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Technical report 06-011

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If you want to cite this report, please use the following reference instead:
I. Necoara, B. De Schutter, T. van den Boom, and H. Hellendoorn, "Min-max model predictive control for uncertain max-min-plus-scaling systems," *Proceedings of the 8th International Workshop on Discrete Event Systems (WODES'06)*, Ann Arbor, Michigan, pp. 439–444, July 2006.

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Min-Max Model Predictive Control for Uncertain Max-Min-Plus-Scaling Systems

Ion Necoara, Bart De Schutter, Ton van den Boom and Hans Hellendoorn

***Abstract*— We extend the model predictive control (MPC) framework that has been developed previously to a class of uncertain discrete event systems that can be modeled using the operations maximization, minimization, addition and scalar multiplication. This class encompasses max-plus-linear systems, min-max-plus systems, bilinear max-plus systems and polynomial max-plus systems. We first consider open-loop min-max MPC and we show that the resulting optimization problem can be transformed into a set of linear programming problems. Then, min-max feedback model predictive control using disturbance feedback policies is presented, which leads to improved performance compared to the open-loop approach.**

I. Introduction

An important class of discrete event systems is the class of max-min-plus-scaling (MMPS) systems, the evolution of which can be described using the operations maximization, minimization, additions and scalar multiplication. Using the results of [1], [2], we can prove that MMPS systems are equivalent with continuous piecewise affine (PWA) systems. PWA systems are defined by partitioning the state space of the system in a finite number of polyhedral regions and associating to each region a different affine dynamic. The relation between PWA and MMPS systems is useful for the investigation of structural properties of PWA systems such as observability and controllability but also in designing controller schemes like model predictive control (MPC).

MPC [3] is a popular control methodology in the process industry. MPC provides many attractive features: it is an easy-to-tune method, it can handle constraints in a systematic way, it is applicable to multi-variable systems, and is capable of tracking pre-scheduled reference signals. In MPC at each sample step the optimal control inputs that minimize a given performance criterion over a given prediction horizon are computed, and applied using a receding horizon approach.

Using the work of [4] in which MPC for MMPS (and equivalently for continuous PWA) systems for the deterministic case without disturbances is proposed, we further extend MPC for the cases with bounded disturbances. The disturbances perturb the system by introducing uncertainty in the system equations. Ignoring the disturbance can lead to a bad tracking or even to unstable closed-loop behavior. The disturbances must thus also be taken into account in MPC. We model disturbances by including extra additive terms in the system equations for MMPS systems.

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Note that there are some results in the literature on specific classes of uncertain discrete event systems (see [5]–[7]) but to the authors' best knowledge this is the first time that such an approach is used for the MMPS framework. Most of the papers [5]–[7] focus on worst-case problems, which basically involves finding the maximum of the cost criterion over some bounded disturbance set and then minimization over the feasible input set. In [5] dynamic programming was used to solve the min-max problem for continuous PWA systems with bounded disturbance. The core difficulty with the dynamic programming approach is that optimizing over feedback policies with arbitrary non-linear functions is difficult, in general. In [6] it is proved that the worst-case feedback MPC for max-plus systems is a convex problem if some assumptions about the cost function and constraints are fulfilled. In [7] we use multi-parametric tools in order to solve the open-loop worst-case problem. Of course the open-loop solution leads to poor performance in general.

This paper is organized as follows. First a brief review of PWA and MMPS systems is given, and MPC for them as it was developed in [4], [7] is presented in Section II. In Section III we discuss open-loop MPC for uncertain MMPS systems. We obtain an efficient MPC method that is based on minimizing the worst-case cost criterion. We prove that the optimization problem at each step of MPC can be transformed into a set of linear programming problems, for which efficient solution methods exist. It is well-known [8] that in the presence of disturbance, a feedback controller performs better than open-loop controller. Therefore, in Section IV we introduce feedback in the worst-case MPC optimization problem, optimizing over disturbance feedback policies. We conclude with a worked example in Section VI.

II. Preliminaries

A. Continuous PWA and MMPS systems

Definition 1: $f:\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be a *continuous* PWA function if there exists a finite family $\mathscr{C}_1,\dots,\mathscr{C}_N$ of closed polyhedral regions that covers $\mathbb{R}^n$ and for each $i \in \{1,\dots,N\}$, $j \in \{1,\dots,m\}$, the component f_j of f can be expressed as $f_j(x) = \alpha_{i,j}^T\, x + \beta_{i,j}$ for any $x \in \mathscr{C}_i$, with $\alpha_{i,j} \in \mathbb{R}^n$, $\beta_{i,j} \in \mathbb{R}$ and f is continuous on the boundary between any two regions.

A *continuous* PWA system in state space representation is a system of the form:

$$x(k+1) = \mathscr{P}_x(x(k),u(k)) \tag{1}$$

$$y(k) = \mathscr{P}_y(x(k),u(k)), \tag{2}$$

where $\mathscr{P}_x$ and $\mathscr{P}_y$ are continuous PWA functions, with input u, output y, state x.

