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Stabilizing model predictive controllers for randomly switching max-plus-linear systems

Ton van den Boom* and Bart De Schutter

***Abstract*— Randomly switching max-plus-linear (RSMPL) systems are discrete event systems that can switch between different modes of operation, where the switching is a stochastic process. In each mode the system is described by a max-plus-linear state equation and a max-plus-linear output equation, with different system matrices for each mode. We discuss conditions for stability and derive a stabilizing model predictive controller for RSMPL systems.**

I. INTRODUCTION

The class of discrete event systems (DES) essentially consists of man-made systems that contain a finite number of resources that are shared by several users all of which contribute to the achievement of some common goal [1]. In general, models that describe the behavior of a discrete event system are nonlinear in conventional algebra.

In this paper we will consider randomly switching max-plus-linear (RSMPL) systems, discrete event systems that can switch between different modes of operation, in which the mode switching depends on a stochastic sequence. In each mode the system is described by a max-plus-linear state equation and a max-plus-linear output equation, with different system matrices for each mode. The class of RSMPL systems contains discrete event systems with synchronization but no concurrency, in which the order of synchronization of the event steps may vary randomly, or cannot be determined a priori. Typical examples of RSMPL systems are flexible manufacturing systems, telecommunication networks, traffic signal controlled urban traffic networks, The random switching between different MPL modes is then due to e.g. randomly changing production recipes, varying customer demands or traffic demands, or failures in production unit, transmission lines or traffic links.

In [10] we have already discussed (ordinary) SMPL systems. The main difference with RSMPL systems is that in [10] the switching was a function of the previous state, the previous mode and the input, whereas in RSMPL systems the switching is a random process.

The paper is organized as follows. In Section II we introduce the max-plus algebra and the concept of RSMPL systems. Section III gives conditions for a stabilizing controller for RSMPL systems. In Section IV we derive a stabilizing model predictive controller for RSMPL systems, and in Section VI we give a worked example.

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II. MAX-PLUS ALGEBRA AND RSMPL SYSTEMS

A. Max-plus algebra

In this section we give the basic definition of the max-plus algebra [1], [3].
Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:

$$x \oplus y = \max(x, y) \qquad x \otimes y = x + y$$

for any $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n}(a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$. The matrix $\mathcal{E}$ is the max-plus-algebraic zero matrix: $[\mathcal{E}]_{ij} = \varepsilon$ for all i, j.

A max-plus diagonal matrix $S = \mathrm{diag}_\oplus(s_1, \ldots, s_n)$ has elements $S_{ij} = \varepsilon$ for $i \neq j$ and diagonal elements $S_{ii} = s_i$ for $i = 1, \ldots, n$. If all s_i are finite we find that the inverse of S is equal to $S^{\otimes^{-1}} = \mathrm{diag}_\oplus(-s_1, \ldots, -s_n)$. There holds $S \otimes S^{\otimes^{-1}} = S^{\otimes^{-1}} \otimes S = E$, where $E = \mathrm{diag}_\oplus(0, \ldots, 0)$ is the max-plus identity matrix.

B. SMPL and RSMPL systems

In [10] we introduced Switching Max-Plus-Linear (SMPL) systems, i.e. discrete event systems that can switch between different modes of operation. For SMPL systems the switching is a function of the previous state. This is in contrast to RSMPL systems, in which the mode switching depends on a stochastic sequence. In each mode $\ell = 1, \ldots, L$, the system is described by a max-plus-linear state equation and a max-plus-linear output equation:

$$x(k) = A^{(\ell(k))} \otimes x(k-1) \oplus B^{(\ell(k))} \otimes u(k) \qquad (1)$$

$$y(k) = C^{(\ell(k))} \otimes x(k) \qquad (2)$$

in which the matrices $A^{(\ell)} \in \mathbb{R}_\varepsilon^{n_x\times n}$, $B^{(\ell)} \in \mathbb{R}_\varepsilon^{n_x\times n_u}$, $C^{(\ell)} \in \mathbb{R}_\varepsilon^{n_y\times n_x}$ are the system matrices for the ℓth mode. The index k is called the event counter. For discrete event systems the state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, the input $u(k)$ contains the time instants at which the input events occur for the kth time, and the output $y(k)$ contains the time instants at which the output events occur for the kth time[1].

[1]More specifically, for a manufacturing system, $x(k)$ contains the time instants at which the processing units start working for the kth time, $u(k)$ the time instants at which the kth batch of raw material is fed to the system, and $y(k)$ the time instants at which the kth batch of finished product leaves the system.

For the RSMPL system (1)-(2), the mode switching variable $\ell(k)$ is a stochastic process. For a system with L possible modes, we assume the probability of a switching from mode i to a mode j to be given by $P_{\mathrm{s}}(i,j)$ for all $i = 1,\dots,L$, $j = 1,\dots,L$.

