Delft University of Technology

Delft Center for Systems and Control

Technical report 06-024

Robust control of constrained max-plus-linear systems*

I. Necoara, B. De Schutter, T.J.J. van den Boom, and H. Hellendoorn

If you want to cite this report, please use the following reference instead:
I. Necoara, B. De Schutter, T.J.J. van den Boom, and H. Hellendoorn, "Robust control of constrained max-plus-linear systems," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 2, pp. 218–242, Jan. 2009.

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.51.19 (secretary)
fax: +31-15-278.66.79
URL: http://www.dcsc.tudelft.nl

Robust control of constrained max-plus-linear systems

Ion Necoara, Bart De Schutter, Ton J.J. van den Boom, Hans Hellendoorn

Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands
Phone:+31-15-278.71.71, Fax: +31-15-278.66.79
{*i.necoara, b.deschutter, t.j.j.vandenboom, j.hellendoorn*}*@dcsc.tudelft.nl*

SUMMARY

Max-plus-linear (MPL) systems are a class of nonlinear systems that can be described by models that are "linear" in the max-plus algebra. We provide here solutions to three types of finite-horizon min-max control problems for uncertain MPL systems, depending on the nature of the control input over which we optimize: open-loop input sequences, disturbances feedback policies, and state feedback policies. We assume that the uncertainty lies in a bounded polytope, and that the closed-loop input and state sequence should satisfy a given set of linear inequality constraints for all admissible disturbance realizations. Despite the fact that the controlled system is nonlinear, we provide sufficient conditions that allow to preserve convexity of the optimal value function and its domain. As a consequence, the min-max control problems can be either recast as a linear program or solved via N parametric linear programs, where N is the prediction horizon. In some particular cases of the uncertainty description (e.g. interval matrices), by employing results from dynamic programming, we show that a min-max control problem can be recast as a deterministic optimal control problem.

KEY WORDS: Discrete event systems, max-plus-linear systems, robust control, bounded disturbances, linear constraints, dynamic programming.

1. INTRODUCTION

Discrete event systems (DES) are event-driven dynamical systems (i.e. the state transitions are initiated by events, rather than a clock) and they often arise in the context of manufacturing systems, telecommunication networks, parallel computing, supply chain systems, etc. DES with synchronization but no concurrency can be described by nonlinear models called max-plus-linear (MPL) systems, i.e. systems that are "linear" in the max-plus algebra whose basic operations are maximization and addition. Among different methods for designing a controller for an MPL system, the class of optimal controllers is the most studied (see [1–7] and the references therein). However, the robust optimal

Contract/grant sponsor: Research supported by the STW projects "Model predictive control for hybrid systems" (DMR.5675), and "Multi-agent control of large-scale hybrid systems" (DWV.6188), and by the European IST project "Modelling, Simulation and Control of Nonsmooth Dynamical Systems (SICONOS)" (IST-2001-37172) and the European project "HYbrid CONtrol: Taming Heterogeneity and Complexity of Networked Embedded Systems (HYCON)" (FP6-IST-511368).; contract/grant number:

control counterpart for this class of systems is still an active area of research. Some of the contributions include open-loop min-max model predictive control [6] and closed-loop control without constraints based on residuation theory [2, 7, 8]. In [7] a feedback controller is derived using residuation theory that also guarantees stability. But the residuation approach used in [7] does not cope with input–state constraints and moreover the uncertainty is not taken into account. In [8] uncertainty is considered in terms of interval transfer functions, which is a particular case of our uncertainty description considered in this paper. In [2] an adaptive control method is derived that takes into account possible mismatch between the system and its model, but without input-state constraints.

The main advantage of this paper compared to existing results on robust control of MPL systems [2, 6–8] is the fact that we also optimize over feedback policies, not only over open-loop input sequences, and that we incorporate state and input constraints directly into the problem formulation. In general, this results in increased feasibility and a better performance. Because MPL systems are nonlinear, non-convexity is clearly a problem if one seeks to develop "efficient" methods for solving min-max control problems for MPL systems. One of the key contributions of this paper is therefore to provide sufficient conditions, which are often satisfied in practice, such that one can employ results from convex analysis and parametric linear programming to compute robust optimal controllers for MPL systems. It is important to note that we require the stage cost to have a particular representation in which the coefficients corresponding to the state vector are nonnegative *and* that the matrix associated with the state constraints is also nonnegative. However, these conditions are often satisfied in practice.

This section proceeds by introducing some notation specific to max-plus algebra and the class of discrete event MPL systems with disturbances. In Section 2 we define three finite-horizon min-max control problems, depending on the nature of the control input: open-loop input sequences, disturbance feedback policies and state feedback policies. We will show that the corresponding open-loop and disturbance feedback min-max control problem can be recast as a linear program while the state feedback min-max control problem can be solved exactly, without gridding, via N parametric linear programs, where N is the prediction horizon. Finally, for particular cases of the uncertainty (such as interval matrices) we show, using the principle of optimality in dynamic programming, that all three min-max control problems are equivalent with a deterministic optimal control problem.

1.1. Definitions and Notation

Define $\varepsilon := -\infty$, $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$ and two operations [9]: $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$, for $x, y \in \mathbb{R}_\varepsilon$. The triple $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ forms the so-called max-plus algebra. We also denote with $x \oplus' y := \min\{x, y\}$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ one can extend these operations as follows: $(A\oplus B)_{ij} := A_{ij} \oplus B_{ij} = \max\{A_{ij}, B_{ij}\}$, $(A\otimes C)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kj} = \max_{k\in\mathbb{N}_{[1,n]}}\{A_{ik} + C_{kj}\}$ for all i,j. E is the identity matrix in the max-plus algebra: $E_{ii} := 0$, for all i and $E_{ij} := \varepsilon$, for all $i \neq j$ and the zero matrix is $\mathcal{E}$: $\mathcal{E}_{ij} := \varepsilon$, for all $i,\ j$, where their dimensions are derived from the context. For any matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, the k^{th} max-plus power of A is denoted with $A^{\otimes^k} := A \otimes A \otimes \cdots \otimes A$ (k times), and we define A^*, whenever it exists, by $A^* := \lim_{k\to\infty} E \oplus A \oplus \cdots \oplus A^{\otimes^k}$. We denote with λ^* the maximal max-plus eigenvalue of the matrix A, i.e. there exists $v \in \mathbb{R}_\varepsilon^n,\ v \neq \mathcal{E}$ such that $A \otimes v = \lambda^* \otimes v$ (see e.g. [9] for how to compute λ^*).

We use $\mathbb{N}_{[k,l]}$ to represent the set of integers $\{k, k+1, \cdots, l\}$. By $H \geq 0$ we mean that the matrix H is nonnegative, i.e. $H_{ij} \geq 0$ for all i, j. Moreover, $H_{i.}$ and $H_{.j}$ denote the i^{th} row and j^{th} column of H, respectively while n_H denotes the number of rows of H. Given a set $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the projection of $\mathcal{Z}$ on $\mathbb{R}^n$ is denoted by $\mathrm{Proj}_n\mathcal{Z} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ s. t. } (x, y) \in \mathcal{Z}\}$. A polyhedron is the intersection of a finite number of closed half-spaces.

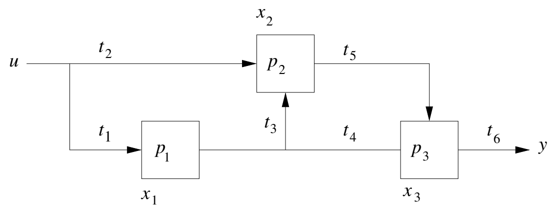

Figure 1. A manufacturing system.

A function $J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ is called *proper* if $J(x) > -\infty$ everywhere and $\{x \in \mathbb{R}^n : J(x) < +\infty\} \neq \emptyset$ [10]. The epigraph of a function $J : \mathcal{X} \to \mathbb{R}$ with $\mathcal{X} \subseteq \mathbb{R}^n$ is defined as $\operatorname{epi} J := \{(x,t) \in \mathcal{X} \times \mathbb{R} : J(x) \leq t\}$. A function J is piecewise affine (PWA) if its epigraph is a finite union of polyhedra [10]. Let $\mathcal{F}_{\text{mps}}$ denote the set of *max-plus-scaling* functions, i.e. functions $J : \mathbb{R}^n \to \mathbb{R}$ such that $J(x) = \max_{j \in \mathbb{N}_{[1,l]}} \{\alpha_j^T x + \beta_j\}$ for all $x \in \mathbb{R}^n$, $\alpha_j \in \mathbb{R}^n$ and $\beta_j \in \mathbb{R}$. Let $\mathcal{F}_{\text{mps}}^+$ denote the set of *max-plus-nonnegative-scaling* functions, i.e. functions defined by $J(x) = \max_{j \in \mathbb{N}_{[1,l]}} \{\alpha_j^T x + \beta_j\}$ with $\alpha_j \geq 0$ for all $j \in \mathbb{N}_{[1,l]}$.

1.2. MPL systems with bounded disturbances

Before introducing the class of (uncertain) MPL systems we provide an example in order to illustrate how disturbances affect an MPL system. Consider the manufacturing system of Figure 1. It consists of three processing units, each unit can only start working on a new product if it has finished processing the previous product (i.e. the unit is idle). We assume that each processing unit starts working as soon as all parts are available. Moreover, at the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. We denote with $u(k)$ the time at which a batch of raw material is fed to the system for the $(k+1)^{th}$ cycle, $x_i(k+1)$ the time at which unit i starts working for the $(k+1)^{th}$ cycle and $y(k)$ the time at which the k^{th} product leaves the system. We also denote with $p_i(k)$ and $t_j(k)$ the transportation and processing times for the $(k+1)^{th}$ cycle. Let us write down the equations corresponding to this manufacturing system:

$$\begin{cases} x_1(k+1) &= \max\{x_1(k) + p_1(k-1),\ u(k) + t_1(k)\} \\ x_2(k+1) &= \max\big\{x_1(k){+}p_1(k{-}1){+}p_1(k){+}t_3(k),\ x_2(k){+}p_2(k{-}1), \\ &\qquad u(k) + \max\{t_2(k),\ p_1(k) + t_1(k) + t_3(k)\}\big\} \\ x_3(k+1) &= \max\big\{x_1(k){+}\max\{p_1(k-1) + p_1(k){+}t_4(k), \\ &\qquad p_1(k-1) + p_1(k) + p_2(k) + t_3(k) + t_5(k)\}, \\ &\qquad x_2(k) + p_2(k-1) + p_2(k) + t_5(k),\ x_3(k) + p_3(k-1), \\ &\qquad u(k){+}\max\{p_1(k){+}t_1(k){+}t_4(k), p_2(k) + t_2(k) + t_5(k), \\ &\qquad p_1(k) + p_2(k) + t_1(k) + t_3(k) + t_5(k)\}\big\} \\ y(k) &= x_3(k) + p_3(k-1) + t_6(k-1). \end{cases} \tag{1}$$

In general, the transportation and processing times may vary from one cycle to another. Therefore, it is clear that the uncertainty comes from the variation of these parameters p_i and t_i. We gather in the vector w all the uncertainty caused by disturbances and errors in the estimation of the parameters p_i and t_i, i.e.

$$w(k) := [p_1(k) \ldots p_l(k)\ t_1(k) \ldots t_{\tilde{l}}(k)]^T.$$

This example can be generalized. We consider the following uncertain MPL system*:

$$\begin{aligned} x(k+1) &= A(w(k-1), w(k)) \otimes x(k) \oplus B(w(k-1), w(k)) \otimes u(k) \\ y(k) &= C(w(k-1)) \otimes x(k). \end{aligned} \tag{2}$$

Since the system matrices of a DES modeled as an MPL system usually consist of sums or maxima of internal process and transportation times, it follows that $A \in \mathcal{F}_{\rm mps}^{n\times n}, B \in \mathcal{F}_{\rm mps}^{n\times m}$ and $C \in \mathcal{F}_{\rm mps}^{p\times n}$ (it is important to note that these matrix functions are nonlinear). We frequently use the short-hand notation

$$f_{\rm MPL}(x, u, w_{\rm p}, w_{\rm c}) := A(w_{\rm p}, w_{\rm c}) \otimes x \oplus B(w_{\rm p}, w_{\rm c}) \otimes u,$$

Clearly, $f_{\rm MPL} \in \mathcal{F}_{\rm mps}^n$ and $f_{\rm MPL}(\cdot, u, w_{\rm p}, w_{\rm c}) \in (\mathcal{F}_{\rm mps}^+)^n$ for any fixed $(u, w_{\rm p}, w_{\rm c})$.

