\ a_6(k,t) &= d_6(k,t)+\tau_6(k,t) \\ a_7(k,t) &= \max(d_7(k,t)+\tau_7(k,t), a_1(k,t)+4) \\ a_8(k,t) &= d_8(k,t)+\tau_8(k,t) \\ a_9(k,t) &= d_9(k,t)+\tau_9(k,t) \end{aligned}$$

for $k=0,1,2,\ldots$ with $d_j(-1)=\varepsilon$, $a_j(-1)=\varepsilon$ for all j. By defining $x(k,t) = \begin{bmatrix} d_1(k,t) & \ldots & d_9(k,t) & a_1(k,t) & \ldots & a_9(k,t) \end{bmatrix}^T$ and $\bar{r}(k) = \begin{bmatrix} r^T(k) & \varepsilon \end{bmatrix}^T$ we can rewrite this system as

$$\begin{aligned} x(k,t) = A_0^{(1)}(k,t) \otimes x(k,t) \oplus \\ A_1^{(1)}(k,t) \otimes x(k-1,t) \oplus \bar{r}(k) \end{aligned} \qquad (14)$$

which is of the form (2) with $m=1$ and $u(k,t)=\bar{r}(k)$.

In the nominal operation we have assumed that some trains should give pre-defined connections to other trains, and the order of trains on the same track is fixed. However, if one of the preceding trains has a too large delay, then it is sometimes better — from a global performance viewpoint — to let a connecting train depart anyway or to change the departure order on a specific track. This is done in order to prevent an accumulation of delays in the network. In this paper we consider the switching between different operation modes, where each mode corresponds to a different set of pre-defined or broken connections and a specific order of train departures. We allow the system to switch between different modes, allowing us to break train connections and to change the order of trains. Note that any broken connection or change of train order leads to a new model, similar to the nominal equation (14), but now with adapted system matrix $A^{(\ell)}$ for the ℓ-th model. We have the following system equation for the perturbed operation for $\ell=2,\ldots,n_{\mathrm{m}}$:

$$\begin{aligned} x(k,t) = A_0^{(\ell(k,t))}(k,t) \otimes x(k,t) \oplus \\ A_1^{(\ell(k,t))}(k,t) \otimes x(k-1,t) \oplus \bar{r}(k) \end{aligned}$$

In this railway network the switching variable $z(k,t)$ is equal to the control vector $v(k,t)$, and each entry of $v(k,t)$ corresponds to a specific control action, so a specific (scheduled) synchronization or specific (scheduled) event order. We assume $v(k,t)$ to be binary, where $v_i(k,t)=0$ corresponds to the nominal case, and $v_i(k,t)=1$ to a perturbed case (a synchronization is broken or the order of two events is switched). Each combination $v_1(k,t),\ldots,v_{n_v}(k,t)$ corresponds to a fixed routing schedule with a specific train order and specific connections.

If for example the order of departure (and thus arrival) of train 1 and 7 is changed in cycle k, the equations for $d_1(k,t)$,

$d_7(k,t)$, $a_1(k,t)$, and $a_7(k,t)$ are replaced by:

$$
\begin{aligned}
d_1(k,t) &= \max(a_3(k-1,t)+1, d_7(k,t)+4, \\
&\qquad\qquad a_9(k-1,t)+3, k60\,) \\
d_7(k,t) &= \max(a_9(k-1,t)+1, d_1(k-1,t)+4, 4+k60) \\
a_1(k,t) &= \max(d_1(k,t)+\tau_1(k,t), a_7(k,t)+4) \\
a_7(k,t) &= \max(d_7(k,t)+\tau_7(k,t), a_1(k-1,t)+4)
\end{aligned}
$$

We now assume $u(k,t) = \bar{r}(k)$ is fixed and $v(k,t)$ is a binary parameter vector. We solve the optimal control problem of solving cost function

$$
J(k,t) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{9} d_i(k+j) - r_i(k+j) + \sum_{l=1}^{n_v} \lambda_l \, v_l(k+j)
$$

subject to constraints (8)–(13), where N_p is chosen sufficiently large. This results in an integer optimization problem. We assume the system is at nominal schedule for $k < 0$. At time $t = 15$ a measurement is received for a delay of train 7 in cycle $k = 0$, and at time $t = 125$ a measurement is received for a delay of train 1 in cycle $k = 2$, so

$$
\tau_7(0,t) = \begin{cases} 12 & \text{for } t < 15 \\ 35 & \text{for } t \geq 15 \end{cases},
$$

$$
\tau_1(2,t) = \begin{cases} 12 & \text{for } t < 125 \\ 32 & \text{for } t \geq 125 \end{cases}
$$

We choose $\lambda_l = 500$ for inputs v_l related to connections and $\lambda_l = 10$ for the other inputs. The input signal is optimized with a branch-and-bound algorithm and we obtain the optimal sequence (for more details about an good initial guess for the optimization, see [14]).

In Figure 2 the maximum delay $e_{\max}(k,t) = \max(e(k,t))$ in each cycle k is given for both the uncontrolled case (so $v(k+j) = 0$ for all $j > 0$) and for the controlled case (with $v^*(k+j)$). We see that the delay in the MPC controlled case decays much faster than the uncontrolled case.

V. Discussion

We have introduced implicit max-plus-linear systems, a class of discrete event systems that can operate in different modes, for which in each mode the dynamics can be described by a model that is "linear" in the max-plus algebra. The implicitness of the system equations is appropriate if we consider systems for which the system matrices may be uncertain or vary in time. It is straightforward to derive the matrices $(A_i(k,t), B_i(k,t))$, $i = 1, \ldots, m$ and the timing aspects in MPC when measurements and estimations become available in time, is easier and transparent. The MPC design technique has been discussed for the implicit switching max-plus-linear systems, and we have applied the control design method to a railway system.

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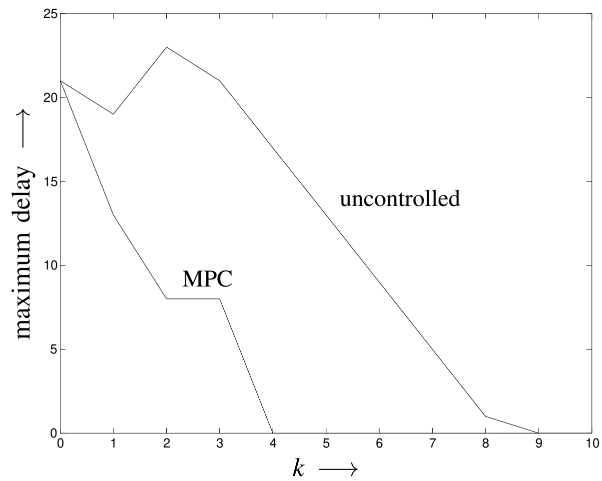

Fig. 2. Maximum delay for an uncontrolled railway system and a railway system using MPC