III. CONDITIONS FOR STABILITY

Just like in [9], we adopt the notion of stability for DES from [8], in which a DES is called stable if all its buffer levels remain bounded. All the buffer levels in DES are bounded if the dwelling times of the parts or batches in the system remain bounded. This implies for the RSMPL system with a due date signal $r(k)$ that closed-loop stability is achieved if there exist finite constants k_0, M_{yr}, M_{yx} and M_{xu} such that

$$|\, y_i(k) - r_i(k)\,| \le M_{yr}, \qquad \forall i \tag{3}$$

$$|\, y_i(k) - x_j(k)\,| \le M_{yx}, \qquad \forall i,j \tag{4}$$

$$|\, x_j(k) - u_m(k)\,| \le M_{xu}, \qquad \forall j,m \tag{5}$$

for all $k > k_0$. Condition (3) means that the delay between the actual output date $y(k)$ and the due date $r(k)$ remains bounded (for $y - r < \infty$), and on the other hand, that the stock time will remain bounded (for $r - y < \infty$). Conditions (4) and (5) mean that the throughput time (i.e. the time between the starting date $u(k)$ and the output date $y(k)$) is bounded. For a due date defined as

$$r(k) = \rho\, k + d(k), \quad \text{where } |d_i(k)| \le d_{\max}, \forall i \tag{6}$$

where r and d are vectors and ρ is a scalar, satisfying $\rho > 0$, this implies finite buffer levels.

Similar to max-plus-linear systems, stability is not an intrinsic feature of the RSMPL system, but it also depends on the due dates (i.e., the reference signal) of the system. In [9] we already observed that for max-plus-linear systems, the max-plus-algebraic eigenvalue of the system matrix A gives an upper bound on the asymptotic slope of the due date sequence. For a strongly connected max-plus-linear system[2] the A-matrix only has one max-plus-algebraic eigenvalue λ and one max-plus-algebraic eigenvector $v \neq \varepsilon$, such that $A \otimes v = v \otimes \lambda$. For RSMPL systems we cannot use the max-plus-algebraic eigenvalue, but we use the concept of maximum growth rate:

Definition 1: Consider an RSMPL the matrices $A_\alpha^{(\ell)}$ with $[A_\alpha^{(\ell)}]_{ij} = [A^{(\ell)}]_{ij} - \alpha$. The maximum growth rate λ of the RSMPL system is the smallest α for which there exists a max-plus diagonal matrix $S = \mathrm{diag}_\oplus(s_1,\dots,s_n)$ with finite diagonal elements s_i, such that

$$[\, S \otimes A_\alpha^{(\ell)} \otimes S^{\otimes^{-1}}\,]_{ij} \le 0, \;\; \forall\, i,j,\ell \tag{7}$$

Remark 1: Note that for any RSMPL system the maximum grow rate λ is finite, or more precisely:

$$\lambda \le \max_{i,j,\ell} [A^{(\ell)}]_{ij}.$$

This fact is easily verified by noting that for $\lambda' = \max_{i,j,\ell}[A^{(\ell)}]_{ij}$, and using the max-plus identity matrix $S = \mathrm{diag}_\oplus(0,\dots,0)$ gives us

$$[S \otimes A_{\lambda'}^{(\ell)} \otimes S^{\otimes^{-1}}]_{ij} = [A_{\lambda'}^{(\ell)}]_{ij} = [A^{(\ell)}]_{ij} - \lambda' \le 0, \;\; \forall i,j,\ell.$$

The maximum growth rate λ can be easily computed by solving a linear programming problem.

Remark 2: For a max-plus-linear system (so $L = 1$), the maximum growth rate λ is equivalent to the max-plus-linear eigenvalue of the matrix $A^{(1)}$.

The set $\mathcal{L}_N = \{\, [\, \ell_1 \;\cdots\; \ell_N\,]^T \mid \ell_m \in \{1,\dots,L\},\, m = 1,\dots,N\}$ is the set of all possible consecutive mode switchings vectors.

Definition 2: An RSMPL system is controllable if there exists a finite positive integer N such that for all $\tilde{\ell} \in \mathcal{L}_N$ the matrices

$$\Gamma_\rho^N(\tilde{\ell}) = \Big[B^{(\ell_N)} \quad A_\rho^{(\ell_N)} \otimes B^{(\ell_{N-1})} \quad A_\rho^{(\ell_N)} \otimes A_\rho^{(\ell_{N-1})} \otimes B^{(\ell_{N-2})} \quad \dots \quad A_\rho^{(\ell_N)} \otimes \cdots \otimes A_\rho^{(\ell_2)} \otimes B^{(\ell_1)} \;\Big]$$

are row-finite, i.e. in each row there is at least one entry larger then ε.

Theorem 3: Consider a switching MPL system with random mode switching and due-date signal (6), and a maximum grow rate λ. Define the matrices $A_\rho^{(\ell)}$ with $[A_\rho^{(\ell)}]_{ij} = [A^{(\ell)}]_{ij} - \rho$. Further assume $C^{(\ell)}$ to be row-finite. Now if

1) $\quad \rho < \lambda \qquad (8)$

2) the system is controllable,

then any input signal

$$u(k) = \rho\, k + \mu(k), \;\; \text{where } \; \mu_{\min} \le \mu_i(k) \le \mu_{\max}, \; \forall i, \tag{9}$$

and $\mu_{\min}$ and $\mu_{\max}$ are finite, will stabilize the SMPL system.