In the context of DES, k is an event counter while $x(k) \in \mathbb{R}_\varepsilon^n$, $y(k) \in \mathbb{R}_\varepsilon^p$ and $u(k) \in \mathbb{R}_\varepsilon^m$ represent times, i.e. starting times, finishing times and feeding times, respectively. Since the states $x(k)$ represent times, we assume they can always be measured or estimated. For a method to estimate or compute $x(k)$ for MPL systems and in the context of model predictive control the reader is referred to [22]. At each step k, the value of the disturbance $w(k)$ is unknown, but it is assumed to be time-varying and to take on values from a polytope $W = \{w \in \mathbb{R}^q : \Omega w \leq s\}$, where $\Omega \in \mathbb{R}^{n_\Omega \times q}$ and $s \in \mathbb{R}^{n_\Omega}$. Therefore, $l + \tilde{l} = q$. We consider that $w(k-1)$ and $w(k)$ are independent. Moreover, at event step k we assume that the disturbance $w(k-1)$ can be computed or measured. Note that since the state $x(k)$ is assumed to be available, $w(k-1)$ can also be computed. For more details about timing issues see also Remark 2.14.

We consider a reference (due date) signal $\{r(k) \in \mathbb{R}^p\}_{k\geq 0}$ which the output of the system (2) may be required to "track", in the sense that, for instance, the tardiness $\max\{y - r, 0\}$ is penalized. The system is assumed to be subject to hard control and state linear constraints over a finite horizon N [21]:

$$H_k x(k) + G_k u(k) + F_k r(k) \leq h_k, \quad k \in \mathbb{N}_{[0,N-1]}, \tag{3}$$

with the *terminal constraint*

$$H_N x(N) + F_N r(N) \leq h_N, \tag{4}$$

where $H_k \in \mathbb{R}^{n_k \times n}, G_k \in \mathbb{R}^{n_k \times m}, F_k \in \mathbb{R}^{n_k \times p}, h_k \in \mathbb{R}^{n_k}$.

2. ROBUST CONTROL FOR UNCERTAIN MPL SYSTEMS

In this section we analyze the solutions to three classes of finite horizon min-max control problems for uncertain MPL systems, each class depending on the nature of the control sequence over which we optimize: open-loop control sequences, disturbance feedback policies, and state feedback policies. Robust performance and robust constraint fulfillment are considered with respect to all possible realizations of the disturbance in a worst-case approach.

---

*Note that the model (2) resembles the conventional nonlinear models although in the context of DES many researchers use the following model for a deterministic MPL system $x(k) = A \otimes x(k-1) \oplus B \otimes u(k)$.

2.1. Open-loop input sequences

Open-loop worst-case control for uncertain MPL systems, applied in a receding horizon fashion, was also considered in [6]. Here we discuss a rather more general control problem: we include mixed state-inputs constraints and we show that the optimal control input can be computed by solving a single linear program, without having to resort to computation of vertices of $\mathcal{W} := W^N$, as was done in [6].

Let $\mathbf{u} := [u_0^T\ u_1^T \ldots u_{N-1}^T]^T$ be an open-loop input sequence and $\mathbf{w} := [w_0^T\ w_1^T \ldots w_{N-1}^T]^T$ denote a realization of the disturbance over the prediction horizon N. Also, let $\phi(i; x, w, \mathbf{u}, \mathbf{w})$ denote the state solution of (2) at event step i when the initial state is x at event step 0, the initial value of the disturbance is w (i.e. $w(-1) = w$), the control is determined by $\mathbf{u}$ (i.e. $u(i) = u_i$) and the disturbance sequence is $\mathbf{w}$. By definition, $\phi(0; x, w, \mathbf{u}, \mathbf{w}) := x$.

Given the initial condition x, the initial disturbance w, the reference signal $\mathbf{r} := [r_0^T\ r_1^T \cdots r_N^T]^T$, the control sequence $\mathbf{u}$, and the disturbance realization $\mathbf{w}$, the cost function $V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w})$ is defined as:

$$V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}) := \sum_{i=0}^{N-1} \ell_i(x_i, u_i, r_i) + V_{\mathrm{f}}(x_N, r_N),$$

where $x_i := \phi(i; x, w, \mathbf{u}, \mathbf{w})$ (and thus $x_0 := x$) and the *terminal cost* is defined as $V_{\mathrm{f}}(x_N, r_N) := \ell_N(x_N, r_N)$. We usually denote with X_{f} the terminal set, i.e.

$$X_{\mathrm{f}} := \{(x, r) : H_N x + F_N r \leq h_N\}.$$

The following key assumptions will be used throughout the paper:

- **A1**: The matrices H_i in (3)–(4) are nonnegative for all $i \in \mathbb{N}_{[0,N]}$.
- **A2**: The stage costs ℓ_i satisfy $\ell_i(\cdot, u, r) \in \mathcal{F}^+_{\mathrm{mps}}$, $\forall (u, r)$ and $\ell_i \in \mathcal{F}_{\mathrm{mps}}$ for all $i \in \mathbb{N}_{[0,N-1]}$. Moreover, $\ell_N(\cdot, r) \in \mathcal{F}^+_{\mathrm{mps}}$, $\forall r$ and $\ell_N \in \mathcal{F}_{\mathrm{mps}}$.

The conditions from assumptions **A1**-**A2** are not too restrictive and are usually met in applications. Note that typical constraints for MPL systems satisfy assumption **A1** (see [21, 22]). A typical example of a stage cost satisfying assumption **A2** is [22]:

$$\ell_i(x_i, u_i, r_i) = \sum_{j=1}^{p} \max\{[y_i - r_i]_j, 0\} - \gamma \sum_{j=1}^{m} [u_i]_j, \tag{5}$$

where $[v_i]_j$ denotes the j^{th} component of a vector v_i, $y_i := C(w_{i-1}) \otimes x_i$ denotes the output at event step i of system (2) and $\gamma \geq 0$. In the context of manufacturing systems the first term of (5) penalizes the delay of the finishing products with respect to the due dates, while the second term tries to feed raw material as late as possible.

For each initial condition x, initial disturbance w and due dates $\mathbf{r}$ we define the set of feasible open-loop input sequences $\mathbf{u}$:

$$\Pi_N^{\mathrm{ol}}(x, w, \mathbf{r}) := \{\mathbf{u} : H_i x_i + G_i u_i + F_i r_i \leq h_i,\ (x_N, r_N) \in X_{\mathrm{f}}, i \in \mathbb{N}_{[0,N-1]},\ \forall \mathbf{w} \in \mathcal{W}\}.$$

Also, let X_N^{ol} denote the set of initial states and reference signals for which a feasible input sequence exists:

$$X_N^{\mathrm{ol}} := \{(x, w, \mathbf{r}) : \Pi_N^{\mathrm{ol}}(x, w, \mathbf{r}) \neq \emptyset\}.$$

The *finite-horizon open-loop min-max* control problem† is defined as:

$$\mathbb{P}_N^{\rm ol}(x,w,\mathbf{r}): \qquad V_N^{0,\rm ol}(x,w,\mathbf{r}) := \inf_{\mathbf{u}\in\Pi_N^{\rm ol}(x,w,\mathbf{r})} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w}). \tag{6}$$

Let $\mathbf{u}_N^0(x,w,\mathbf{r}) = [(u_0^0(x,w,\mathbf{r}))^T \cdots (u_{N-1}^0(x,w,\mathbf{r}))^T]^T$ be an optimizer of (6) if the infimum is attained, i.e. $\mathbf{u}_N^0(x,w,\mathbf{r}) \in \arg\min_{\mathbf{u}\in\Pi_N^{\rm ol}(x,w,\mathbf{r})} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w})$. Standard optimal control implements the control sequence $u(k) = u_k^0(x(0),w(-1),[r^T(0)\ r^T(1)\cdots r^T(N)]^T)$, while model predictive control [11] involves an iterative moving horizon approach in which at each event step k the optimal control sequence is recomputed and only the first sample of the control sequence, i.e. $u_0^0\big(x(k),w(k-1),[r^T(k)\ r^T(k+1)\cdots r^T(k+N)]^T\big)$, is applied to the system, for $k=0,1,\cdots$. Robust stability of the model predictive control is studied in [21]. Note that a stabilizing controller based on residuation theory was derived also in [7] for deterministic MPL systems.

If we denote with $\mathbf{x} := [x_0^T\ x_1^T \cdots x_N^T]^T$ then it follows that:

$$\mathbf{x} = \begin{bmatrix} E \\ \Theta(1,1;w,\mathbf{w}) \\ \vdots \\ \Theta(N,1;w,\mathbf{w}) \end{bmatrix} \otimes x \oplus \begin{bmatrix} \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} \\ B(w,w_0) & \mathcal{E} & \cdots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(N,1;w,\mathbf{w}) & \Phi(N,2;w,\mathbf{w}) & \cdots & B(w_{N-2},w_{N-1}) \end{bmatrix} \otimes \mathbf{u},$$

where $\Theta(k,1;w,\mathbf{w}) := A(w_{k-2},w_{k-1}) \otimes \cdots \otimes A(w,w_0)$ and $\Phi(k,j;w,\mathbf{w}) := A(w_{k-2},w_{k-1}) \otimes \cdots \otimes A(w_{j-1},w_j) \otimes B(w_{j-2},w_{j-1})$ (note that we consider $w_{-1} = w$). Therefore, $\mathbf{x}$ can be written more compactly as:

$$\mathbf{x} = \Theta(w,\mathbf{w}) \otimes x \oplus \Phi(w,\mathbf{w}) \otimes \mathbf{u}, \tag{7}$$

where $\Theta(w,\mathbf{w})$ and $\Phi(w,\mathbf{w})$ are appropriately defined. Similarly, the inequalities (3)–(4) can be written as:

$$\mathbf{Hx} + \mathbf{Gu} + \mathbf{Fr} \le \mathbf{h},$$

for some matrices $\mathbf{H},\mathbf{G},\mathbf{F}$ and a vector $\mathbf{h}$ of appropriate dimensions. Note that $\mathbf{H}$ has nonnegative entries according to assumption **A1**. Now, the set of admissible open-loop input sequences $\Pi_N^{\rm ol}(x,w,\mathbf{r})$ can be rewritten more compactly as:

$$\Pi_N^{\rm ol}(x,w,\mathbf{r}) = \{\mathbf{u} : \mathbf{H}(\Theta(w,\mathbf{w})\otimes x \oplus \Phi(w,\mathbf{w})\otimes\mathbf{u}) + \mathbf{Gu} + \mathbf{Fr} \le \mathbf{h},\ \forall \mathbf{w}\in\mathcal{W}\}. \tag{8}$$

After some manipulations we obtain that the set of feasible $\mathbf{u}$ is given by:

$$\Pi_N^{\rm ol}(x,w,\mathbf{r}) = \{\mathbf{u} : F\mathbf{u} + \Psi\mathbf{w} \le c(x,w,\mathbf{r}),\ \forall \mathbf{w}\in\mathcal{W}\}, \tag{9}$$

where $F \in \mathbb{R}^{n_F\times Nm}$, $\Psi \in \mathbb{R}^{n_F\times Nq}$ and $c(x,w,\mathbf{r}) \in \mathbb{R}^{n_F}$ is an affine expression in $(x,w,\mathbf{r})$.

Lemma 2.1. *The sets* $X_N^{\rm ol}$ *and* $\Pi_N^{\rm ol}(x,w,\mathbf{r})$ *are polyhedra.*

Proof: Note that $\Pi_N^{\rm ol}(x,w,\mathbf{r}) = \{\mathbf{u} : F\mathbf{u} \le c(x,w,\mathbf{r}) - \psi^0\}$, where the i^{th} component of the vector ψ^0 is given by $\psi_i^0 := \max_{\mathbf{w}\in\mathcal{W}} \Psi_{i.}\mathbf{w}$ (recall that $\Psi_{i.}$ denotes the i^{th} row of Ψ). Since W is a compact set it follows that ψ^0 is a finite vector. Therefore, $\Pi_N^{\rm ol}(x,w,\mathbf{r})$ is a polyhedron.

---

†Since V_N is continuous and $\mathcal{W}$ is compact, "sup" of V_N over $\mathcal{W}$ is attained and thus we can use directly "max" instead of sup.