Definition 2: A scalar-valued MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by the recursive relation:

$$\begin{aligned} f(x) = & x_i | \alpha | \max(f_k(x), f_l(x)) | \min(f_k(x), f_l(x)) | \\ & f_k(x) + f_l(x) | \beta f_k(x), \end{aligned}$$

where $i \in \{1, \dots, n\}$, $\alpha, \beta \in \mathbb{R}$ and $f_k, f_l : \mathbb{R}^n \to \mathbb{R}$ are again MMPS functions, and | stands for *"or"*. For vector-valued MMPS functions the above statements hold component-wise.

An MMPS system is written in the following form:

$$x(k+1) = \mathscr{M}_x(x(k), u(k)) \tag{3}$$

$$y(k) = \mathscr{M}_y(x(k), u(k)), \tag{4}$$

where $\mathscr{M}_x$, $\mathscr{M}_y$ are vector-valued MMPS functions.

Proposition 1 ([4]): Any scalar-valued MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ can be written into *min-max canonical form*

$$f(x) = \min_{j \in \{1, \cdots, \hat{l}\}} \max_{i \in T_j} (\alpha_{i,j}^T x + \beta_{i,j}), \tag{5}$$

or into *max-min canonical form*

$$f(x) = \max_{j \in \{1, \cdots, l\}} \min_{i \in S_j} (\gamma_{i,j}^T x + \delta_{i,j}), \tag{6}$$

for some integers $\hat{l}$, l, N; $\{S_j\}_{j=1}^{l}$ and $\{T_j\}_{j=1}^{\hat{l}}$ are families subsets of $\{1, \dots, N\}$ and $\alpha_{i,j}, \gamma_{i,j} \in \mathbb{R}^n$, $\beta_{i,j}, \delta_{i,j} \in \mathbb{R}$.

Proposition 2 ([2]): Any continuous PWA function can be written as an MMPS function and vice versa.

Corollary 1: Continuous PWA systems and MMPS systems are equivalent in the sense that for a given continuous PWA model there exists an MMPS model (and vice versa) such that the input-output behavior of both models coincides.

B. MPC for MMPS systems

Now we give a short overview of the main results of [4], [7] about MPC for systems of the form (3)-(4). Note that in [4] disturbances in the model are not included.

In MMPS-MPC we define at each sample step k a cost criterion $J(k) = J_{\text{out}}(k) + \lambda J_{\text{in}}(k)$ over the period $[k, k + N_{\text{p}} - 1]$, where N_{p} is the prediction horizon and $\lambda > 0$. By optimizing $J(k)$ we obtain an optimal input sequence $u^*(k), \dots, u^*(k + N_{\text{p}} - 1)$, but we apply only the first input sample $u^*(k)$ according to a receding horizon strategy. At the next sample step the whole procedure is repeated.

Now we explain in more detail how MPC for MMPS systems can be implemented efficiently in the case when $J(k)$ is an MMPS function of the input. Assuming that at each step k, the state $x(k)$ can be measured or predicted, we can make an estimation of the output of the model (3)–(4):

$$\hat{y}(k + j|k) = \mathscr{M}_j(x(k), u(k), \dots, u(k + j)) \tag{7}$$

at sample step $k + j$ for $j = 0, \dots, N_{\text{p}} - 1$ using the information available up to sample step k. It is easy to verify that $\mathscr{M}_j$ is an MMPS function of $x(k), u(k), \dots, u(k + j)$. Our goal is to track a reference signal r. We define the vectors $\tilde{u}(k) = [u^T(k), \dots, u^T(k + N_{\text{p}} - 1)]^T$, $\tilde{y}(k) = [\hat{y}^T(k|k), \dots, \hat{y}^T(k + N_{\text{p}} - 1|k)]^T$, and $\tilde{r}(k) = [r^T(k), \dots, r^T(k + N_{\text{p}} - 1)]^T$.

We consider only linear constraints on the input

$$P(k)\tilde{u}(k) + q(k) \leq 0. \tag{8}$$

In practical situations, such constraints occur when we have to guarantee that the input signal or the rate of variation of the input signal must stay within certain bounds. As output cost function, just as in [4] we could take:

$$J_{\text{out},1}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_1, \; J_{\text{out},\infty}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_\infty, \tag{9}$$

which reflect the tracking error, and are MMPS functions of $x(k), \tilde{u}(k), \tilde{r}(k)$. As input cost function one could take:

$$J_{\text{in},1}(k) = \|\tilde{u}(k)\|_1, \quad J_{\text{in},\infty}(k) = \|\tilde{u}(k)\|_\infty, \tag{10}$$

which are also MMPS functions of $\tilde{u}(k)$. We introduce a control horizon N_{c} such that

$$u(k + j) = u(k + N_{\text{c}} - 1) \quad \text{for} \quad j = N_{\text{c}}, \dots, N_{\text{p}} - 1, \tag{11}$$

to decrease the number of degrees of freedom for $\tilde{u}(k)$ and thus also the computational effort. Note that (11) can also be expressed in the form (8).