Proof : First note that condition (8) holds if and only if

$$[\, S \otimes A_\rho^{(\ell)} \otimes S^{\otimes^{-1}}\,]_{ij} < 0, \;\; \forall\, i,j,\ell\,.$$

Let S be the diagonal matrix with finite diagonal elements, such that (8) is satisfied, and define the signals

$$z(k) = S \otimes (x(k) - \rho\, k)$$
$$w(k) = S \otimes (y(k) - \rho\, k)$$
$$\mu(k) = S \otimes (u(k) - \rho\, k)$$

and the matrices $\bar{A}_\rho^{(\ell)}$, $\bar{B}^{(\ell)}$, and $\bar{C}^{(\ell)}$ with

$$\bar{A}_\rho^{(\ell)} = S \otimes A_\rho^{(\ell)} \otimes S^{\otimes^{-1}},$$
$$\bar{B}^{(\ell)} = S \otimes B^{(\ell)},$$
$$\bar{C}^{(\ell)} = C^{(\ell)} \otimes S^{\otimes^{-1}}.$$

To every ρ we can associate a shifted system [11]

$$z(k) = \bar{A}_\rho^{(\ell)} \otimes z(k-1) \oplus \bar{B}^{(\ell)} \otimes \mu(k) \tag{10}$$

$$w(k) = \bar{C}^{(\ell)} \otimes z(k). \tag{11}$$

[2] A max-plus-linear system is called strongly connected if its graph is strongly connected. This means that for any two nodes i, j of the graph, node j is reachable from node i [5].

Stability means that all signals in this system should remain bounded, as will be proven later. In other words, we are looking for finite values $z_{\max}$, $w_{\max}$, such that

$$|z(k)| \le z_{\max} \quad , \quad |w(k)| \le w_{\max}.$$

Now consider RSMPL system (10)–(11) for the input signal $\mu_i(k) \le \mu_{\max}$, $\forall i, k$. Let $z_{\max}(k) = \max_i z_i(k)$ and $\bar{b}_{\max} = \max_{\ell,i,j}([\bar{B}^{(\ell)}]_{ij})$, then

$$\begin{aligned}
z_i(k) &= \max\Big(\max_j([\bar{A}_\rho^{(\ell)}]_{ij} + z_j(k-1)),\\
&\qquad \max_m([\bar{B}^{(\ell)}]_{im} + \mu_m(k))\Big)\\
&\le \max\Big(\max_j([\bar{A}_\rho^{(\ell)}]_{ij}) + z_{\max}(k-1),\\
&\qquad \max_{\ell,i,m}([\bar{B}^{(\ell)}]_{im}) + \mu_{\max}\Big)\\
&\le \max\Big(z_{\max}(k-1), \bar{b}_{\max} + \mu_{\max}\Big)
\end{aligned}$$

where we use the fact that $[\bar{A}_\rho^{(\ell)}]_{ij} < 0$ because of (8). We find

$$z_{\max}(k) \le \max\Big(z_{\max}(k-1), \bar{b}_{\max} + \mu_{\max}\Big)$$

This means that all entries of the shifted state $z(k)$ have a non-increasing function $z_{\max}(k)$ as an upper bound.

Now again consider RSMPL system (10)–(11) for the input signal $\mu_i(k) \ge \mu_{\min}$, $\forall i, k$, and let N be such that $\gamma_{i,\max}(\tilde{\ell}) = \max_j([\Gamma_\rho^N(\tilde{\ell})]_{ij}) > \varepsilon$ for all $i, \tilde{\ell}$. By successive substitution we find that for any $m \ge N$ there holds

$$\begin{aligned}
[z(k+m)]_i &= [\bar{A}_\rho^{(\ell(k+m))} \otimes z(k+m-1)]_i \oplus\\
&\quad \bar{B}_\rho^{(\ell(k+m))} \otimes \mu(k+m)]_i\\
&= [\bar{A}_\rho^{(\ell(k+m))} \otimes \bar{A}_\rho^{(\ell(k+m-1))} \otimes z(k+m-2) \oplus\\
&\quad \bar{B}_\rho^{(\ell(k+m))} \otimes \mu(k+m-N)\\
&\quad \bar{A}_\rho^{(\ell(k+m))} \otimes \bar{B}_\rho^{(\ell(k+m))} \otimes \mu(k+m-N)]_i\\
&= [\bar{A}_\rho^{(\ell(k+m))} \otimes \bar{A}_\rho^{(\ell(k+m-1))} \otimes \ldots \otimes\\
&\quad \bar{A}_\rho^{(\ell(k+m-N+1))} \otimes z(k+m-N)]_i \oplus\\
&\quad [\bar{B}_\rho^{(\ell(k+m))} \otimes \mu(k+m)\\
&\quad \bar{A}_\rho^{(\ell(k+m))} \otimes \bar{B}_\rho^{(\ell(k+m-1))} \otimes \mu(k+m-1) \oplus\\
&\quad \bar{A}_\rho^{(\ell(k+m))} \otimes \bar{A}_\rho^{(\ell(k+m-1))} \otimes \ldots \otimes\\
&\quad \bar{B}_\rho^{(\ell(k+m-N+1))} \otimes \mu(k+m-N+1)]_i\\
&= [\bar{A}_\rho^{(\ell(k+m))} \otimes \bar{A}_\rho^{(\ell(k+m-1))} \otimes \ldots\\
&\quad \otimes \bar{A}_\rho^{(\ell(k+m-N+2))} \otimes z(k+m-N)]_i \oplus\\
&\quad [\bigoplus_{t=1}^{N} \bar{A}_\rho^{(\ell(k+m))} \otimes \bar{A}_\rho^{(\ell(k+m-1))} \otimes \ldots\\
&\quad \otimes \bar{A}_\rho^{(\ell(k+m-N+t+1))} \otimes \bar{B}^{(\ell(k+m-N+t))} \otimes\\
&\quad \otimes \mu(k+m-N+t)]_i\\
&\ge [\bigoplus_{t=1}^{N} \bar{A}_\rho^{(\ell(k+m-1))} \otimes \bar{A}_\rho^{(\ell(k+m-2))} \otimes \ldots\\
&\quad \otimes \bar{A}_\rho^{(\ell(k+m-N+t-1))} \otimes \bar{B}^{(\ell(k+m-N+t))} \otimes\\
&\quad \otimes \mu(k+m-N+t)]_i
\end{aligned}$$