Similarly $X_N^{\rm ol} = \{(x, w, \mathbf{r}) : \exists \mathbf{u}$ such that $F\mathbf{u} \leq c(x, w, \mathbf{r}) - \psi^0\}$ and since $c(x, w, \mathbf{r})$ is an affine expression in $(x, w, \mathbf{r})$ it follows that $X_N^{\rm ol}$ is the projection of the polyhedron $\{(x, w, \mathbf{r}, \mathbf{u}) : F\mathbf{u} - c(x, w, \mathbf{r}) \leq \psi^0\}$ onto a suitably-defined subspace. Therefore, $X_N^{\rm ol}$ is a polyhedron. ♢

Since $\ell_i(\cdot, u, r) \in \mathcal{F}^+_{\rm mps}, \forall (u, r)$ it follows that:

$$V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}) = \max_{j \in \mathcal{J}} \{\alpha_j^T \mathbf{x} + \beta_j^T \mathbf{u} + \gamma_j^T \mathbf{w} + \delta_j(x, w, \mathbf{r})\}, \tag{10}$$

where α_j are nonnegative vectors (i.e. $\alpha_j \geq 0$ for all $j \in \mathcal{J}$) and $\delta_j(x, w, \mathbf{r})$ are affine expressions in $(x, w, \mathbf{r})$.

Remark 2.2 Note that if the entries of matrix functions A, B and C are max-plus-nonnegative-scaling functions (i.e. A_{ij}, B_{il} and C_{kl} are in $\mathcal{F}^+_{\rm mps}$ for all i, j, l and k) then the vectors γ_j are also nonnegative. We will make use of this property in Section 4. ♢

Equivalently, we can write $V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w})$ as:

$$V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}) = \max_{i \in \mathcal{I}} \{\tilde{p}_i^T \mathbf{u} + q_i^T \mathbf{w} + s_i(x, w, \mathbf{r})\}, \tag{11}$$

for some vectors $\tilde{p}_i, q_i$ and $s_i(x, w, \mathbf{r})$ are affine expressions in $(x, w, \mathbf{r})$. We define:

$$J_N(x, w, \mathbf{r}, \mathbf{u}) := \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}). \tag{12}$$

Lemma 2.3. *The function* $(x, w, \mathbf{r}, \mathbf{u}) \mapsto J_N(x, w, \mathbf{r}, \mathbf{u})$ *is convex and PWA.*

Proof: From (11) we remark that $V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w})$ is a convex function in $(x, w, \mathbf{r}, \mathbf{u})$ since $z \mapsto \max_i\{z_i\}$ is a convex map and convexity is preserved under composition of a convex function with affine maps. Since the point-wise supremum of an arbitrary, infinite set of convex functions is convex [10], it follows that $J_N(x, w, \mathbf{r}, \mathbf{u})$ is a convex function. Using similar arguments we can prove that J_N is also PWA (see also the explanations below). ♢

If we denote with $q_i^0 = \max_{\mathbf{w} \in \mathcal{W}} q_i^T \mathbf{w}$ (since $\mathcal{W}$ is compact, the q_i^0's are finite and can be computed by solving a linear program), then the open-loop min-max optimization problem (6) can be recast as a *linear program*:

$$\min_{\mu, \mathbf{u}} \{\mu : \ F\mathbf{u} \leq c(x, w, \mathbf{r}) - \psi^0, \ \ \tilde{p}_i^T \mathbf{u} - \mu \leq -q_i^0 - s_i(x, w, \mathbf{r}), \ \forall i \in \mathcal{I}\}. \tag{13}$$

Note that in [6] a solution of (6) is obtained by first computing the vertices of $\mathcal{W}$. Let $\mathcal{L}_v$ be the number of vertices of W. In the worst-case the number of vertices of $\mathcal{W}$ may be exponential: $\mathcal{L}_v^N \geq 2^{qN}$. So, the computational complexity of our approach is better than the approach of [6] because in the corresponding linear program of [6] we have $|\mathcal{I}|(\mathcal{L}_v^N - 1)$ more inequalities and also more variables than in our linear program (13).

2.2. *Disturbance feedback policies*

Effective control in the presence of disturbance requires one to optimize over *feedback policies* [12,13], rather than open-loop input sequences. A feedback controller prevents the trajectory from diverging excessively and also the performance is improved compared to the open-loop case. One way of including feedback is to consider semi-feedback control sequences, i.e. to search over the set of time-varying max-plus-scaling (i.e. convex piecewise affine) state feedback control policies with memory of prior states [1, 2, 7]:

$$u_i = \bigoplus_{j=0}^{i} L_{i,j} \otimes x_j \oplus g_i, \ \forall i \in \mathbb{N}_{[0,N-1]}, \tag{14}$$

where each $L_{i,j} \in \mathbb{R}_{\varepsilon}^{m\times n}$ and $g_i \in \mathbb{R}_{\varepsilon}^{m}$. We can also consider the affine approximation of (14), i.e time-varying affine state feedback control policies with memory of prior states:

$$u_i = \sum_{j=0}^{i} \tilde{L}_{i,j} x_j + \tilde{g}_i, \ \forall i \in \mathbb{N}_{[0,N-1]}, \tag{15}$$

where each $\tilde{L}_{i,j} \in \mathbb{R}^{m\times n}$ and $\tilde{g}_i \in \mathbb{R}^m$,

It is known, even for linear systems [14–16], that given an initial state x and an initial disturbance w, the set of gains $\tilde{L}_{i,j}$ and $\tilde{g}_i$ such that the control sequence given by (15) satisfies the constraints (3)–(4) is a non-convex set (and thus a similar result holds for max-plus-scaling state feedback control policies (14)). Therefore, finding admissible $\tilde{L}_{i,j}$ and $\tilde{g}_i$ ($L_{i,j}$ and g_i, respectively) given the current state x and current disturbance w is a very difficult problem. The state feedback policy (14) can be written more compactly as $\mathbf{u} = \mathbf{L} \otimes \mathbf{x} \oplus \mathbf{g}$, where $\mathbf{L}$ and $\mathbf{g}$ have appropriate dimensions. Replacing the expression of $\mathbf{u}$ in (7) ones gets that: $\mathbf{x} = \Phi(w, \mathbf{w}) \otimes \mathbf{L} \otimes \mathbf{x} \oplus \Theta(w, \mathbf{w}) \otimes x \oplus \Phi(w, \mathbf{w}) \otimes \mathbf{g}$. Using the fact (see [9]) that the equation $z = D \otimes z \oplus c$ has a solution in max-plus algebra given by $x = D^* \otimes c$, we can easily derive that $\mathbf{x} = \big(\Phi(w, \mathbf{w}) \otimes \mathbf{L}\big)^* \otimes \big(\Theta(w, \mathbf{w}) \otimes x \oplus \Phi(w, \mathbf{w}) \otimes \mathbf{g}\big)$ and after some long but straightforward computations $\mathbf{u}$ can be rewritten as:

$$\mathbf{u} = \big(\mathbf{L} \otimes \Phi(w, \mathbf{w})\big)^* \otimes \big(\mathbf{L} \otimes \Theta(w, \mathbf{w}) \otimes x \oplus \mathbf{g}\big).$$

Define $\mathbf{u}(w, \mathbf{w}) := (\mathbf{L} \otimes \Phi(w, \mathbf{w}))^* \otimes (\mathbf{L} \otimes \Theta(w, \mathbf{w}) \otimes x \oplus \mathbf{g})$ then the function $(w, \mathbf{w}) \mapsto \mathbf{u}(w, \mathbf{w})$ is in $\mathcal{F}_{\text{mps}}^{Nm}$ (i.e. a convex piecewise affine function). Recall that we assume at each step k the previous disturbances $w, w_0 \ldots w_{k-1}$ are known (they can be computed or measured). Since $\mathbf{L}$ and $\Phi(w, \mathbf{w})$ are lower triangular matrices, it can be proved after some long but straightforward computations that $u_i(w, \mathbf{w})$ is a max-plus-scaling function depending only on the previous disturbances $w, w_0 \ldots w_{i-1}$, for all $i \in \mathbb{N}_{[0,N-1]}$. It follows that the class of time-varying max-plus-scaling state feedback policies with memory of the prior states defined in (14) is included in the class of *max-plus-scaling disturbance feedback policies with memory of the prior disturbances*. Therefore, an alternative approach to state feedback policies (14) is to parameterize the control policy as a max-plus-scaling function of the previous disturbances. Unfortunately, this parametrization of the control will lead to non-convex inequalities as well. As an alternative, we propose to approximate the convex piecewise affine function $\mathbf{u}(w, \mathbf{w})$ with an affine one, i.e. to parameterize the controller as an affine function of the past disturbances [14, 16]:

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \ \forall i \in \mathbb{N}_{[0,N-1]}, \tag{16}$$

where $M_{i,j} \in \mathbb{R}^{m\times q}$ and $v_i \in \mathbb{R}^m$. We now show that contrary to state feedback policies (14) or (15), the set of gains M_{ij} and v_i such that the control sequence (16) satisfies the constraints (3)–(4) is a convex set.

Let us denote with $\mathbf{v} := [v_0^T \ v_1^T \cdots v_{N-1}^T]^T$ and

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1,0} & M_{N-1,1} & \cdots & 0 \end{bmatrix} \tag{17}$$

so that the disturbance feedback policy becomes $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$. Note that for $\mathbf{M} = 0$, (15) reduces to an open-loop control sequence.

For each x, w and $\mathbf{r}$ we define the set of feasible pairs $(\mathbf{M}, \mathbf{v})$:

$$\Pi_N^{\mathrm{df}}(x, w, \mathbf{r}) = \{(\mathbf{M}, \mathbf{v}) : \mathbf{M} \text{ as in (17)},\ u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, H_i x_i + G_i u_i + F_i r_i \le h_i,$$
$$(x_N, r_N) \in X_{\mathrm{f}}, \forall i \in \mathbb{N}_{[0,N-1]},\ \forall \mathbf{w} \in \mathcal{W}\}.$$

The *finite-horizon disturbance feedback min-max* control problem becomes:

$$\mathbb{P}_N^{\mathrm{df}}(x, w, \mathbf{r}) : \qquad V_N^{0,\mathrm{df}}(x, w, \mathbf{r}) := \inf_{(\mathbf{M},\mathbf{v}) \in \Pi_N^{\mathrm{df}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \mathbf{M}\mathbf{w} + \mathbf{v}, \mathbf{w}). \tag{18}$$

We denote with $(\mathbf{M}_N^0(x, w, \mathbf{r}), \mathbf{v}_N^0(x, w, \mathbf{r}))$ an optimizer of (18), whenever the infimum is attained. Let X_N^{df} denote the set of initial states for which a solution to the optimization problem (18) exists, i.e.

$$X_N^{\mathrm{df}} = \{(x, w, \mathbf{r}) : \Pi_N^{\mathrm{df}}(x, w, \mathbf{r}) \neq \emptyset\}.$$

We will show in the sequel that the set $\Pi_N^{\mathrm{df}}(x, w, \mathbf{r})$ is polyhedral and moreover the optimization problem (18) is a linear program, for all $(x, w, \mathbf{r}) \in X_N^{\mathrm{df}}$. ¿From (7) it follows that $\mathbf{x}$ can be written as $\mathbf{x} = \Theta(w, \mathbf{w}) \otimes x \oplus \Phi(w, \mathbf{w}) \otimes (\mathbf{M}\mathbf{w} + \mathbf{v})$, Using (9), the set of admissible affine disturbance feedback parameters $\Pi_N^{\mathrm{df}}(x, w, \mathbf{r})$ can be rewritten more compactly as follows:

$$\Pi_N^{\mathrm{df}}(x, w, \mathbf{r}) = \{(\mathbf{M}, \mathbf{v}) : \mathbf{M} \text{ as in (17)}, F\mathbf{v} + (F\mathbf{M} + \Psi)\mathbf{w} \le c(x, w, \mathbf{r}),\ \forall \mathbf{w} \in \mathcal{W}\}.$$

Lemma 2.4. *The sets X_N^{df} and $\Pi_N^{\mathrm{df}}(x, w, \mathbf{r})$ are polyhedra.*

Proof: From previous formula it follows that we can rewrite $\Pi_N^{\mathrm{df}}(x, w, \mathbf{r})$ equivalently as:

$$\Pi_N^{\mathrm{df}}(x, w, \mathbf{r}) = \big\{(\mathbf{M}, \mathbf{v}) : \mathbf{M} \text{ as in (17)},\ F\mathbf{v} + \max_{\mathbf{w} \in \mathcal{W}}\{(F\mathbf{M} + \Psi)\mathbf{w}\} \le c(x, w, \mathbf{r})\big\},$$

where $\max_{\mathbf{w} \in \mathcal{W}}\{(F\mathbf{M} + \Psi)\mathbf{w}\}$ is the vector defined as follows

$$\max_{\mathbf{w} \in \mathcal{W}}\{(F\mathbf{M} + \Psi)\mathbf{w}\} := \big[\max_{\mathbf{w} \in \mathcal{W}}\{(F\mathbf{M} + \Psi)_{1\cdot}\mathbf{w}\} \cdots \max_{\mathbf{w} \in \mathcal{W}}\{(F\mathbf{M} + \Psi)_{n_F\cdot}\mathbf{w}\}\big]^T,$$

and where $(F\mathbf{M} + \Psi)_{i\cdot}$ denotes the i^{th} row of the matrix $F\mathbf{M} + \Psi$. Since $\mathcal{W}$ is a polytope, we can compute an admissible pair $(\mathbf{M}, \mathbf{v})$ using duality for linear programming [17], by solving a single linear program. It is clear that

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^{Nq} : \mathbf{\Omega}\mathbf{w} \le \mathbf{s}\}, \tag{19}$$

where‡ $\mathbf{\Omega} = \mathrm{diag}(\Omega)$ and $\mathbf{s} = [s^T \cdots s^T]^T$. The dual problem of the linear program

$$\max_{\mathbf{w}}\{(F\mathbf{M} + \Psi)_{i\cdot}\mathbf{w} : \mathbf{\Omega}\mathbf{w} \le \mathbf{s}\}$$

is the following linear program

$$\min_{d_i}\{\mathbf{s}^T d_i : \mathbf{\Omega}^T d_i = (F\mathbf{M} + \Psi)_{i\cdot}^T,\ d_i \ge 0\}.$$

‡ $\mathrm{diag}(\Omega)$ denotes the block diagonal matrix having the entries on the diagonal equal to Ω and the rest equal to 0.