Since after substitution of $\tilde{y}(k)$ using (7), the cost function $J(k)$ is an MMPS function of $\tilde{u}(k)$ which can be written in min-max canonical form, it follows that at each sample step k we have to solve an optimization problem of the form

$$\begin{aligned} & \min_{\tilde{u}(k)} \min_{j \in \{1, \dots, \hat{l}\}} \max_{i \in T_j} (\alpha_{i,j}^T \tilde{u}(k) + \beta_{i,j}(k)) \\ & \text{subject to: } P(k)\tilde{u}(k) + q(k) \leq 0, \end{aligned} \tag{12}$$

and thus for any $j \in \{1, \dots, \hat{l}\}$ we obtain a linear programming (LP) problem:

$$\min_{\tilde{u}(k), t(k)} t(k) \tag{13}$$

$$\text{subject to: } \begin{cases} P(k)\tilde{u}(k) + q(k) \leq 0 \\ t(k) \geq \alpha_{i,j}^T \tilde{u}(k) + \beta_{i,j}(k)), \text{ for all } i \in T_j. \end{cases}$$

The LP problems are easy to solve using the simplex method or an interior point algorithm. Let $[t^*(k) \; \tilde{u}_{(j)}^{*T}(k)]^T$ be the optimal solution of (13). To obtain the solution of (12), we solve (13) for $j = 1, \dots, \hat{l}$ and afterward we select the $\tilde{u}_{(j)}^*(k)$ for which $\max_{i \in T_j} (\alpha_{i,j}^T(k) \tilde{u}_{(j)}^*(k) + \beta_{i,j}(k))$ is the smallest.

C. Uncertain continuous PWA or MMPS systems

In this section we extend the *continuous PWA* (or equivalently the *MMPS*) deterministic model (1)–(2) or (3)–(4), without disturbances, to take also the uncertainty into account (see also [7]). If we ignore the disturbance in the plant, this can lead to errors in the system equations and even an unstable closed-loop behavior. The MPC method is based on a model of the system; the prediction of the future behavior is made using the model. Therefore we must also take into account the uncertainty when we implement MPC.

As in conventional linear systems, we model the disturbances by including an additive term in the system equations for continuous PWA systems. Hence, we consider the *uncertain continuous PWA* model:

$$x(k+1) = \mathscr{P}_x(x(k), u(k), e(k)) \tag{14}$$

$$y(k) = \mathscr{P}_y(x(k), u(k), e(k)), \tag{15}$$

where $\mathscr{P}_x$ and $\mathscr{P}_y$ are continuous vector-valued PWA functions and the uncertainty caused by disturbances in the estimation of the real system is gathered in the uncertainty vector $e(k)$. We assume that this uncertainty is included in a bounded polyhedral set.

Using the link between continuous PWA and MMPS systems, the uncertain continuous PWA model (14)–(15) can be also written as an MMPS system:

$$x(k+1) = \mathscr{M}_x(x(k), u(k), e(k)) \tag{16}$$

$$y(k) = \mathscr{M}_y(x(k), u(k), e(k)), \tag{17}$$

where $\mathscr{M}_x$, $\mathscr{M}_y$ are vector-valued MMPS functions.

We assume that at each step k of MPC, the state $x(k)$ is available (can be measured or estimated) and we gather the uncertainty over the interval $[k, k+N_\mathrm{p}-1]$ in the vector $\tilde{e}(k) = [e^T(k), \ldots, e^T(k+N_\mathrm{p}-1)]^T \in \tilde{\mathscr{E}}$. We assume that $\tilde{\mathscr{E}}$ is a bounded polyhedral set. Then it is easy to see that the prediction $\hat{y}(k+j|k)$ of the future output for the system (16)–(17) can be written in MMPS form, for $j = 0, \ldots, N_\mathrm{p}-1$.

Using as cost criterion a combination of (9) and (10): $J(k) = J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k)$, and keeping in mind that all these cost criteria are MMPS expressions, we get a min-max canonical form of $J(k)$:

$$J(\tilde{e}(k), \tilde{u}(k), x(k)) = \min_{j \in \{1,\ldots,\hat{l}\}} \max_{i \in T_j} (\alpha_{i,j}^T x(k) + \beta_{i,j}^T \tilde{u}(k) + \gamma_{i,j}^T \tilde{e}(k) + \delta_{i,j}), \tag{18}$$

or a max-min canonical representation:

$$J(\tilde{e}(k), \tilde{u}(k), x(k)) = \max_{j \in \{1,\ldots,l\}} \min_{i \in S_j} (\bar{\alpha}_{i,j}^T x(k) + \bar{\beta}_{i,j}^T \tilde{u}(k) + \bar{\gamma}_{i,j}^T \tilde{e}(k) + \bar{\delta}_{i,j}). \tag{19}$$

If the reference signal r depends on k then $\delta_{i,j}, \bar{\delta}_{i,j}$ will depend also on k (i.e., $\delta_{i,j}, \bar{\delta}_{i,j}$ are affine expressions in $\tilde{r}$).