$$\begin{aligned}
&\ge \max_j\Big([\bigoplus_{t=1}^{N} \bar{A}_\rho^{(\ell(k+m-1))} \otimes \bar{A}_\rho^{(\ell(k+m-2))} \otimes \ldots\\
&\quad \otimes \bar{A}_\rho^{(\ell(k+m-N+t-1))} \otimes \bar{B}^{(\ell(k+m-N+t))}]_{ij}\\
&\quad + \mu_j(k+m-N+t)\Big)\\
&\ge \max_j\Big([S \otimes \bigoplus_{t=1}^{N} A_\rho^{(\ell(k+m-1))} \otimes A_\rho^{(\ell(k+m-2))} \otimes \ldots\\
&\quad \otimes A_\rho^{(\ell(k+m-N+t-1))} \otimes B^{(\ell(k+m-N+t))}]_{ij}\Big) + \mu_{\min}\\
&\ge \max_p([S]_{pp}) + \max_j\Big([\bigoplus_{t=1}^{N} A_\rho^{(\ell(k+m-1))}\\
&\quad \otimes A_\rho^{(\ell(k+m-2))} \otimes \ldots \otimes A_\rho^{(\ell(k+m-N+t+1))}\\
&\quad \otimes B^{(\ell(k+m-N-t))}]_{ij}\Big) + \mu_{\min}\\
&\ge \max_p(s_p) + \max_j([\Gamma_\rho^N(\tilde{\ell})]_{ij}) + \mu_{\min}\\
&\ge \max_p(s_p) + \gamma_{i,\max}(\tilde{\ell}) + \mu_{\min}\\
&\ge s_{\max} + \gamma_{\min} + \mu_{\min}
\end{aligned}$$

where $s_{\max} = \max_p(s_p)$ and $\gamma_{\min} = \min_{\tilde{\ell},i} \gamma_{i,\max}(\tilde{\ell})$. We conclude that after N event steps we have a lower bound for our shifted state $z(k)$. Let $\bar{c}_{\max}$ and $\bar{c}_{\min}$ be the largest and the smallest finite values of $\bar{C}^{(\ell)}, \forall \ell$, respectively. Now from (11) it follows that

$$w_i(k) = \max_j([\bar{C}_\rho^{(\ell)}]_{i,j} + z_j(k)])$$

and so after N event steps $w(k)$ will be bounded by

$$\begin{aligned}
s_{\max} + \gamma_{\min} + \mu_{\min} + \bar{c}_{\min} &\le w_i(k)\\
&\le \max(z_{\max}(k-1), \bar{b}_{\max} + \mu_{\max}) + \bar{c}_{\max}
\end{aligned}$$

where $z_{\max}(k)$ is a non-increasing signal.

Define $\zeta(k) = S \otimes (r(k) - \rho\, k) = S \otimes d(k)$. Then $|\zeta_i(k)| \le \zeta_{\max} = s_{\max} + d_{\max}$. For any $k > N$, there holds

$$\begin{aligned}
&y_i(k) - r_i(k) =\\
&\quad = [(S^{\otimes^{-1}} \otimes w(k)) + \rho\, k]_i - [(S^{\otimes^{-1}} \otimes \zeta(k)) + \rho\, k]_i\\
&\quad = (-s_i + w_i(k) + \rho\, k) - (-s_i + \zeta_i(k) + \rho\, k)\\
&\quad = w_i(k) - \zeta_i(k)\\
&\quad \le \max(z_{\max}(0), \bar{b}_{\max} + \mu_{\max}) + \bar{c}_{\max} + s_{\max} + d_{\max}\\
&\quad = M_{yr1},\\
&r_i(k) - y_i(k) = \zeta_i(k) - w_i(k)\\
&\quad \le d_{\max} - \gamma_{\min} - \mu_{\min} - \bar{c}_{\min}\\
&\quad = M_{yr2},\\
&|y_i(k) - r_i(k)| = |w_i(k) - \zeta_i(k)|\\
&\quad \le \max(M_{yr1}, M_{yr2}) = M_{yr} < \infty,
\end{aligned}$$