In conclusion, we can write:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r}) = \{(\mathbf{M},\mathbf{v}) : \exists \mathbf{D}\geq 0, \mathbf{M} \text{ as in (17)},\ F\mathbf{v}+\mathbf{D}^T\mathbf{s}\leq c(x,w,\mathbf{r}), F\mathbf{M}+\Psi=\mathbf{D}^T\mathbf{\Omega}\},$$

where $\mathbf{D}\in\mathbb{R}^{Nn_\Omega\times n_F}$ is defined as $\mathbf{D}_{.j}=d_j$ for all $j\in\mathbb{N}_{[1,n_F]}$ (recall that $\mathbf{D}_{.j}$ denotes the j^{th} column of $\mathbf{D}$).

It is clear that $\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})$ is a polyhedron, since it is the projection of the polyhedron

$$\{(\mathbf{M},\mathbf{v},\mathbf{D}) : \mathbf{M} \text{ as in (17)},\ \mathbf{D}\geq 0,\ F\mathbf{v}+\mathbf{D}^T\mathbf{s}\leq c(x,w,\mathbf{r}),\ F\mathbf{M}+\Psi=\mathbf{D}^T\mathbf{\Omega}\}$$

onto a suitably defined subspace. Similarly $X_N^{\mathrm{df}} = \{(x,w,\mathbf{r}) : \exists(\mathbf{M},\mathbf{v}),\ \mathbf{M} \text{ as in (17)},\ \mathbf{D}\geq 0,\ F\mathbf{v}+\mathbf{D}^T\mathbf{s}\leq c(x,w,\mathbf{r}),\ F\mathbf{M}+\Psi=\mathbf{D}^T\mathbf{\Omega}\}$ and since $c(x,w,\mathbf{r})$ is an affine expression in $(x,w,\mathbf{r})$ it follows that X_N^{df} is also the projection of a polyhedron onto a suitably-defined subspace and thus X_N^{df} is a polyhedron. ♢

¿From (11) it follows that, as a function of $(\mathbf{M},\mathbf{v})$, V_N can be expressed as:

$$V_N(x,w,\mathbf{r},\mathbf{M}\mathbf{w}+\mathbf{v},\mathbf{w}) = \max_{i\in\mathcal{I}}\{\tilde{p}_i^T\mathbf{v}+(\tilde{p}_i^T\mathbf{M}+q_i^T)\mathbf{w}+s_i(x,w,\mathbf{r})\}.$$

We define:

$$J_N(x,w,\mathbf{r},\mathbf{M},\mathbf{v}) := \max_{\mathbf{w}\in\mathcal{W}} V_N(x,w,\mathbf{r},\mathbf{M}\mathbf{w}+\mathbf{v},\mathbf{w}).$$

Using the same arguments as in the proof of Lemma 2.3 we obtain:

Lemma 2.5. *The function* $(x,w,\mathbf{r},\mathbf{M},\mathbf{v})\mapsto J_N(x,w,\mathbf{r},\mathbf{M},\mathbf{v})$ *is convex.* ♢

Theorem 2.6. *The robust optimal control problem* (18) *can be recast as a linear program.*

Proof: Note that $J_N(x,w,\mathbf{r},\mathbf{M},\mathbf{v}) = \max_{i\in\mathcal{I}}\left\{\tilde{p}_i^T\mathbf{v}+\max_{\mathbf{w}\in\mathcal{W}}\{(\tilde{p}_i^T\mathbf{M}+q_i^T)\mathbf{w}\}+s_j(x,w,\mathbf{r})\right\}$. Using again duality for linear programming (see also [16, 17]) it follows that

$$\max_{\mathbf{w}\in\mathcal{W}}\{(\tilde{p}_i^T\mathbf{M}+q_i^T)\mathbf{w}\} = \min_{z_i}\{\mathbf{s}^Tz_i : \mathbf{\Omega}^Tz_i = (\tilde{p}_i^T\mathbf{M}+q_i^T)^T,\ z_i\geq 0\}.$$

So, the robust optimal control problem (18) can be recast as a linear program:

$$\min_{\mu,\mathbf{M},\mathbf{v},\mathbf{D},\mathbf{Z}}\{\mu :\ \mathbf{M} \text{ as in (17)},\ F\mathbf{v}+\mathbf{D}^T\mathbf{s}\leq c(x,w,\mathbf{r}),\ F\mathbf{M}+\Psi=\mathbf{D}^T\mathbf{\Omega}, P^T\mathbf{M}+Q^T=\mathbf{Z}^T\mathbf{\Omega}, \mathbf{Z}\geq 0,$$
$$P^T\mathbf{v}+\mathbf{Z}^T\mathbf{s}+S(x,w,\mathbf{r})\leq\bar{\mu},\ \mathbf{D}\geq 0,\ \bar{\mu}=[\mu\ldots\mu]^T,\ \mathbf{D}\in\mathbb{R}^{Nn_\Omega\times n_F},\ \mathbf{Z}\in\mathbb{R}^{Nn_\Omega\times|\mathcal{I}|}\}. \qquad (20)$$

where $P_{.j}=p_j,\ Q_{.j}=q_j,\ S_j(x,w,\mathbf{r})=s_j(x,w,\mathbf{r})$ and $Z_{.j}=z_j$, for all $j\in\mathcal{I}$. ♢

In the particular case when $\mathbf{M}=0$ we obtain the open-loop controller derived in Section 2.1 and thus

$$X_N^{\mathrm{ol}}\subseteq X_N^{\mathrm{df}},\quad V_N^{0,\mathrm{df}}(x,w,\mathbf{r})\leq V_N^{0,\mathrm{ol}}(x,w,\mathbf{r})\ \ \forall(x,w,\mathbf{r})\in X_N^{\mathrm{ol}}.$$

2.3. State feedback policies

In this section we consider full state feedback policies. Therefore, we will define the decision variable in the optimal control problem, for a given initial condition x, initial disturbance w and the reference signal $\mathbf{r}$ as a control policy $\pi := (\mu_0,\mu_1,\ldots,\mu_{N-1})$, where each $\mu_i:\mathbb{R}^n\times\mathbb{R}^q\times\mathbb{R}^{Np}\to\mathbb{R}^m$ is a state feedback control *law*. Also, let $x_i=\phi(i;x,w,\pi,\mathbf{w})$ denote the state solution of (2) at step i when the initial state is x at step 0, the initial disturbance is w, the control is determined by the policy π, i.e. $u(i)=\mu_i(\phi(i-1;x,w,\pi,\mathbf{w}),w_{i-1},\mathbf{r})$, and the disturbance sequence is $\mathbf{w}$ (where $w_{-1}:=w$).

For each initial condition x, initial disturbance w and due dates $\mathbf{r}$ we define the set of feasible policies π:

$$\Pi_N^{\mathrm{sf}}(x, w, \mathbf{r}) := \{\pi : \ H_i x_i + G_i \mu_i(x_{i-1}, w_{i-1}, \mathbf{r}) + F_i r_i \le h_i, \ i \in \mathbb{N}_{[0,N-1]}, \\ (x_N, r_N) \in X_{\mathrm{f}}, \ \forall \mathbf{w} \in \mathcal{W}\}. \tag{21}$$

Also, let X_N^{sf} denote the set of initial states *and* reference signals for which a feasible policy exists, i.e.

$$X_N^{\mathrm{sf}} := \{(x, w, \mathbf{r}) : \Pi_N^{\mathrm{sf}}(x, w, \mathbf{r}) \neq \emptyset\}. \tag{22}$$

The *finite-horizon state feedback min-max* control problem considered here is:

$$\mathbb{P}_N^{\mathrm{sf}}(x, w, \mathbf{r}) : \qquad V_N^{0,\mathrm{sf}}(x, w, \mathbf{r}) := \inf_{\pi \in \Pi_N^{\mathrm{sf}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \pi, \mathbf{w}). \tag{23}$$

Let $\pi_N^0(x, w, \mathbf{r}) \in \arg\min_{\pi \in \Pi_N^{\mathrm{sf}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \pi, \mathbf{w})$ be an optimizer, whenever the infimum is attained. We will proceed to show how for the assumptions **A1** and **A2**, in conjunction with the convexity and monotonicity of the system dynamics (2), an explicit expression of the solution to the state feedback problem (23) can be computed via dynamic programming (DP), using results from polyhedral algebra and parametric linear programming.

DP is a well-known method for solving sequential, or multi-stage, decision problems [12,13]. More specifically, one computes sequentially the partial return functions $\{V_i^0\}_{i=1}^N$, the associated set-valued optimal control laws $\{\kappa_i\}_{i=1}^N$ (such that $\mu_{N-i}^0(x, w_{\mathrm{p}}, \mathbf{r}) \in \kappa_i(x, w_{\mathrm{p}}, \mathbf{r})$) and their domains $\{X_i\}_{i=1}^N$. If we define

$$J_i(x, w_{\mathrm{p}}, \mathbf{r}, u) := \max_{w_{\mathrm{c}} \in W} \{\ell_{N-i}(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), u, r_{N-i}) + \\ V_{i-1}^0(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), w_{\mathrm{c}}, \mathbf{r})\}, \tag{24a}$$

for all $(x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z_i$,, where

$$Z_i := \{(x, w_{\mathrm{p}}, \mathbf{r}, u) \colon H_{N-i} f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}) + G_{N-i} u + F_{N-i} r_{N-i} \le h_{N-i}, \\ w_{\mathrm{p}} \in W, \ (f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), w_{\mathrm{c}}, \mathbf{r}) \in X_{i-1}, \ \forall w_{\mathrm{c}} \in W\}, \tag{24b}$$

then we can compute $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ recursively, as follows [13]:

$$X_i := \{(x, w_{\mathrm{p}}, \mathbf{r}) : (x, \mathbf{r}) \in X_i^{(x,\mathbf{r})}, \ w_{\mathrm{p}} \in W\}, \ X_N := \mathrm{Proj}_{n+q+pN} Z_N, \tag{24c}$$

where for all $i \in \mathbb{N}_{[1,N-1]}$ (since for X_i must hold that the previous disturbance is in the set W)

$$X_i^{(x,\mathbf{r})} := \{(x, \mathbf{r}) : (x, w_{\mathrm{p}}, \mathbf{r}) \in \mathrm{Proj}_{n+q+pN} Z_i, \forall w_{\mathrm{p}} \in W\}, \tag{24d}$$

$$V_i^0(x, w_{\mathrm{p}}, \mathbf{r}) := \min_u \{J_i(x, w_{\mathrm{p}}, \mathbf{r}, u) : (x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z_i\}, \ \forall (x, w_{\mathrm{p}}, \mathbf{r}) \in X_i, \tag{24e}$$

$$\kappa_i(x, w_{\mathrm{p}}, \mathbf{r}) := \arg\min_u \{J_i(x, w_{\mathrm{p}}, \mathbf{r}, u) : (x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z_i\}, \tag{24f}$$

with the boundary conditions

$$X_0 := \{(x, w_{\mathrm{p}}, \mathbf{r}) : \ (x, r_N) \in X_{\mathrm{f}}, \ w_{\mathrm{p}} \in W\}, \tag{24g}$$

$$V_0^0(x, w_{\mathrm{p}}, \mathbf{r}) := V_{\mathrm{f}}(x, r_N), \ \forall (x, w_{\mathrm{p}}, \mathbf{r}) \in X_0. \tag{24h}$$