D. Worst-case MMPS-MPC

The *worst-case MMPS-MPC problem* at step k is then defined as in [7]:

$$J^*(x(k)) = \min_{\tilde{u}(k)} \max_{\tilde{e}(k) \in \tilde{\mathscr{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)) \tag{20}$$

$$\text{subject to: } P(k)\tilde{u}(k) + q(k) \leq 0, \tag{21}$$

where $J(\cdot)$ is given by (18) or (19).

For a given $\tilde{u}(k), x(k)$ we define the *inner* worst-case MMPS-MPC problem

$$\max_{\tilde{e}(k) \in \tilde{\mathscr{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)). \tag{22}$$

We denote

$$\tilde{e}^*(\tilde{u}(k), x(k)) = \arg \max_{\tilde{e}(k) \in \tilde{\mathscr{E}}} J(\tilde{e}(k), \tilde{u}(k), x(k)), \tag{23}$$

$$J^*(\tilde{u}(k), x(k)) = J(\tilde{e}^*(\tilde{u}(k), x(k)), \tilde{u}(k), x(k)). \tag{24}$$

Proposition 3 ([7]): For a given $\tilde{u}(k)$ and $x(k)$, $\tilde{e}^*(\tilde{u}(k), x(k))$ given by (23) can be computed using a set of LP problems.

III. Open-loop MPC for uncertain MMPS systems

In [7], $J^*(\cdot)$ as defined in (24) was determined explicitly using multi-parametric LP (MP-LP) tools. In the case when the reference signal r is a non-zero sequence we cannot solve the inner-worst case problem off-line, using MP-LP, because the cost function depends also on $\tilde{r}(k)$, unless we include $\tilde{r}(k)$ as additive parameters in the MP-LP program. Of course the computational complexity increases in that case because the vector of parameters ($[x(k)^T \ \tilde{u}(k)^T \ \tilde{r}(k)^T]^T$) has dimension much larger than $\theta = [x(k)^T \ \tilde{u}(k)^T]^T$, corresponding to the case $r = 0$. We will show in this section that using the duality theory of LP [9] we can avoid this drawback.

Note that the inner problem can be written equivalently as

$$\max_{j=1,\ldots,l} \max_{\tilde{e}(k)} \min_{i \in S_j} (\bar{\alpha}_{i,j}^T x(k) + \bar{\beta}_{i,j}^T \tilde{u}(k) + \bar{\gamma}_{i,j}^T \tilde{e}(k) + \bar{\delta}_{i,j})$$
$$\text{subject to: } \tilde{S}\tilde{e}(k) \leq \tilde{q}. \tag{25}$$

and then according to Proposition 3 for each $j \in \{1, \ldots, l\}$ we must solve an LP problem

$$\max_{\tilde{e}(k), t_{(j)}(k)} t_{(j)}(k) \tag{26}$$

subject to:

$$\begin{cases} t_{(j)}(k) \leq \bar{\alpha}_{i,j}^T x(k) + \bar{\beta}_{i,j}^T \tilde{u}(k) + \bar{\gamma}_{i,j}^T \tilde{e}(k) + \bar{\delta}_{i,j}, i \in S_j \\ \tilde{S}\tilde{e}(k) \leq \tilde{q}. \end{cases} \tag{27}$$

Note that the primal problem (26)-(27) can be written (for simplicity we drop the index k):

$$\text{(P):} \begin{cases} \max_{\tilde{e}, t_{(j)}} t_{(j)} \\ \text{subject to: } \begin{cases} t_{(j)} - \bar{\gamma}_{i,j}^T \tilde{e} \leq \bar{\alpha}_{i,j}^T x + \bar{\beta}_{i,j}^T \tilde{u} + \bar{\delta}_{i,j}, \forall i \in S_j \\ \tilde{S}\tilde{e} \leq \tilde{q}. \end{cases} \end{cases}$$

We define $c_{i,j}(x, \tilde{u}) = \bar{\alpha}_{i,j}^T x + \bar{\beta}_{i,j}^T \tilde{u} + \bar{\delta}_{i,j}$, which is an affine expression in $(x, \tilde{u})$ and $\bar{\delta}_{i,j}$ depends affinely on $\tilde{r}$ which varies with k. In matrix notation the primal problem becomes:

$$\text{(P):} \begin{cases} \max_{\tilde{e}, t_{(j)}} t_{(j)} \\ \text{subject to: } \begin{bmatrix} 1 & -\bar{\gamma}_{i,j}^T \\ 0 & \tilde{S} \end{bmatrix} \begin{bmatrix} t_{(j)} \\ \tilde{e} \end{bmatrix} \leq \begin{bmatrix} c_{i,j}(x, \tilde{u}) \\ \tilde{q} \end{bmatrix} \\ \quad \text{for each } i \in S_j \end{cases}$$

Note that in primal problem (P) the variables $t_{(j)}, \tilde{e}$ are free. The dual problem then has the following form:

$$\text{(D):} \begin{cases} \min_{y_j} [c_{1,j}(x, \tilde{u}), \ldots, c_{\#S_j,j}(x, \tilde{u}), \tilde{q}_1, \ldots, \tilde{q}_{n_{\tilde{S}}}]^T y_j \\ \text{subject to: } \begin{bmatrix} 1 & 0 \\ -\bar{\gamma}_{i,j} & \tilde{S}^T \end{bmatrix} y_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \forall i \in S_j \\ \quad y_j \geq 0, \end{cases}$$

where $\#S_j$ denotes the cardinality of the set S_j and $n_{\tilde{S}}$ denotes the number of rows of the matrix $\tilde{S}$.