$$
\begin{aligned}
y_i(k) - x_j(k) &= \\
&= [(S^{\otimes^{-1}} \otimes w(k)) + \rho\, k]_i - [(S^{\otimes^{-1}} \otimes z(k)) + \rho\, k]_j \\
&= (-s_i + w_i(k) + \rho\, k) - (-s_j + z_j(k) + \rho\, k) \\
&= w_i(k) - z_j(k) + (s_j - s_i) \\
&\leq \max(z_{\max}(0), \bar{b}_{\max} + \mu_{\max}) + \bar{c}_{\max} \\
&\quad - s_{\max} - \gamma_{\min} - \mu_{\min} + s_{\max} - s_{\min} \\
&= M_{yx} < \infty, \\
x_j(k) - u_m(k) &= \\
&= [(S^{\otimes^{-1}} \otimes z(k)) + \rho\, k]_j - [(S^{\otimes^{-1}} \otimes \mu(k)) + \rho\, k]_m \\
&= (-s_j + z_j(k) + \rho\, k) - (-s_m + \mu_m(k) + \rho\, k) \\
&= z_j(k) - \mu_m(k) + (s_m - s_j) \\
&\leq \max(z_{\max}(0), \bar{b}_{\max} + \mu_{\max}) - \mu_{\min} + s_{\max} - s_{\min} \\
&= M_{xu} < \infty \ \ ,
\end{aligned}
$$

which proves stability for the RSMPL system (1)–(2). $\diamond$

Remark 4: For a max-plus-linear system (so $L = 1$), condition (8) is equivalent to the condition that the production rate ρ should be larger than the max-plus-linear eigenvalue λ of the matrix $A^{(1)}$.

IV. A STABILIZING MODEL PREDICTIVE CONTROLLER

Model predictive control (MPC) [2], [7] is a model-based predictive control approach that has its origins in the process industry and that has mainly be developed for linear or nonlinear time-driven systems. Its main ingredients are: a prediction model, a performance criterion to be optimized over a given horizon, constraints on inputs and outputs, and a receding horizon approach. In [4], [10] we have extended this approach to MPL systems and switching MPL systems and shown that the resulting optimization problem can be solved efficiently. In this section we show that also for RSMPL systems the MPC optimization problem can be solved efficiently.

In MPC we use predictions of future signals based on the RSMPL model. Define the prediction vectors

$$
\tilde{y}(k) = \begin{bmatrix} \hat{y}(k|k) \\ \vdots \\ \hat{y}(k+N_\mathrm{p}-2|k) \\ \hat{y}(k+N_\mathrm{p}-1|k) \end{bmatrix}, \quad \tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_\mathrm{p}-2) \\ u(k+N_\mathrm{p}-1) \end{bmatrix},
$$

$$
\tilde{\ell}(k) = \begin{bmatrix} \ell(k) \\ \vdots \\ \ell(k+N_\mathrm{p}-2) \\ \ell(k+N_\mathrm{p}-1) \end{bmatrix},
$$

where $\hat{y}(k+j|k)$ denotes the prediction of $y(k+j)$ based on knowledge at event step k, $u(k+j)$ denotes the future inputs, $\ell(k+j)$ denotes the future modes, and N_p is the prediction horizon (so it determines how many cycles we look ahead in our control law design).

Now for any mode sequence $\tilde{\ell}(k)$ the prediction model for (1)–(2) is given by:

$$
\tilde{y}(k) = \tilde{C}(\tilde{\ell}(k)) \otimes x(k-1) \oplus \tilde{D}(\tilde{\ell}(k)) \otimes \tilde{u}(k) \qquad (12)
$$

in which $\tilde{C}(\tilde{\ell}(k))$ and $\tilde{D}(\tilde{\ell}(k))$ are given by

$$
\tilde{C}(\tilde{\ell}(k)) = \begin{bmatrix} \tilde{C}_1(\tilde{\ell}(k)) \\ \vdots \\ \tilde{C}_{N_\mathrm{p}}(\tilde{\ell}(k)) \end{bmatrix}
$$

$$
\tilde{D}(\tilde{\ell}(k)) = \begin{bmatrix} \tilde{D}_{11}(\tilde{\ell}(k)) & \cdots & \tilde{D}_{1N_\mathrm{p}}(\tilde{\ell}(k)) \\ \vdots & \ddots & \vdots \\ \tilde{D}_{N_\mathrm{p}1}(\tilde{\ell}(k)) & \cdots & \tilde{D}_{N_\mathrm{p}N_\mathrm{p}}(\tilde{\ell}(k)) \end{bmatrix}
$$

where

$$
\tilde{C}_m(\tilde{\ell}(k)) = C^{(\ell(k+m-1))} \otimes A^{(\ell(k+m-1))} \otimes \ldots \otimes A^{(\ell(k))}
$$

and

$$
\tilde{D}_{mn}(\tilde{\ell}(k)) = \begin{cases} C^{(\ell(k+m-1))} \otimes A^{(\ell(k+m-1))} & \\ \quad \otimes A^{(\ell(k+n))} \otimes B^{(\ell(k+n-1))} & \text{if } m > n \\ C^{(\ell(k+m-1))} \otimes B^{(\ell(k+m-1))} & \text{if } m = n \\ \mathcal{E} & \text{if } m < n \end{cases}
$$

The probability for the switching sequence $\tilde{\ell}(k) \in \mathcal{L}_{N_\mathrm{p}}$, given the previous mode $\ell(k-1)$, is given by

$$
\begin{aligned}
P(\tilde{\ell}(k)|\ell(k-1)) &= P_\mathrm{s}(\ell(k-1), \ell(k)) \cdot \\
&\quad P_\mathrm{s}(\ell(k), \ell(k+1)) \cdots P_\mathrm{s}(\ell(k+N_\mathrm{p}-2), \ell(k+N_\mathrm{p}-1))
\end{aligned}
$$

where P_s denotes the switching probability (see Section II-B).