¿From the principle of optimality of DP [12, 13] it follows that

$$X_N^{\mathrm{sf}} = X_N,\ V_N^{0,\mathrm{sf}}(x, w, \mathbf{r}) = V_N^0(x, w, \mathbf{r}),\ \forall (x, w, \mathbf{r}) \in X_N.$$

To simplify notation in the rest of the paper, we define two prototype problems and we study their properties. The prototype maximization problem $\mathbb{P}_{\max}$ is defined as:

$$\mathbb{P}_{\max}:\ J(x, w_{\mathrm{p}}, \mathbf{r}, u) := \max_{w_{\mathrm{c}} \in W} \{\ell(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), u, r) + V(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), w_{\mathrm{c}}, \mathbf{r})\}, \tag{25}$$

for all $(x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z$, where the domain of J is

$$Z := \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : H f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}) + Gu + Fr \le h,\ w_{\mathrm{p}} \in W, (f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), w_{\mathrm{c}}, \mathbf{r}) \in \Omega, \forall w_{\mathrm{c}} \in W\}, \tag{26a}$$

$$X := \{(x, w_{\mathrm{p}}, \mathbf{r}) : (x, \mathbf{r}) \in X^{(x,\mathbf{r})},\ w_{\mathrm{p}} \in W\} \quad \text{or} \quad X := \mathrm{Proj}_{n+q+pN} Z, \tag{26b}$$

with $X^{(x,\mathbf{r})} := \{(x, \mathbf{r}) : (x, w_{\mathrm{p}}, \mathbf{r}) \in \mathrm{Proj}_{n+q+pN} Z, \forall w_{\mathrm{p}} \in W\}$, $\ell : \mathbb{R}^{n+m+p} \to \mathbb{R}$, $V : \Omega \to \mathbb{R}$, $\mathbf{r}$ has the form $\mathbf{r} = [\ldots r^T \ldots]^T$ (i.e., $\exists k : r_k = r$). The prototype minimization problem $\mathbb{P}_{\min}$ is defined as:

$$\mathbb{P}_{\min}:\ V^0(x, w_{\mathrm{p}}, \mathbf{r}) := \min_u \{J(x, w_{\mathrm{p}}, \mathbf{r}, u) : (x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z\}, \tag{27a}$$

$$\kappa(x, w_{\mathrm{p}}, \mathbf{r}) := \arg\min_u \{J(x, w_{\mathrm{p}}, \mathbf{r}, u) : (x, w_{\mathrm{p}}, \mathbf{r}, u) \in Z\}, \tag{27b}$$

for all $(x, w_{\mathrm{p}}, \mathbf{r}) \in X$.

In terms of these prototype problems, it is easy to identify the DP recursion (24) by setting $r \leftarrow r_{N-i}$, $\ell \leftarrow \ell_{N-i}$, $V \leftarrow V_{i-1}^0$, $V^0 \leftarrow V_i^0$, $X \leftarrow X_i$, $Z \leftarrow Z_i$ and $\Omega \leftarrow X_{i-1}$. Moreover, H, G, F are identified with $H_{N-i}, G_{N-i}, F_{N-i}$, respectively.

Clearly, we can now proceed to show, via induction, that a certain set of properties is possessed by each element in the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ by showing that if $\{V, \Omega\}$ has a given set of properties, then $\{V^0, X\}$ also has these properties, with the properties of κ being the same as those of each of the elements in the sequence $\{\kappa_i\}_{i=1}^N$. In the sequel, constructive proofs of the main results are presented, so that the reader can develop a prototype algorithm for computing the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$.

2.3.1. *Properties of X*

The following lemma is a simple consequence of the basic properties of the max operator:

Lemma 2.7. *The set $\mathcal{Z} = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \bar{H} f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}) + \bar{G}u + \bar{F}w_{\mathrm{p}} + \bar{E}\mathbf{r} \le \bar{h}, \forall w_{\mathrm{c}} \in W\}$ with $\bar{H} \ge 0$, can be written equivalently as $\mathcal{Z} = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\mathrm{p}} + \tilde{E}\mathbf{r} \le \tilde{h}\}$ with $\tilde{H} \ge 0$.* ♢

The next lemma shows that some useful properties of a class of polyhedra are inherited by its projection.

Lemma 2.8. *[21] Let $\mathcal{Z} = \{(x, r, t, u) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m : \bar{H}x + \bar{F}r + \bar{K}t + \bar{G}u \le \bar{h}\}$ be given, where $\bar{H} \ge 0$ and $\bar{K} \le 0$. The set $\mathcal{X} := \{(x, r, t) : \exists u \text{ s.t. } (x, r, t, u) \in \mathcal{Z}\}$ is a polyhedral set of the form $\mathcal{X} = \{(x, r, t) : \tilde{H}x + \tilde{F}r + \tilde{K}t \le \tilde{h}\}$, where $\tilde{H} \ge 0$ and $\tilde{K} \le 0$.* ♢

We are now in a position to show that X has the same structural properties as Ω.

Lemma 2.9. *Suppose Ω is a polyhedral set given by $\Omega = \{(x, w, \mathbf{r}) : \Gamma x + \Phi\mathbf{r} \leq \gamma,\ w \in W\}$ with $\Gamma \geq 0$, and assume that H in* (26a) *satisfies $H \geq 0$. Then, the set X defined in* (26b) *is a polyhedron given by either $X = \{(x, w_{\rm p}, \mathbf{r}) : \hat{H}x + \hat{E}\mathbf{r} \leq \hat{h},\ w_{\rm p} \in W\}$ or $X = \{(x, w_{\rm p}, \mathbf{r}) : \hat{H}x + \hat{F}w_{\rm p} + \hat{E}\mathbf{r} \leq \hat{h}\}$, where $\hat{H} \geq 0$.*

Proof: The set Z is described as follows:

$$Z=\{(x,w_{\rm p},\mathbf{r},u) : \bar{H}f_{\rm MPL}(x,u,w_{\rm p},w_{\rm c})+\bar{G}u+\bar{F}w_{\rm p}+\bar{E}\mathbf{r} \leq \bar{h},\ \forall w_{\rm c} \in W\},$$

where $\bar{H} = [H^T\ \Gamma^T\ 0]^T \geq 0,\ \bar{G} = [G^T\ 0\ 0]^T,\ \bar{F} = [0\ 0\ \Omega^T]^T, \bar{E} = [(Fr)^T\ (\Phi\mathbf{r})^T\ 0]^T$ and $\bar{h} = [h^T\ \ \gamma^T\ \ s^T]^T$. From Lemma 2.7 it follows that Z can be written equivalently as $Z = \{(x, w_{\rm p}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\rm p} + \tilde{E}\mathbf{r} \leq \tilde{h}\}$ where $\tilde{H} \geq 0$. By applying a particular case of Lemma 2.8 it follows that $\mathrm{Proj}_{n+q+pN} Z = \{(x, w_{\rm p}, \mathbf{r}) : \hat{H}x + \hat{F}w_{\rm p} + \hat{E}\mathbf{r} \leq \hat{h}\},\ \ \bar{H} \geq 0$. The rest follows immediately. ♢

Note that the set X_0 in (24g) is of the form given in Lemma 2.9 (since we assume that assumption **A1** holds).

2.3.2. Properties of $\mathbb{P}_{\max}$

We now derive an invariance property of the prototype maximization problem $\mathbb{P}_{\max}$.

Lemma 2.10. *If $\ell,\ V \in \mathcal{F}_{\rm mps}$ and $\ell(\cdot, u, r),\ V(\cdot, w_{\rm p}, \mathbf{r}) \in \mathcal{F}^+_{\rm mps}$ for any fixed $(w_{\rm p}, \mathbf{r}, u)$, then J possesses the same properties, i.e. $J \in \mathcal{F}_{\rm mps}$ and $J(\cdot, w_{\rm p}, \mathbf{r}, u) \in \mathcal{F}^+_{\rm mps}$, for any fixed $(w_{\rm p}, \mathbf{r}, u)$.*

Proof: Using basic properties of the max operator one can write $\ell(f_{\rm MPL}(x, u, w_{\rm p}, w_{\rm c}), u, r) + V(f_{\rm MPL}(x, u, w_{\rm p}, w_{\rm c}), w_{\rm c}, \mathbf{r}) = \max_{j\in\mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\rm p} + \mu_j^T w_{\rm c} + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}$, where $\alpha_j \geq 0$ for all $j \in \mathcal{J}$, so that

$$\begin{aligned} J(x, w_{\rm p}, \mathbf{r}, u) = \max_{w_{\rm c}\in W} \big\{ \max_{j\in\mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\rm p} + \mu_j^T w_{\rm c} + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}\big\} = \\ \max_{j\in\mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\rm p} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}, \end{aligned}$$

where $\theta_j := \tilde{\theta}_j + \max_{w_{\rm c}\in W}\{\mu_j^T w_{\rm c}\}$, for all $j \in \mathcal{J}$. Note that $\{\theta_j\}_{j\in\mathcal{J}}$ can be computed by solving a sequence of linear programs. Moreover, the coefficients of the variable x in J are the nonnegative vectors, i.e. $\alpha_j \geq 0$. ♢

Recall that ℓ and V_0^0 given in (5) and (24h) satisfy the conditions of Lemma 2.10.

2.3.3. Properties of $\mathbb{P}_{\min}$

This section derives the main properties of V^0 and κ. The following lemma gives a characterization of the solution and of the optimal value of the prototype minimization problem $\mathbb{P}_{\min}$.

Lemma 2.11. *Suppose Ω is a polyhedral set given by $\Omega = \{(x, w, \mathbf{r}) : \Gamma x + \Phi\mathbf{r} \leq \gamma,\ w \in W\}$ with $\Gamma \geq 0$, and assume that H in* (26a) *satisfies $H \geq 0$. Suppose also that $Z \neq \emptyset$, $J \in \mathcal{F}_{\rm mps}$ and*

V^0 is proper[§]. Then, the value function $V^0 \in \mathcal{F}_{\rm mps}$ and has domain X, where X is a polyhedral set. The (set-valued) control law $\kappa(x, w_{\rm p}, \mathbf{r})$ is a polyhedron for a given $(x, w_{\rm p}, \mathbf{r}) \in X$. Moreover, it is always possible to select a continuous and PWA control law μ such that $\mu(x, w_{\rm p}, \mathbf{r}) \in \kappa(x, w_{\rm p}, \mathbf{r})$ for all $(x, w_{\rm p}, \mathbf{r}) \in X$.

Proof: From the proof of Lemma 2.9 it follows that Z is a non-empty polyhedron: $Z = \{(x, w_{\rm p}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\rm p} + \tilde{E}\mathbf{r} \leq \tilde{h}\}$, with $\tilde{H} \geq 0$. Since $J \in \mathcal{F}_{\rm mps}$, we can write $J(x, w_{\rm p}, \mathbf{r}, u) = \max_{j \in \mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\rm p} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$. The prototype minimization problem $\mathbb{P}_{\rm min}(x, \mathbf{r})$ becomes:

$$V^0(x, w_{\rm p}, \mathbf{r}) = \min_{\mu, u}\{\mu : \alpha_j^T x + \beta_j^T w_{\rm p} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \leq \mu, \forall j \in \mathcal{J},\ (x, w_{\rm p}, \mathbf{r}, u) \in Z\}, \quad (28)$$

i.e. we have obtained a feasible linear program for any $(x, w_{\rm p}, \mathbf{r}) \in \text{Proj}_{n+q+pN} Z$. It follows that $\mathbb{P}_{\rm min}$ is a parametric linear program of the type considered in [18, 19]: $\inf_u\{c^T u : \mathcal{F}u \leq \mathcal{G}x + g\}$, where $x \in \mathcal{X}$ is the parameter vector. The properties stated above then follow from the properties of the parametric linear program. ♢

Now we can state the following key result, which, together with Lemma 2.9–2.11, allow one to deduce, via induction, some important properties of the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$:

Theorem 2.12. *Suppose that the same assumptions as in Lemma 2.11 hold. If, in addition, $J(\cdot, w_{\rm p}, \mathbf{r}, u) \in \mathcal{F}_{\rm mps}^+$ for any fixed $(w_{\rm p}, \mathbf{r}, u)$, then the value function $V^0(\cdot, w_{\rm p}, \mathbf{r}) \in \mathcal{F}_{\rm mps}^+$ for any fixed $(w_{\rm p}, \mathbf{r})$.*