There are algorithms (e.g., the double description method of [10]) to compute a compact explicit description of the elements of the polyhedral cone:

$$K_j = \left\{ y_j \geq 0 : \begin{bmatrix} 1 & 0 \\ -\bar{\gamma}_{i,j} & \tilde{S}^T \end{bmatrix} y_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, i \in S_j \right\}$$

These elements can be expressed as follows:

$$y_j = \sum_{i=1}^{N_j} \alpha_{ij} y_j^i + \sum_{i=1}^{M_j} \beta_{ij} z_j^i$$

with $\sum_i \alpha_{ij} = 1, \alpha_{ij} \geq 0$ and $\beta_{ij} \geq 0$. The y_j^i are called finite vertexes and the z_j^i are called extreme rays (using the definitions of [9], [11]). Because we assume that the primal problem (P) has a finite optimum, we are interested only in the finite vertexes (as extreme rays give rise to infinite solutions): $\{y_j^1, \ldots, y_j^{N_j}\}$. Note that the finite vertexes $y_j^1, \ldots, y_j^{N_j}$ do not depend on the reference signal $\tilde{r}(k)$, since $\tilde{r}(k)$ appears linearly in the $\bar{\delta}_{i,j}$ which are present in the expressions of $c_{i,j}$ but not in the expression of the polyhedral cone K_j. According to strong duality theorem for linear programming we have:

$$t^*_{(j)}(x,\tilde{u}) = \min(c_j^T(x,\tilde{u}) y_j^1, \ldots, c_j^T(x,\tilde{u}) y_j^{N_j}) \qquad (28)$$

where $c_j(x,\tilde{u}) = [c_{1,j}(x,\tilde{u}), \cdots, c_{\#S_j,j}(x,\tilde{u}), \tilde{q}_1, \cdots, \tilde{q}_{n_{\tilde{S}}}]^T$ Then

$$\begin{aligned} J^*(x,\tilde{u}) = & \max_{j\in\{1,\cdots,l\}} (t^*_{(j)}(x,\tilde{u})) = \\ & \max_{j\in\{1,\cdots,l\}} \min(c_j(x,\tilde{u}) y_j^1, \ldots, c_j(x,\tilde{u}) y_j^{N_j}) \end{aligned} \qquad (29)$$

So we obtained directly the max-min canonical form of $J^*(\cdot)$.

For a given $x(k)$, the *outer* worst-case MMPS-MPC problem is now defined as:

$$\min_{\tilde{u}(k)} J^*(\tilde{u}(k), x(k)) \qquad (30)$$

$$\text{subject to } P(k)\tilde{u}(k) + q(k) \leq 0. \qquad (31)$$

Proposition 4 ([7]): Given $x(k)$, the outer worst-case MMPS-MPC problem can be solved using a set of LP problems.

Based on the results discussed above we now present an algorithm to solve the worst-case MMPS-MPC problem.

Step 1: Solve *off-line* the inner worst-case MMPS-MPC problem (22) using duality. Then, $J^*(x,u)$ is an MMPS function. Compute also off-line the min-max canonical form of this function.

Step 2: Compute *on-line* (at each step k) the solution of the outer worst-case MMPS-MPC problem (30)-(31) according to Proposition 4.

According to this algorithm, the open-loop worst-case MMPS-MPC problem can be solved using a set of LP problems. Moreover the associated controller is a PWA function of $x(k)$. An advantage of this approach in comparison with the algorithm from [7] is that the computations of the finite vertexes does not depend on the reference signal r. Therefore, we can compute off-line the expression of J^* even when $r \neq 0$, keeping the computations low.

IV. Disturbance feedback MPC for uncertain MMPS systems

It is well-known [8] that in the presence of disturbances, the MPC controller performs better if we optimize over feedback policies in the worst-case optimization problem (20)–(21). Another approach to controlling an uncertain MMPS system is to include feedback by searching over the set of affine functions of the past disturbances [12], [13]. Since full measurements of the state are assumed (in the case of discrete event systems this assumption is not restrictive since the states represent times and therefore they can be easily measured), it follows that the past disturbance sequence is easily calculated as the difference between the actual state and the state predicted with the nominal system (i.e., in the absence of disturbance).Therefore, we consider *disturbance feedback policies* of the form:

$$u(k+i) = \sum_{j=0}^{i-1} M_{i,j} e(k+j) + v(k+i), \qquad (32)$$

for all $i \in \{0, \cdots, N_\mathrm{p}-1\}$, where each $M_{i,j} \in \mathbb{R}^{m\times s}$ and $v(k+i) \in \mathbb{R}^m$. Let us denote with $\tilde{u} = [u^T(k)\; u^T(k+1) \cdots u^T(k+N_\mathrm{p}-1)]^T$, $\tilde{v} = [v^T(k)\; v^T(k+1) \cdots v^T(k+N_\mathrm{p}-1)]^T$ and