In MPC we aim at computing the optimal $\tilde{u}(k)$ that minimizes the expectation of a cost criterion $J(k)$, subject to linear constraints on the inputs. The cost criterion reflects the input and output cost functions (J_in and J_out, respectively) in the event period $[k, k+N_\mathrm{p}-1]$:

$$
J(k) = J_\mathrm{out}(k) + \beta J_\mathrm{in}(k) \quad , \qquad (13)
$$

where $\beta \geq 0$ is a tuning parameter, chosen by the user. The output cost function is defined by

$$
\begin{aligned}
J_\mathrm{out}(k) &= \mathbb{E}\left\{ \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} \max(y_i(k+j) - r_i(k+j), 0) \right\} \\
&= \mathbb{E}\left\{ \sum_{i=1}^{n_y N_\mathrm{p}} \max(\tilde{y}_i(k) - \tilde{r}_i(k), 0) \right\} \\
&= \mathbb{E}\left\{ \sum_{i=1}^{n_y N_\mathrm{p}} [(\tilde{y}(k) - \tilde{r}(k)) \oplus \bar{0}]_i \right\} \\
&= \mathbb{E}\left\{ \sum_{i=1}^{n_y N_\mathrm{p}} \Big[\Big((\tilde{C}(\tilde{\ell}(k)) \otimes x(k-1) \right. \\
&\qquad \left. \oplus \tilde{D}(\tilde{\ell}(k)) \otimes \tilde{u}(k)) - \tilde{r}(k) \Big) \oplus \bar{0} \Big]_i \right\}
\end{aligned}
$$

$$= \sum_{\tilde{\ell}\in\mathcal{L}_N} \left\{ \sum_{i=1}^{n_y N_\mathrm{p}} \Big[\Big(\tilde{C}(\tilde{\ell}) \otimes x(k-1) \oplus \tilde{D}(\tilde{\ell}) \otimes \tilde{u}(k)) - \tilde{r}(k) \Big) \oplus \bar{0} \Big]_i \right\} P(\tilde{\ell}|\ell(k-1)) \tag{14}$$

where $I\!\!E$ stands for the expectation over all possible switching sequences, and $\bar{0}$ is a zero column vector. The input cost function is chosen as

$$J_{\mathrm{in},u}(k) = -\sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_u} u_i(k+j) = -\sum_{i=1}^{n_u N_\mathrm{p}} [\tilde{u}(k)]_i \ . \tag{15}$$

The MPC problem for RSMPL systems with due-date signal (6) can be defined at event step k as

$$\min_{\{\tilde{u}(k)\}} J(k) \tag{16}$$

subject to

$$u(k+j) - u(k+j-1) \geq 0, \ \ j=0,\ldots,N_\mathrm{p}-1 \tag{17}$$

$$\mu_{\min} \leq u_i(k) - \rho\, k \leq \mu_{\max}, \ i=1,\ldots,n_u, \tag{18}$$

where (17) guarantees a non-decreasing input sequence, and (18) guarantees stability (cf. Theorem 3).

MPC uses a receding horizon strategy. So after computation of the optimal control sequences $\tilde{u}^*(k)$, only the first control sample $u(k) = u^*(k)$ will be implemented, subsequently the horizon is shifted and the model and the initial state estimate can be updated if new measurements are available, then the new MPC problem is solved, etc.

Theorem 4: Assume that $\mathcal{L}_{N_\mathrm{p}}$ can be rewritten as $\mathcal{L}_{N_\mathrm{p}} = \{\tilde{\ell}^1, \tilde{\ell}^2, \ldots, \tilde{\ell}^M\}$ for $M = L^{N_\mathrm{p}}$. The MPC problem (16)-(18) can be recast as a linear programming problem:

$$\min_{\{\tilde{u}(k), t_{i,m}\}} \sum_{i=1}^{n_y N_\mathrm{p}} \sum_{m=1}^{N} t_{i,m} P(\tilde{\ell}^m|\ell(k-1)) - \beta \sum_{i=1}^{n_u N_\mathrm{p}} \tilde{u}_i(k) \tag{19}$$

subject to

$$t_{i,m} \geq [\tilde{C}(\tilde{\ell}^m)]_{i,l} + x_l(k-1) - \tilde{r}_i(k) \ , \ \ \forall i,m,l \tag{20}$$

$$t_{i,m} \geq [\tilde{D}(\tilde{\ell}^m)]_{i,l} + \tilde{u}_l(k) - \tilde{r}_i(k) \ , \ \ \forall i,m,l \tag{21}$$

$$t_{i,m} \geq 0 \ , \ \ \forall i,m \tag{22}$$

$$u_i(k+j) - u_i(k+j-1) \geq 0, \ \ \forall i,j \tag{23}$$

$$\mu_{\min} \leq u_i(k+j) - \rho\, k \leq \mu_{\max}, \ \ \forall i,j \tag{24}$$