Proof: ¿From Lemma 2.11 and the fact that V^0 is proper, it follows that $V^0 \in \mathcal{F}_{\rm mps}$ and its domain is $\text{Proj}_{n+q+pN} Z$. Using the proof of Lemma 2.11, the epigraph of V^0 has the following form:

$$\begin{aligned}\text{epi}\, V^0 :=& \{(x, w_{\rm p}, \mathbf{r}, t) : V^0(x, w_{\rm p}, \mathbf{r}) \leq t,\ (x, w_{\rm p}, \mathbf{r}) \in \text{Proj}_{n+q+pN} Z\} = \\ & \{(x, w_{\rm p}, \mathbf{r}, t) : \exists u \ \text{ s.t. } \ (x, w_{\rm p}, \mathbf{r}, u) \in Z,\ J(x, w_{\rm p}, \mathbf{r}, u) \leq t\} = \\ & \{(x, w_{\rm p}, \mathbf{r}, t) : \exists u \ \text{ s.t. } \ \bar{H}x + \bar{G}u + \bar{F}w_{\rm p} + \bar{E}\mathbf{r} + \bar{K}t \leq \bar{h}\},\end{aligned}$$

where $\bar{H} = [\tilde{H}^T \ \alpha_1^T \ \cdots \ \alpha_l^T]^T \geq 0$ and $\bar{K} = [0 \ -1 \ldots -1]^T \leq 0$. From Lemma 2.8 we obtain that the epigraph of V^0 is a polyhedral set given by epi $V^0 = \{(x, w_{\rm p}, \mathbf{r}, t) : \hat{H}x + \hat{F}w_{\rm p} + \hat{E}\mathbf{r} + \hat{K}t \leq \hat{h}\}$, where $\hat{H} \geq 0, \hat{K} \leq 0$. Let $l = n_{\hat{H}}$ be the number of inequalities describing epi V^0. We arrange the indexes $i \in \mathbb{N}_{[1,l]}$ such that $\hat{K}_i < 0$ for $i \in \mathbb{N}_{[1,v]}$ but $\hat{K}_i = 0$ for $i \in \mathbb{N}_{[v+1,l]}$ (possibly $v = 0$, i.e. $\hat{K}_i = 0$ for all i). Taking $a_i = -\hat{H}_{i\cdot}/\hat{K}_i, b_i = -\hat{F}_{i\cdot}/\hat{K}_i, c_i = -\hat{E}_{i\cdot}/\hat{K}_i$ and $d_i = -\hat{h}_i/\hat{K}_i$ for all $i \in \mathbb{N}_{[1,v]}$, we get that the epigraph of V^0 is expressed as:

$$\begin{aligned}\text{epi}\, V^0 = \{(x, w_{\rm p}, \mathbf{r}, t) :& a_i x + b_i w_{\rm p} + c_i \mathbf{r} - d_i \leq t, \forall i \in \mathbb{N}_{[1,v]} \\ & \hat{H}_{i\cdot} x + \hat{F}_{i\cdot} w_{\rm p} + \hat{E}_{i\cdot}\mathbf{r} \leq \hat{h}_i,\ \forall i \in \mathbb{N}_{[v+1,l]}\}.\end{aligned} \quad (29)$$

But V^0 is proper and thus $v > 0$. Since $V^0 \in \mathcal{F}_{\rm mps}$, (29) gives us a representation of V^0 as $V^0(x, w_{\rm p}, \mathbf{r}) = \max_{i \in \mathbb{N}_{[1,v]}}\{a_i x + b_i w_{\rm p} + c_i \mathbf{r} - d_i\}$, where $a_i = -\hat{H}_{i\cdot}/\hat{K}_i \geq 0$, for all $i \in \mathbb{N}_{[1,v]}$, i.e. $V^0(\cdot, w_{\rm p}, \mathbf{r}) \in \mathcal{F}_{\rm mps}^+$ for any fixed $(w_{\rm p}, \mathbf{r})$. ♢

[§]Note that since we always have that u_0 should be larger than the current time instant, i.e. the time instant at which we start performing the computations, u_0 is bounded from below and V^0 will always be proper.

¿From Theorem 2.12 it follows that V^0 is a continuous and convex function. Based on the invariance properties of the two prototype problems $\mathbb{P}_{\max}$ and $\mathbb{P}_{\min}$, we can now derive the properties of V_i^0, κ_i and X_i for all $i \in \mathbb{N}_{[1,N]}$. The following follows by applying Lemmas 2.9–2.11 and Theorem 2.12 to the DP equations (24):

Theorem 2.13. *Suppose that **A1** and **A2** hold, Z_i is non-empty and V_i^0 is proper for all $i \in \mathbb{N}_{[1,N]}$. The following holds for each $i \in \mathbb{N}_{[1,N]}$:*

(i) X_i is a non-empty polyhedron,

(ii) V_i^0 is a convex, continuous PWA function with domain X_i,

(iii) $V_i^0(\cdot, w_\mathrm{p}, \mathbf{r}) \in \mathcal{F}^+_\mathrm{mps}$ for any fixed $(w_\mathrm{p}, \mathbf{r})$.

(iv) There exists a continuous PWA function μ^0_{N-i} such that $\mu^0_{N-i}(x, w_\mathrm{p}, \mathbf{r}) \in \kappa_i(x, w_\mathrm{p}, \mathbf{r})$ for all $(x, w_\mathrm{p}, \mathbf{r}) \in X_i$.

(v) The sequences $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ and $\{\mu_i^0\}_{i=1}^N$ can be computed by solving exactly N parametric linear programs.

Since the proofs of all the above results are constructive, it follows that the sequences $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ and $\{\mu_i^0\}_{i=1}^N$ can be computed iteratively, without gridding by solving N parametric linear programs as in the linear case (see [20]). It is clear that:

$$
\begin{aligned}
&X_N^{\mathrm{ol}} \subseteq X_N^{\mathrm{df}} \subseteq X_N^{\mathrm{sf}}, \\
V_N^{0,\mathrm{sf}}(x, w, \mathbf{r}) \quad &\leq V_N^{0,\mathrm{df}}(x, w, \mathbf{r}) \leq \quad V_N^{0,\mathrm{ol}}(x, w, \mathbf{r}),\ \forall (x, w, \mathbf{r}) \in X_N^{\mathrm{ol}}.
\end{aligned} \tag{30}
$$

The results developed in this paper are valid for a class of systems for which the MPL systems are a subclass, namely for continuous PWA systems $x(k+1) = f(x(k), y(k), w(k)),\ y(k) = h(x(k), w(k))$ (i.e. f and h are continuous PWA functions) satisfying $f(\cdot, u, w) \in (\mathcal{F}^+_\mathrm{mps})^n$ and $h(\cdot, w) \in (\mathcal{F}^+_\mathrm{mps})^p$ for any fixed (u, w).

Remark 2.14 MPL systems are DES and they thus differ from conventional time driven systems in the sense that the event counter k is not directly related to a specific time. Note that in practical applications the entries of the system matrices are nonnegative or take the value ε. It follows that if $x(k)$ is completely available, then $u(k-1)$ and $w(k-1)$ are also available. The reader might ask how we determine the cycle k. Let t_0 be the time when one of the optimization problems discussed in the previous sections is solved. Let us assume a sampling time T and for simplicity we take $t_0 = 0$. Then, at each time jT, where $j \geq 0$, we can define the cycle k as follows: $k = \arg\max\{l : x_i(l) \leq jT\ \forall i \in \mathbb{N}_{[1,n]}\}$. This means that at time jT the state $x(k)$ is completely available and also $w(k-1), u(k-1)$ are completely known. However, at time jT also some future components of the inputs and of the disturbances might be known. Therefore, these constraints on the inputs and on the disturbances must be taken into account and (due to causality) can be recast as linear equality or inequality constraints, which thus fit in the framework presented in this paper.

In the open-loop case at time $t = u_i(k)$, where $u_i(k) = [u_k^0(x, w, \mathbf{r})]_i$, the i^{th} input is activated for the k^{th} cycle. In the disturbance or state feedback case $u_i(k) = [\mu_k^0(x(k), w, w(0), \cdots, w(k-1), \mathbf{r})]_i$, where μ_k^0 is either the disturbance feedback policy computed in Section 2.2 or the state feedback policy computed in Section 2.3. Therefore, we can compute at time jT the optimal input $u(k)$ as explained above and we take into account the new information available as equality constraints on the input and on the disturbance. At time $(j+1)T$ the whole procedure is repeated.

For future research we want to investigate in more depth the timing issues that appears in a moving horizon framework, this being one of our current research topics.

3. "TIME–OPTIMAL" CONTROL

As an application of the three robust control problems discussed in the previous section, we consider the MPL counterpart of the conventional time-optimal control problem: Given a maximum horizon length $N_{\max}$ we consider the problem of ensuring that the completion times after N events (with $N \in \mathbb{N}_{[1,N_{\max}]}$) are less than or equal to a specified target time **T** (i.e. $y(N) \leq \mathbf{T}$), using the latest controller that satisfies the input and state constraints (3). Note that such a problem, but without considering constraints and disturbances, was considered also in [9] in terms of lattice theory. The time-optimal control problem in our setting is different from the classical one (we want to *maximize*¶ N instead of minimizing it; so in fact a better term would be "throughput-optimal" control). Since we want the maximal N, the robust time-optimal control problem can be posed in the framework of the finite-horizon min-max control problems considered in the previous section.

One proceeds by defining

$$N^0(x, w, \mathbf{T}) := \max_{N,\pi}\{N \in \mathbb{N}_{[1,N_{\max}]} : \pi \in \Pi_N^{\mathbf{T}}(x, w, [0 \ \ldots 0 \ \mathbf{T}^T]^T)\}, \tag{31}$$

where $\Pi_N^{\mathbf{T}}(\cdot)$ is either $\Pi_N^{\mathrm{ol}}(\cdot)$ or $\Pi_N^{\mathrm{df}}(\cdot)$ or $\Pi_N^{\mathrm{sf}}(\cdot)$ depending whether π is an open-loop input sequence or a disturbance feedback policy or a state feedback policy, respectively, but with the substitutions $H_N \leftarrow [H_N^T \ I]^T \geq 0$, $F_N \leftarrow [F_N^T \ 0]^T$ and $h_N \leftarrow [h_N^T \ ((-C^T) \otimes' \mathbf{T})^T]^T$ (note that $F_i r_i = 0$, for all $i \in \mathbb{N}_{[0,N-1]}$ and $r_N = \mathbf{T}$). It follows that

$$N^0(x, w, \mathbf{T}) = \max_{N}\{N \in \mathbb{N}_{[1,N_{\max}]} : (x, w, \mathbf{r}) \in X_N^{\mathbf{T}}\}, \tag{32}$$

where $\mathbf{r} = [0 \ 0 \cdots 0 \ \mathbf{T}^T]^T$ and $X_N^{\mathbf{T}} = \{(x, w, \mathbf{r}) : \Pi_N^{\mathbf{T}}(x, w, \mathbf{r}) \neq \emptyset\}$. Since we want to feed the raw material as late as possible [9], a suitable choice of stage cost is $\ell_i(x_i, u_i, r_i) := -\sum_{j=1}^{m}[u_i]_j$. The robust time-optimal controller is implemented as follows:

1. For each $N \in \mathbb{N}_{[1,N_{\max}]}$, solve problem (6) or (18) or (23) where $\mathbf{r}$ is defined as $\mathbf{r} = [0 \ 0 \cdots 0 \ \mathbf{T}^T]^T \in \mathbb{R}^{pN}$.
2. Determine $N^0(x, w, \mathbf{T})$ according to (32).
3. Let $\mathbf{r}^0 := [0 \ 0 \cdots 0 \ \mathbf{T}^T]^T \in \mathbb{R}^{pN}$, with $N = N^0(x, w, \mathbf{T})$.
4. Let the control policy be given by π_N^0, with $N = N^0(x, w, \mathbf{T})$.
5. Apply the control policy $u(k) = \mu_k^0(x(k-1), w_{\mathrm{p}}, \mathbf{r}^0)$ for $k = 1, 2, \ldots,$ $N^0(x, w, \mathbf{T})$, where at step $k, w(k-1) = w_{\mathrm{p}}$.

The robust time-optimal control problem involves solving either a set of linear programs or parametric linear programs.

4. "DETERMINISTIC" MIN-MAX CONTROL

The main drawback of the min-max optimization problems described in Section 2 is the computational complexity. Although the open-loop control problem (6) can be recast as a linear program (13) with

¶For a manufacturing system this requirement corresponds to producing as many products as possible by the target time.