$$\tilde{M} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N_\mathrm{p}-1,0} & M_{N_\mathrm{p}-1,1} & \cdots & 0 \end{bmatrix} \qquad (33)$$

so that the disturbance feedback policy becomes

$$\tilde{u} = \tilde{M}\tilde{e} + \tilde{v} \qquad (34)$$

Under this type of policy, the worst case MMPS-MPC problem becomes:

$$J^*(x(k)) = \min_{\tilde{M},\tilde{v}} \max_{\tilde{e}\in\tilde{\mathscr{E}}} J(\tilde{e}, \tilde{M}\tilde{e}+\tilde{v}, x(k)) \qquad (35)$$

$$\text{subject to: } P(k)(\tilde{M}\tilde{e}+\tilde{v}) + q(k) \leq 0,\ \forall \tilde{e} \in \tilde{\mathscr{E}} \qquad (36)$$

The inner worst-case problem is formulated as:

$$\begin{aligned} J^*(\tilde{M},\tilde{v},x(k)) = \max_{\tilde{e}} \max_{j\in\{1,\ldots,l\}} \min_{i\in S_j} (& \bar{\alpha}_{i,j}^T x(k) + \\ & (\bar{\beta}_{i,j}^T \tilde{M} + \bar{\gamma}_{i,j}^T)\tilde{e} + \bar{\beta}_{i,j}^T \tilde{v} + \bar{\delta}_{i,j}) \\ & \text{subject to: } \tilde{S}\tilde{e}(k) \leq \tilde{q}, \end{aligned}$$

Using similar arguments as in Proposition 3, we conclude that for a given $(\tilde{M},\tilde{v})$, $J^*(\tilde{M},\tilde{v},x(k))$ can be computed efficiently using a set of LP problems. Note that in this particular case we cannot obtain an explicit expression for $J^*(\tilde{M},\tilde{v},x(k))$ as in (29).

The outer worst-case problem becomes:

$$\min_{\tilde{M},\tilde{v}} J^*(\tilde{M},\tilde{v},x(k)) \qquad (37)$$

$$\text{subject to } P(k)(\tilde{M}\tilde{e}+\tilde{v}) + q(k) \leq 0,\ \forall \tilde{e} \in \tilde{\mathscr{E}} \qquad (38)$$

Note that the constraints (38) are nonlinear in the variables $\tilde{M}$ and $\tilde{e}$ but we write them as $P(k)\tilde{M}\tilde{e} \leq -P(k)\tilde{v} - q(k)$ for all $\tilde{e} \in \tilde{\mathscr{E}}$ or $[\max_{\tilde{e}\in\tilde{\mathscr{E}}} (P(k)\tilde{M})_1 \tilde{e} \cdots \max_{\tilde{e}\in\tilde{\mathscr{E}}} (P(k)\tilde{M})_{n_P} \tilde{e}]^T \leq -P(k)\tilde{v} - q(k)$, where $(P(k)\tilde{M})_i$ denotes the i^{th} row of the matrix $P(k)\tilde{M}$. Therefore, using duality for LP problems and the fact that $\tilde{\mathscr{E}} = \{\tilde{e} : \tilde{S}\tilde{e} \leq \tilde{q}\}$ is a polytope and thus compact, it follows that the constraint (38) are equivalent with:

$$P(k)\tilde{M} = Z^T \tilde{S},\ Z^T \tilde{q} + P(k)v + q(k) \leq 0,\ Z \geq 0$$

where by $Z \geq 0$ we mean a matrix with all entries non-negative (i.e., $Z_{ij} \geq 0$ for all i, j). It follows that the outer worst-case problem is written as:

$$\min_{\tilde{M},\tilde{v},Z} J^*(\tilde{M},\tilde{v},x(k))$$

$$\text{subject to: } P(k)\tilde{M} = Z^T\tilde{S},\ Z^T\tilde{q} + P(k)v + q(k) \leq 0, Z \geq 0$$

Now the constraints are linear in $\tilde{M}$, $\tilde{v}$ and Z. This problem can be solved using a gradient projection algorithm. In each iteration step ℓ of the algorithm for the outer problem the function values of J^* (and its gradient, which can be obtained using numerical approximation) have to be computed in the current iteration point (M_ℓ, v_ℓ). This involves solving the inner problem for the given M_ℓ, and v_ℓ, which can be done efficiently by solving a set of LP problems as shown before.

Note that in the case when $M_{i,j} = 0$ for all i, j we obtain the open-loop controller derived in the previous section.