Proof: From (14) we derive:

$$\begin{aligned} J_\mathrm{out}(k) &= \sum_{i=1}^{n_y N_\mathrm{p}} \sum_{m=1}^{M} \max\Big\{ \Big[\tilde{C}(\tilde{\ell}^m(k)) \otimes x(k-1) \\ &\quad \oplus \tilde{D}(\tilde{\ell}^m(k)) \otimes \tilde{u}(k) \Big]_i - \tilde{r}_i(k), 0 \Big\} P(\tilde{\ell}^m|\ell(k-1)) \\ &= \sum_{i=1}^{n_y N_\mathrm{p}} \sum_{m=1}^{M} \max\Big\{ \max_l \Big([\tilde{C}(\tilde{\ell}^m(k))]_{i,l} \\ &\quad + x_l(k-1) - \tilde{r}_i(k) \Big), \max_j \Big([\tilde{D}(\tilde{\ell}^m(k))]_{i,j} \\ &\quad + \tilde{u}_j(k) - \tilde{r}_i(k) \Big), 0 \Big\} P(\tilde{\ell}^m|\ell(k-1)) \\ &= \sum_{i=1}^{n_y N_\mathrm{p}} \sum_{m=1}^{M} t_{i,m} P(\tilde{\ell}^m|\ell(k-1)) \end{aligned}$$

where

$$t_{i,m} = \max\Big(\max_l ([\tilde{C}(\tilde{\ell}^m(k))]_{i,l} + x_l(k-1) - \tilde{r}_i(k)), \max_j ([\tilde{D}(\tilde{\ell}^m(k))]_{i,j} + \tilde{u}_j(k)) - \tilde{r}_i(k+j)), 0 \Big) \tag{25}$$

If we would minimize J_out subject to (20)-(24) then, given the fact that the coefficients $P(\tilde{\ell}^m|\ell(k-1))$ are nonnegative, and the variables $t_{i,m}$ only appear in the left-hand side of the inequalities (20)-(24), the inequality indeed becomes an equality for at least one of the indices and so $t_{i,m}$ will be equal to the maximum (25).

This implies that the MPC problem (16)-(18) can indeed be written as the linear programming problem (19)-(24). $\diamond$

So the optimization in the MPC algorithm boils down to a linear programming problem, which is polynomially solvable [6] and for which efficient algorithms are available.

V. TIMING ISSUES

Discrete event MPL systems are different from conventional time-driven systems in the sense that the event counter k is not directly related to a specific time. In the previous we use the assumption that at event step k the state $x(k)$ is available. However, in general not all components of $x(k)$ are known at the same time instant (recall that $x(k)$ contains the time instants at which the internal activities or processes of the system start for the kth cycle). Therefore, we will present a method to address the availability issue of the state at a certain time t_0. We consider the case of full state information. Note that in practical applications the entries of the system matrices are nonnegative or take the value ε. Since the components of x correspond to event times, they are in general easy to measure. Also note that measurements of occurrence times of events are in general not as susceptible to noise and measurement errors as measurements of continuous-time signals involving variables such as temperature, speed, pressure, etc. Let t_0 be the time when an optimal control problem is performed. We can define the initial cycle k_0 as follows:

$$k_0 = \arg\max \Big\{ l : x_i(l) \leq t_0 \, \forall i \{1, 2, \ldots, n\} \Big\}$$

Hence, the state $x(k_0)$ is completely known at time t_0 and thus $u(k_0-1)$ is also available. Note that at time t_0 some components of the forthcoming states and of the forthcoming inputs might be known (so $x_i(k_0+l) \leq t_0$ and $u_j(k_0+l) \leq t_0$ for some $l > 0$). Due to causality, these states are completely determined by the known forthcoming inputs. During the optimization at time t_0 the known values of the input have to be fixed by equality constraints, which fits perfectly in the linear programming problem. Due to the information at time t_0 it might be possible to conclude that certain forthcoming modes ($\ell(k_0+l)$ for $l > 0$) are not feasible any more.

In that case we can set the switching probabilities for this mode at zero, and normalize the switching probabilities of the other modes. With these new probabilities we can do the optimization at time t_0.

VI. EXAMPLE

A. Production system

Consider the production system of Figure 1. This system consists of three machines M_1, M_2, and M_3. Three products (A,B,C) can be made with this system, each with its own recipe, meaning that the order in the production sequence is different for every product.

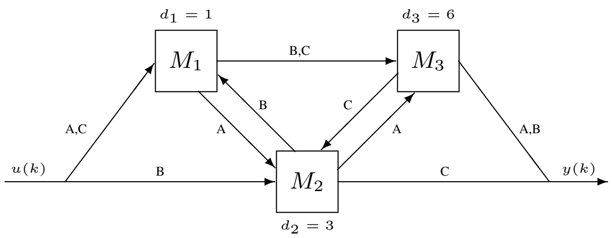

Fig. 1. A production system.

For product A the production order is M_1-M_2-M_3, which means that the raw material is fed to machine M_1 where it is processed. The intermediate product is sent to machine M_2 for further processing, and finally the product is finished in machine M_3. Similarly, for product B the processing order is M_2-M_1-M_3, and for product C the processing order is M_1-M_3-M_2. We assume that the type of the kth product (A, B, or C) only becomes available at the start of the production, so that we do not know $\ell(k)$ when computing $u(k)$.