$Nm+1$ variables, the number of inequalities that describe the feasible set in this linear program is $|\mathcal{I}| + n_F$, which, in general, may be very large. In the disturbance feedback approach (18), we still have to solve a linear program (20), as in the open-loop case, but the improvement in performance and feasibility compared to the open-loop case is obtained at the expense of introducing $N(N-1)mq/2 + Nn_F n_\Omega + |\mathcal{I}|Nn_\Omega$ extra variables and $n_F + |\mathcal{I}|$ extra inequalities. For the state feedback approach (23), the solution is computed off-line, but the number of regions generated by the parametric linear programming algorithm is also, in general, large. In this section we show that the computational complexity of the three min-max control problems considered in Section 2 can be reduced significantly if the disturbance has a certain description.

We assume a particular description of the uncertainty for an MPL system. From example (1) we have seen that the system matrices of an MPL system (2) depend on the consecutive disturbances $w(k-1)$ and $w(k)$. However, there are situations when the system matrices depend only on the disturbance $w(k)$. One possibility is described next. We could redefine the uncertainty as

$$w(k) := [p_1(k-1) \ldots p_l(k-1)\ p_1(k) \ldots p_l(k)\ t_1(k) \ldots t_{\bar{l}}(k)]^T, \tag{33}$$

but in this case we introduce some conservatism since we do not take into account that some components of $w(k-1)$ and $w(k)$ coincide.

A second possibility is the following. Note that in the context of MPL systems, the uncertainty comes from the parameters p_i and t_i. Moreover, only the parameters p_i depend on $k-1$. So. let us consider the case where the parameters p_i are known and only the parameters t_i are uncertain. In this situation the uncertainty vector becomes

$$w(k) := [t_1(k) \cdots t_q(k)]^T. \tag{34}$$

In these two situations it follows that the MPL system (2) can be rewritten as

$$\begin{aligned} x(k+1) &= A(w(k)) \otimes x(k) \oplus B(w(k)) \otimes u(k) \\ y(k) &= C(w(k)) \otimes x(k). \end{aligned} \tag{35}$$

Moreover, we assume that there exists a $\overline{w} \in W$ such that

$$A(w) \leq \overline{A},\ B(w) \leq \overline{B},\ C(w) \leq \overline{C},\ \forall w \in W. \tag{36}$$

where $\overline{A} := A(\overline{w}),\ \overline{B} := B(\overline{w}),\ \overline{C} := C(\overline{w})$.

In the previous two situations described above, the inequalities in (36) typically hold since the parameters p_i and t_j denote processing times and transportation times and thus we can assume that each of them varies in some intervals: $p_i \in [\underline{p}_i\ \overline{p}_i]$ and $t_j \in [\underline{t}_j\ \overline{t}_j]$. Then, the uncertainty set W is given by a box in $\mathbb{R}^q$, $W := [\underline{w}\ \ \overline{w}]$ where $W = \left([\underline{p}_1\ \overline{p}_1] \times \cdots \times [\underline{p}_l\ \overline{p}_l]\right)^2 \times [\underline{t}_1\ \overline{t}_1] \times \ldots \times [\underline{t}_{\bar{l}}\ \overline{t}_{\bar{l}}]$ corresponds to the case (33) and $W = [\underline{t}_1\ \overline{t}_1] \times \ldots \times [\underline{t}_q\ \overline{t}_q]$ corresponds to the case (34). Moreover, the entries of the system matrices corresponding to an MPL system are given by sums or maxima of processing times (p_i) and transportation times (t_j) and thus the entries of matrices A, B and C are max-plus-nonnegative-scaling functions:

$$A_{ij},\ B_{il},\ C_{kl} \in \mathcal{F}^{+}_{\mathrm{mps}},\ \forall\, i, j, l, k \tag{37}$$

i.e. each entry is a function defined as $w \mapsto \max_j\{\alpha_j^T w + \beta_j\}$, where α_j are vectors with entries 0 and 1 (and thus $\alpha_j \geq 0$) and $\beta_j \geq 0$. Since for any vector $\alpha \geq 0$ it follows that $\alpha^T \underline{w} \leq \alpha^T w \leq \alpha^T \overline{w}$ for all $w \in W (= [\underline{w}\ \ \overline{w}])$, we can conclude that the inequalities (36) hold. Note that interval transfer functions for DES were also considered in [8].

We will show in the sequel a quite interesting result, namely that under the previous hypothesis (i.e. we assume that (35), (36) and (37) are valid) the finite-horizon min-max control problems discussed in Section 2 reduce to an optimal control problem for a particular deterministic MPL system. It is straightforward to show that the following inequality holds in the max-plus algebra:

$$C_1 \leq D_1,\ C_2 \leq D_2 \quad \Rightarrow \quad C_1 \otimes C_2 \leq D_1 \otimes D_2, \tag{38}$$

for any matrices C_1, C_2, D_1 and D_2 of appropriate dimensions.

First let us consider the open-loop min-max case from Section 2.1. For an uncertain MPL system in the form (35), we do not have dependence on w anymore (e.g. $\Theta(w, \mathbf{w})$ becomes in this new settings $\Theta(\mathbf{w})$, etc). Let us define $\overline{\mathbf{w}} := [\overline{w}^T \dots \overline{w}^T]^T \in \mathcal{W}$, $\overline{\Theta} := \Theta(\overline{\mathbf{w}})$ and $\overline{\Phi} := \Phi(\overline{\mathbf{w}})$. From (38) it follows that

$$\Theta(\mathbf{w}) \otimes x \oplus \Phi(\mathbf{w}) \otimes \mathbf{u} \leq \Theta(\overline{\mathbf{w}}) \otimes x \oplus \Phi(\overline{\mathbf{w}}) \otimes \mathbf{u} = \overline{\Theta} \otimes x \oplus \overline{\Phi} \otimes \mathbf{u}, \ \ \forall \mathbf{w} \in \mathcal{W}.$$

Since $\mathbf{H} \geq 0$, it follows from (8) that

$$\Pi_N^{\text{ol}}(x, \mathbf{r}) = \{\mathbf{u} :\ \mathbf{H}(\overline{\Theta} \otimes x \oplus \overline{\Phi} \otimes \mathbf{u}) + \mathbf{G}\mathbf{u} + \mathbf{F}\mathbf{r} \leq \mathbf{h}\},$$

which coincides with the set of feasible input sequences over the horizon N corresponding to the deterministic MPL system

$$\overline{x}(k+1) = \overline{A} \otimes \overline{x}(k) \oplus \overline{B} \otimes u(k), \ \ \overline{y}(k) = \overline{C} \otimes \overline{x}(k). \tag{39}$$

Moreover, in (10) we have $\alpha_j, \gamma_j \geq 0$ (see Remark 2.2 and (37)). It follows from (12) that

$$J_N(x, \mathbf{r}, \mathbf{u}) = V_N(x, \mathbf{r}, \mathbf{u}, \overline{\mathbf{w}}).$$

We now consider an optimal control problem for the deterministic system (39) over a horizon window of length N:

$$\mathbb{P}_N^{\text{upper}}(x, \mathbf{r}) : \qquad V_N^{0,\text{upper}}(x, \mathbf{r}) := \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(x,\mathbf{r})} V_N(x, \mathbf{u}, \mathbf{r}, \overline{\mathbf{w}}). \tag{40}$$

¿From the previous discussion it follows that:

Lemma 4.1. *Suppose that* (35)*,* (36) *and* (37) *hold. Then, the open-loop min-max control problem* $\mathbb{P}_N^{\text{ol}}(x, \mathbf{r})$ *is equivalent with the deterministic optimal control problem* $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$*, for all* $(x, \mathbf{r}) \in X_N^{\text{ol}}$*.*

Let us now show that the state feedback min-max control problem $\mathbb{P}_N^{\text{sf}}(x, \mathbf{r})$ from Section 2.3 is equivalent with the same deterministic optimal control problem $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$. Indeed, since $\ell_i(\cdot, u, r) \in \mathcal{F}_{\text{mps}}^+$ and using also Theorem 2.13 (iii) it follows that:

$$V_i^0(f_{\text{MPL}}(x, u, w), \mathbf{r}) \leq V_i^0(A(\overline{w}) \otimes x \oplus B(\overline{w}) \otimes u, \mathbf{r}) = V_i^0(\overline{A} \otimes x \oplus \overline{B} \otimes u, \mathbf{r}), \forall w \in W,$$

$$\ell_i(f_{\text{MPL}}(x, u, w), u, r) \leq \ell_i(A(\overline{w}) \otimes x \oplus B(\overline{w}) \otimes u, u, r) = \ell_i(\overline{A} \otimes x \oplus \overline{B} \otimes u, u, r), \forall w \in W.$$

Therefore, $J_i(x, \mathbf{r}, u)$ as defined in (24a) is given by:

$$J_i(x, \mathbf{r}, u) = \ell_{N-i}(\overline{A} \otimes x \oplus \overline{B} \otimes u, u, r_{N-i}) + V_{i-1}^0(\overline{A} \otimes x \oplus \overline{B} \otimes u, \mathbf{r}).$$

and the corresponding feasible set Z_i reduces to

$$Z_i = \{(x, \mathbf{r}, u) : H_{N-i}(\overline{A} \otimes x \oplus \overline{B} \otimes u) + G_{N-i}u + F_{N-i}r_{N-i} \leq h_{N-i},\ \overline{A} \otimes x \oplus \overline{B} \otimes u \in X_{i-1}\}.$$

The next result follows:

Theorem 4.2. *Suppose that* (35), (36) *and* (37) *hold then* $X_N^{\rm ol} = X_N^{\rm df} = X_N^{\rm sf}$ *and the robust control problems considered in Section 2, i.e.* $\mathbb{P}_N^{\rm ol}(x, \mathbf{r})$, $\mathbb{P}_N^{\rm df}(x, \mathbf{r})$ *and* $\mathbb{P}_N^{\rm sf}(x, \mathbf{r})$ *are reduced to the optimal control problem* $\mathbb{P}_N^{\rm upper}(x, \mathbf{r})$ *corresponding to the deterministic system* (39) *for any* $(x, \mathbf{r}) \in X_N^{\rm sf}$.

Proof: From the previous discussion (note that the optimal input sequence of the deterministic optimal control problem (40) can also be computed via dynamic programming approach) and using Bellman's principle of optimality for dynamic programming [13] it follows that the optimal problems $\mathbb{P}_N^{\rm sf}(x, \mathbf{r})$ and $\mathbb{P}_N^{\rm upper}(x, \mathbf{r})$ are equivalent. Therefore, from Lemma 4.1 and the inclusions (30) it follows that $X_N^{\rm ol} = X_N^{\rm df} = X_N^{\rm sf}$ and robust control problems $\mathbb{P}_N^{\rm ol}(x, \mathbf{r})$, $\mathbb{P}_N^{\rm sf}(x, \mathbf{r})$ reduce to $\mathbb{P}_N^{\rm upper}(x, \mathbf{r})$. Let $\mathbf{u}^0(x, \mathbf{r})$ be the optimal solution of these problems for a $(x, \mathbf{r}) \in X_N^{\rm sf}$. Then, using now the inequalities from (30) it follows that the disturbance feedback control problem $\mathbb{P}_N^{\rm df}(x, \mathbf{r})$ reduces to the same deterministic optimal control problem $\mathbb{P}_N^{\rm upper}(x, \mathbf{r})$ (an optimal solution for the disturbance feedback approach is $\mathbf{M}(x, \mathbf{r}) = 0$ and $\mathbf{v}(x, \mathbf{r}) = u^0(x, \mathbf{r})$). ♢

There is a significant advantage in solving the linear program (40) that has fewer constraints and number of variables than the linear program (13).