Remark 1 If we consider a nominal cost, corresponding to a most probable value of the disturbance (without loss of generality we may assume that the nominal value of the disturbance is $e = 0$) then we can replace the worst-case approach with a disturbance feedback MPC scheme with a nominal cost as in [13]:

$$J^*(x(k)) = \min_{\tilde{M},\tilde{v}} J(0, \tilde{M}0 + \tilde{v}, x(k))$$

$$\text{subject to: } P(k)(\tilde{M}\tilde{e} + \tilde{v}) + q(k) \leq 0,\ \forall \tilde{e} \in \tilde{\mathscr{E}}$$

Since $J(0, \tilde{M}0 + \tilde{v}, x(k))$ is an MMPS function it can be written as $J(0, \tilde{M}0 + \tilde{v}, x(k)) = \min_{j\in\{1,\dots,\hat{l}\}} \max_{i\in T_j} (\alpha_{i,j}^T x(k) + \beta_{i,j}^T \tilde{v} + \delta_{i,j})$. In conclusion we have to solve the following optimization problem:

$$\min_{\tilde{M},\tilde{v},Z} \min_{j\in\{1,\dots,\hat{l}\}} \max_{i\in T_j} (\alpha_{i,j}^T x(k) + \beta_{i,j}^T \tilde{v} + \delta_{i,j})$$

$$\text{subject to: } P(k)\tilde{M} = Z^T\tilde{S},\ Z^T\tilde{q} + P(k)v + q(k) \leq 0, Z \geq 0$$

or equivalently for each $j \in \{1, \dots, \hat{l}\}$ we must solve the following LP problem:

$$\min_{\tilde{M},\tilde{v},Z} \max_{i\in T_j} (\alpha_{i,j}^T x(k) + \beta_{i,j}^T \tilde{v} + \delta_{i,j})$$

$$\text{subject to: } P(k)\tilde{M} = Z^T\tilde{S},\ Z^T\tilde{q} + P(k)v + q(k) \leq 0, Z \geq 0.$$

Therefore, in this case we also have to solve on-line a set of LP problems.

V. Computational complexity

From a computational point of view, both approaches that we derived before (open-loop scheme and disturbance feedback scheme) consist of two steps. In first step we have to solve the maximization problem corresponding to the worst-case uncertainty. This can be done computing the vertexes of some polyhedral cones as in Section III, or some LP problems as in Section IV. In the second step we have to solve on-line a set of LP problems or to apply iterative procedures, in order to determine the optimal MPC input. The main advantage of the second approach is that by introducing feedback, the corresponding MPC controller will perform better than the open-loop controller. This improvement in performance is obtained at the expense of introducing $\frac{N_p(N_p-1)}{2}\ m\ s + n_P\ n_{\tilde{S}}$ extra variables and $n_P + n_{\tilde{S}}$ extra inequalities (where n_P and $n_{\tilde{S}}$ denotes the number of rows of the matrices P and $\tilde{S}$, respectively).

Note that the reduction to canonical form is computationally intensive, but can be done off-line (for both the inner and the outer worst-case MMPS-MPC problems).

In the worst-case MMPS-MPC problems (20)–(21) or (35)–(36), considered in this paper state constraints are not taken into account. It is well-known in the literature [8], that in the presence of state constraints, the open-loop formulation is conservative (we can have infeasibility) and we should consider optimization over feedback laws, as it was done in Section IV. However, as we mentioned previously, in this paper we do not consider state constraints. So, in this case both optimization problems (20)–(21) and (35)–(36) will be always feasible. If we consider reference tracking (the reference signal $r \neq 0$) using dynamic programming approach we must include $\tilde{r}$ as a parameter in the multi-parametric program, which increases the computational complexity. Moreover, in the dynamic programming approach [5] we cannot consider variable input constraints (e.g., bounded rate variation $m \leq u(k+1) - u(k) \leq M$). Note that these issues can be easily handled with our approaches (open-loop or disturbance feedback MPC).

VI. Example

Consider a room with a basic heat source and an additional controlled heat source. Let u be the contribution to the increase in room temperature per time unit caused by the controlled heat source (so $u \geq 0$). For the basic heat source, this value is assumed to be constant and equal to 1. The temperature in the room is assumed to be uniform and obeys the first-order differential equation

$$\dot{T}(t) = \alpha(T(t))T(t) + u(t) + 1 + e_1(t)\ ,$$

the disturbance being gathered in the scalar variable e_1. We assume that the temperature coefficient has the following piecewise constant form: $\alpha(T) = 1$ if $T < 0$, and $\alpha(T) = -1$ if $T \geq 0$. We assume that the temperature is measured, but the measurement is noisy: $y(t) = T(t) + e_2(t)$. Using the Euler discretization scheme, with a sample time of 1 time unit and denoting the state $x(k) = T(k \cdot 1)$, we get the following continuous discrete-time PWA system:

$$x(k+1) = \begin{cases} 2x(k) + u(k) + e_1(k) + 1 & \text{if } x(k) < 0 \\ u(k) + e_1(k) + 1 & \text{if } x(k) \geq 0 \end{cases} \tag{39}$$

$$y(k) = x(k) + e_2(k). \tag{40}$$

Assume that we have $-2 \leq e_1(k), e_2(k) \leq 2$, $e_1(k) + e_2(k) \leq 1$. The equivalent MMPS representation of (39)–(40) is

$$x(k+1) = \min(2x(k) + u(k) + e_1(k) + 1, u(k) + e_1(k) + 1)$$
$$y(k) = x(k) + e_2(k).$$

Because at sample step k the input $u(k)$ has no influence on $y(k)$, we take $N_p = 3, N_c = 2$, $\tilde{y}(k) = [\hat{y}(k+1|k)\ \hat{y}(k+2|k)]^T$,

TABLE I

OPEN-LOOP COMPUTES THE CONTROLLER BASED ON SECTION II-D; DIST. FEEDBACK COMPUTES THE CONTROLLER BASED ON SECTION IV; MP-LP USES THE APPROACH FROM [5].

	off-line[1]			on-line		
N_p	2	3	4	2	3	4
No. of LPs	7	12	18	4	8	16
Time Open-loop (s)	0.35	0.9	2	0.06	0.08	0.1
Time Dist. feedback (s)	0.65	1.3	3.5	0.09	0.3	0.95
Time MP-LP (s)	5.7	–	–	0.07	–	–

$\tilde{r}(k) = [r(k+1)\ r(k+2)]^T$, $\tilde{u}(k) = [u(k)\ u(k+1)]^T$. Let the uncertainty vector $e(k)$ be $e(k) = [e_1(k)\ e_2(k+1)]^T$. Therefore, $\tilde{e}(k) = [e^T(k)\ e^T(k+1)]^T$. We consider the following constraints on the input: $-4 \leq \Delta u(k) = u(k+1) - u(k) \leq 4$ and $u(k) \geq 0$ for all k. As cost criterion we take

$$J(k) = J_{\text{out},\infty}(k) + \lambda J_{\text{in},1}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_\infty + \lambda \|\tilde{u}(k)\|_1.$$

The first term of $J(k)$ expresses the fact that we penalize the maximum difference between the reference and the output signal, while the second term penalizes the absolute value of the control effort. Because $u(k) \geq 0$, we have $\|u(k)\|_1 = u(k)$ and therefore we get the following formula for $J(k)$:

$$\begin{aligned} J(k) = \max(&y(k+1) - r(k+1) + \lambda u(k) + \lambda u(k+1), \\ &r(k+1) - y(k+1) + \lambda u(k) + \lambda u(k+1), \\ &y(k+2) - r(k+2) + \lambda u(k) + \lambda u(k+1), \\ &r(k+2) - y(k+2) + \lambda u(k) + \lambda u(k+1)). \end{aligned}$$

Therefore, we can also write $J(k)$ in max-min canonical form. We compute now the closed-loop MPC controller over a simulation period $[1,20]$, with $\lambda = 0.1$, initial state $x(0) = -6$, $u(-1) = 0$ and the reference signal $\{r(k)\}_{k=1}^{20} = -5, -5, -5, -5, -5, -3, -3, 1, 3, 3, 8, 8, 8, 8, 10,$ 10, 10, 7, 7, 7, 4, 3, 1, 1, 6, 7, 8, 9, 11, 11 using the methods given in Section II-D and IV. For $N > 2$ we cannot apply MP-LP approach [5] since we consider variable input constraints.

After off-line computation of the max-min canonical form of $J^*(x, \cdot)$ and elimination of the redundant terms we obtain a min-max canonical form of $J^*(x(k), \cdot)$ that gives rise to only 4 LP problems that must be solved on-line at each sample step k in both cases (open-loop approach and disturbance feedback approach). Table I gives more computation details for the three MPC approaches discussed in this paper.

Figure 1 represents the output of the disturbance feedback approach and the open-loop approach. We see that the MPC controller obtained using disturbance feedback policies performs the tracking better than the open-loop MPC controller.

VII. CONCLUSIONS

We have extended the MPC framework for MMPS (or equivalently for continuous PWA) systems to include also bounded disturbances. We have considered the disturbances as an extra additive term on the system equations. This allowed us to design a worst-case MMPS-MPC controller for such systems based on optimization over open-loop input sequences and disturbance feedback policies. We have shown that the resulting optimization problems can be computed efficiently using a two-level optimization approach.

[1]The off-line computation time does not include the transformation into canonical form and elimination of redundant terms as this operation was done by hand.

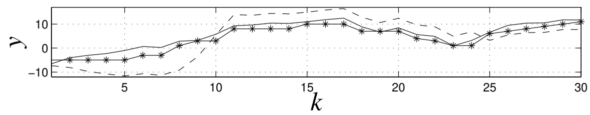

Fig. 1. Worst-case MPC for uncertain MMPS system (39)–(40): disturbance feedback (full), open-loop approach (star line), reference signal r (dashed).

ACKNOWLEDGMENTS

Research supported by STW projects "Model predictive control for hybrid systems" (DMR.5675), and "Multi-agent control of large-scale hybrid systems" (DWV.6188), and European IST project "Modelling, Simulation and Control of Nonsmooth Dynamical Systems (SICONOS)" (IST-2001-37172) and European Framework Network of Excellence "HYbrid CONtrol: Taming Heterogeneity and Complexity of Networked Embedded Systems (HYCON)".

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