5. EXAMPLES

5.1. Example 1

Let us consider the manufacturing system from Figure 1. The dynamical equations are given by (1). We assume that the system is subject to hard constraint: $x_2(k) - u(k) \leq 5$ for all k. We consider that the parameters p_1, p_2, t_2, t_4 and t_6 are fixed at each cycle and taking the values $p_1 = 1, p_2 = 1, t_2 = 1, t_4 = 3$ and $t_6 = 0$. However, the rest of the parameters are assumed to be varying with each cycle: $p_3(k) \in [1.5 \;\; 2.5], t_1(k) \in [0 \;\; 2],\; t_3(k) \in [0 \;\; 1]$ and $t_5(k) \in [0 \;\; 1]$. We define the uncertainty as $w(k) = [p_3(k-1) \; t_1(k) \; t_3(k) \; t_5(k)]^T$, i.e. we employ the conservatism from (33). Then, the uncertainty set is described by the following box $W = [1.5 \;\; 2.5] \times [0 \;\; 2] \times [0 \;\; 1] \times [0 \;\; 1]$. Moreover, the dynamical equations of the process (1) can be written in matrix form as in (35). There exists a feasible value of the uncertainty $\overline{w} = [2.5 \; 2 \; 1 \; 1]^T \in W$ for which the inequalities (36) hold. We can easily verify that the conditions from Theorem 4.2 are fulfilled and thus the robust control problems considered in Section 2, i.e. $\mathbb{P}_N^{\rm ol}(x, \mathbf{r})$, $\mathbb{P}_N^{\rm df}(x, \mathbf{r})$ and $\mathbb{P}_N^{\rm sf}(x, \mathbf{r})$ are reduced to the optimal control problem $\mathbb{P}_N^{\rm upper}(x, \mathbf{r})$ associated to the deterministic system

$$\overline{x}(k+1) = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ 3 & 1 & \varepsilon \\ 5 & 3 & 2.5 \end{bmatrix} \otimes \overline{x}(k) \oplus \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \otimes u(k), \quad \overline{y}(k) = [\varepsilon \;\; \varepsilon \;\; 2.5] \otimes \overline{x}(k).$$

We choose the following reference signal $r(k) = 5 + 1.5\lambda^* k$ (here $\lambda^* = 2.5$), the prediction horizon $N = 4$ and the initial state is $x = [13 \;\; 14.5 \;\; 17]^T$. Using the stage cost defined in (5) with $\gamma = 0.18$, the deterministic optimal control problem (40) yields the following optimal input sequence: $\{u^0(k)\}_{k=0}^9 = 6,\; 11,\; 12,\; 13,\; 14,\; 15.4,\; 19,\; 22.7,\; 26.5,\; 30.2$. The results are displayed in Figure 2 using a feasible sequence of random disturbances.

We observe from the first plot that although we start later than the initial due date, the closed-loop output is able to track the due dates signal after a finite transient behavior. The second plot displays the optimal input. The input-state constraints $x_2(k) - u(k) \leq 5$ are depicted in the third plot. Note that sometimes the constraints are indeed active.

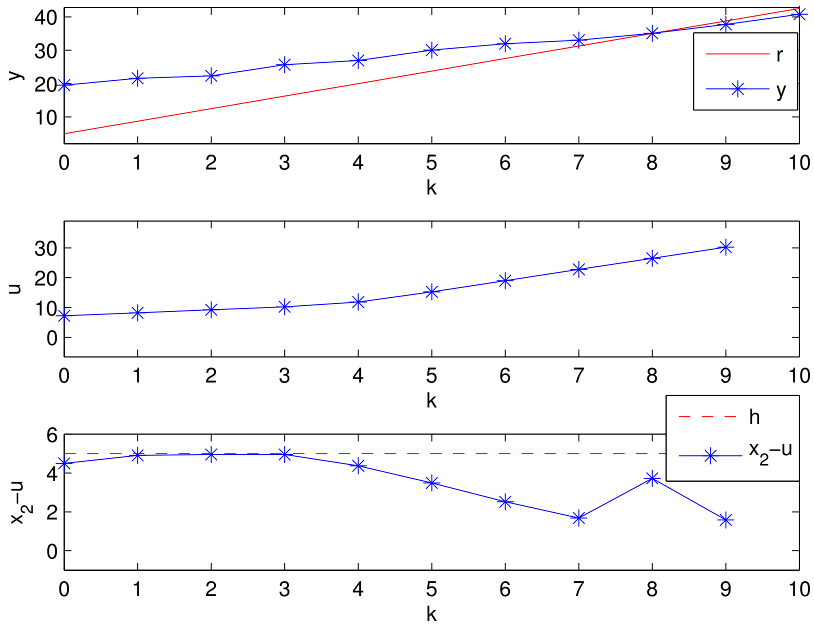

Figure 2. The closed-loop simulations.

Let us now compare our method with the other control design methods mentioned in Section 1. The adaptive control approach proposed in [2] has the most features in common with our approach in the sense that the approach of [2] allow violations of the due dates and tries to minimize these violations by updating the model at each step of the computation of the optimal control sequence. However, the approach in [2] cannot cope with more complex state and input constraints. For instance, using the same disturbance realization as in our method and the adaptive control approach of [2] we obtain the following optimal input sequence $\{u(k)\}_{k=0}^{9} = 6,\ 6,\ 6,\ 9.2,\ 12.3,\ 16,\ 19.9,\ 24.1,\ 28.4,\ 31.3.$ Note that in that case $x_2(1) - u(1) = 9.5 \not\leq 5$. In [6] an open-loop min-max model predictive controller is derived using only input constraints. However, the extension to input-state constraints is straightforward according to Section 2.1. Moreover, from Section 2.1 we see that the optimal input sequence can be found without having to resort to computations of vertices of W, as was done in [6]. Note that in this particular example, the open-loop approach from Section 2.1 is equivalent with the state feedback approach derived in Section 2.3. However, in the next example we will see that the feedback approach outperforms, in general, the open-loop approach.

5.2. *Example 2*

We now consider the following example, for which we compare the three robust optimal control problems presented in Section 2:

$$x(k+1) = \begin{bmatrix} -w_1(k)+w_2(k)+2 & \varepsilon \\ -w_1(k)-w_2(k)+5 & w_1(k)-2 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} -w_1(k)+3 \\ -w_2(k)+2 \end{bmatrix} \otimes u(k)$$

$$y(k) = [0 \quad \varepsilon] \otimes x(k).$$

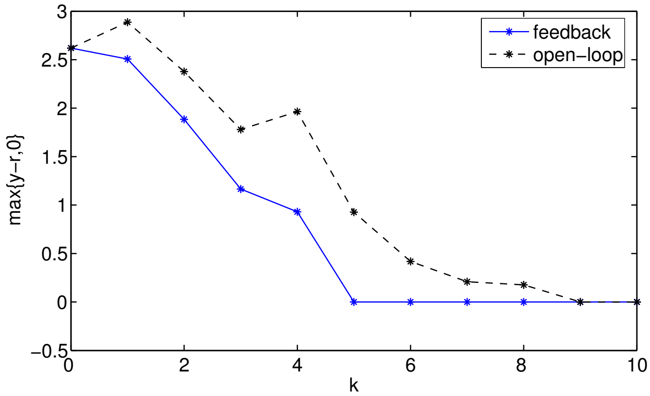

Figure 3. The tardiness $\max\{y(\cdot) - r(\cdot), 0\}$ for the feedback controllers (full) and the open-loop controller (dotted).

N	1	2	3	4	5	6	7	8	9	10
n_R	2	7	7	10	13	15	19	23	25	25

Table I. The number of regions n_R in the parametric linear programs of Section 2.3 as a function of N.

We assume a bounded disturbance set: $W = \{w \in \mathbb{R}^2 : w_1 \in [2\ \ 3],\ w_2 \in [1\ \ 2], w_1 + w_2 \leq 4\}$. We choose $N = 10$, the due date signal sequence is $\mathbf{r} = [3.4\ 5\ 7\ 9.5\ 11.8\ 14\ 16.7\ 19.4\ 21.6\ 23.8\ 26]^T$ and the initial state is $x(0) = [6\ \ 8]^T$. The system is subject to input-state constraints: $x_2(k) - u(k) \leq 2, x_1(N) + x_2(N) \leq 2r_N, -6 + r_k \leq u(k) \leq 6 + r_k$. We use the stage cost defined in (5) with $\gamma = 0.1$ and a feasible random sequence of disturbances.

In this particular example we observe that the disturbance feedback controller from Section 2.2 coincides with the state feedback controller from Section 2.3. Moreover, the number of regions of the computed parametric linear programs corresponding to the state feedback approach, as a function of the prediction horizon N, is given in Table I. Note that these regions increases with the prediction horizon.

Figure 3 shows the tardiness (i.e., the signal $\max\{y(\cdot) - r(\cdot), 0\}$) for the open-loop controller from Section 2.1 and for the feedback controllers derived in Sections 2.2–2.3. Note that the performance of the feedback approaches are better than the open-loop approach. The figure shows that the feedback controllers give a lower tardiness (i.e., better "tracking") than the open-loop controller.

6. Conclusions

We have provided solutions to three finite horizon min-max control problems for constrained MPL systems depending on the nature of the control input over which we optimize: open-loop input sequences, disturbance feedback policies, or state feedback policies. We have shown that the open-loop and the disturbance feedback min-max problem can be recast as linear programs while the state feedback min-max problem can be solved exactly, without gridding, via N parametric linear programs, where N is the prediction horizon. The key assumptions that allow us to preserve convexity in the min-max problems that we have considered, were that the stage cost be a max-plus-nonnegative-scaling expression in the state and the matrices associated with the state constraints have nonnegative entries. Finally, for a particular case of the uncertainty we have proven that all three min-max problems are equivalent with a deterministic one.

ACKNOWLEDGEMENTS

This paper benefited considerably from several stimulating discussions with E.C. Kerrigan.

REFERENCES

1. B. Cottenceau, L. Hardouin, J. Boimond, J. Ferrier, Model reference control for timed event graphs in dioid, Automatica 37 (8) (2001) 1451–1458.
2. E. Menguy, J. Boimond, L. Hardouin, J. Ferrier, A first step towards adaptive control for linear systems in max algebra, Discrete Event Dynamic Systems: Theory and Applications 10 (4) (2000) 347–367.
3. L. Libeaut, J. Loiseau, Model matching for timed event graphs, in: Proceedings of the 13th IFAC World Congress, San Francisco, California, 1996, pp. 383–388.
4. R. Kumar, V.K. Garg. Extremal solutions of inequations over lattices with applications to supervisory control, in: Proceedings of the 33rd IEEE Conference on Decision and Control, pages 3636–3641, Orlando, Florida, December 1994.
5. D.D. Cofer, V.K. Garg. Supervisory control of real-time discrete-event systems using lattice theory. *IEEE Transactions on Automatic Control*, 41(2):199–209, February 1996.
6. T. van den Boom, B. De Schutter, Model predictive control for perturbed max-plus-linear systems, Systems & Control Letters 45 (1) (2002) 21–33.
7. C. A. Maia, L. Hardouin, R. Santos-Mendes, B. Cottenceau, Optimal closed-loop control of timed event graphs in dioids, IEEE Transactions on Automatic Control 48 (12) (2003) 2284–2287.
8. M. Lhommeau, L. Hardouin, B. Cottenceau, L. Jaulin, Interval analysis in dioid: application to robust controller design for timed-event graphs, Automatica 40 (11) (2004) 1923–1930.
9. F. Baccelli, G. Cohen, G. Olsder, J. Quadrat, Synchronization and Linearity, John Wiley & Sons, New York, 1992.
10. R. T. Rockafellar, R. J.-B. Wets, Variational Analysis, Springer-Verlag, Berlin, 2004.
11. D. Mayne, J. Rawlings, C. Rao, P. Scokaert, Constrained model predictive control: Stability and optimality, Automatica 36 (2000) 789–814.
12. D. P. Bertsekas, I. B. Rhodes, On the minmax reachability of target sets and target tubes, Automatica 7 (1971) 233–247.
13. D. Q. Mayne, Control of constrained dynamic systems, European Journal of Control 7 (2001) 87–99, Survey paper.
14. P. Goulart, E. Kerrigan, J. Maciejowski, Optimization over state feedback policies for robust control with constraints, Automatica 42 (4) (2005) 523–533.
15. J. Lofberg, Approximations of closed-loop MPC, in: Proceedings of the 42th IEEE Conference on Decision and Control, Maui, Hawaii, USA, 2003, pp. 1438–1442.
16. A. Ben-Tal, A. Goryashko, E. Guslitzer, A. Nemirovski, Adjustable robust solutions of uncertain linear programs, Tech. rep., Minerva Optimization Center, Technion, Israeli Institute of Technology (2002).
17. S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, UK, 2004.
18. F. Borelli, A. Bemporad, M. Morari, A geometric algorithm for multi-parametric linear programming, Journal of Optimization Theory and Applications 118 (3) (2003) 515–540.
19. J. Spjøtvold, P. Tøndel, T. A. Johansen, Unique polyhedral representations to continuous selections for convex multiparametric quadratic programs, in: Proceedings 2005 American Control Conference, Portland, Oregon USA, 2005.
20. A. Bemporad, A. Borelli, M. Morari, Model predictive control based on linear programming–the explicit solution, IEEE Transactions on Automatic Control 47 (12) (2002) 1974–1984.
21. I. Necoara, Model Predictive Control for Max-Plus-Linear and Piecewise Affine Systems, Ph.D. thesis, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands (2006).
22. T. van den Boom, B. De Schutter, Properties of MPC for max-plus-linear systems, European Journal of Control 8 (5) (2002) 453–462.