Each machine starts working as soon as possible on each batch, i.e., as soon as the raw material or the required intermediate products are available, and as soon as the machine is idle (i.e., the previous batch has been finished and has left the machine). We define $u(k)$ as the time instant at which the system is fed for the kth time, $x_i(k)$ as the time instant at which machine i starts for the kth time, and $y(k)$ as time instant at which the kth product leaves the system. We assume that all the internal buffers are large enough, and no overflow will occur.

We assume the transportation times between the machines to be negligible, and the processing time of the machines M_1, M_2 and M_3 are given by $d_1 = 1$, $d_2 = 2$ and $d_3 = 3$, respectively. The system equations for x_1 and x_2 for recipe A are given by

$$
\begin{aligned}
x_1(k) &= \max(x_1(k-1)+d_1, u(k)) \ , \\
x_2(k) &= \max(x_1(k)+d_1, x_2(k-1)+d_2) \\
&= \max(x_1(k-1)+2d_1, x_2(k-1)+d_2, u(k)+d_1) \ , \\
x_3(k) &= \max(x_2(k)+d_2, x_3(k-1)+d_3) \\
&= \max(x_1(k-1)+2d_1+d_2, x_2(k-1)+2d_2, \\
&\quad x_3(k-1)+d_3, u(k)+d_1+d_2) \ , \\
y(k) &= x_3(k)+d_3 \ ,
\end{aligned}
$$

leading to the systems matrices for recipe A:

$$
A^{(1)} = \begin{bmatrix} d_1 & \varepsilon & \varepsilon \\ 2d_1 & d_2 & \varepsilon \\ 2d_1+d_2 & 2d_2 & d_3 \end{bmatrix} \ , \quad B^{(1)} = \begin{bmatrix} 0 \\ d_1 \\ d_1+d_2 \end{bmatrix} \ ,
$$

$$
C^{(1)} = \begin{bmatrix} \varepsilon & \varepsilon & d_3 \end{bmatrix} .
$$

Similarly we derive for recipe B:

$$
A^{(2)} = \begin{bmatrix} d_1 & 2d_2 & \varepsilon \\ \varepsilon & d_2 & \varepsilon \\ 2d_1 & d_1+2d_2 & d_3 \end{bmatrix} \ , \quad B^{(2)} = \begin{bmatrix} d_2 \\ 0 \\ d_1+d_2 \end{bmatrix} \ ,
$$

$$
C^{(2)} = \begin{bmatrix} \varepsilon & \varepsilon & d_3 \end{bmatrix} \ ,
$$

and for recipe C:

$$
A^{(3)} = \begin{bmatrix} d_1 & \varepsilon & \varepsilon \\ 2d_1+d_3 & d_2 & 2d_3 \\ 2d_1 & \varepsilon & d_3 \end{bmatrix} \ , \quad B^{(3)} = \begin{bmatrix} 0 \\ d_1+d_3 \\ d_1 \end{bmatrix} \ ,
$$

$$
C^{(3)} = \begin{bmatrix} \varepsilon & d_2 & \varepsilon \end{bmatrix} .
$$

The switching probability from one recipe to the next one is assumed to be given by:

$$
\begin{array}{llllll}
P(1,1) = 0.5 & , & P(1,2) = 0.25 & , & P(1,3) = 0.25 & , \\
P(2,1) = 0.25 & , & P(2,2) = 0.5 & , & P(2,3) = 0.25 & , \\
P(3,1) = 0.25 & , & P(3,2) = 0.25 & , & P(3,3) = 0.5 & ,
\end{array}
$$

which means that if we have a specific recipe in cycle k, then the probability of having the same recipe for cycle $k+1$ is 50%, and the probability of a switching to any other recipe is 25%. Note that this system is indeed an RSMPL system.

The maximum growth rate of the system is equal to $\lambda = 11$. We therefore choose a reference signal given by $r(k) = \rho \cdot k$, where $\rho = 12.1 > \lambda$. The initial state is equal to $x(0) = \begin{bmatrix} 4 & 4 & 4 \end{bmatrix}^T$, and J is given by (13) for $N_\mathrm{p} = 3$, and $\beta = 10^{-5}$.

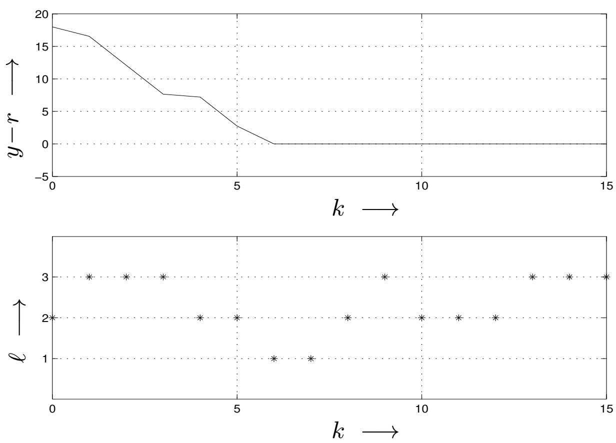

Fig. 2. (a) Tracking error $y(k)-r(k)$ and (b) switching sequence

Figure 2-a gives the tracking error between the reference signal and the output signal $y(k)$, for a switching sequence given in Figure 2-b, when the system is in closed-loop with the receding horizon model predictive controller. It can be observed that $y(k)-r(k)$ is initially larger than zero, which

is due to the initial state. The error decreases very rapidly and for $k \geq 6$ the error is always equal to zero, which means that the the product is always delivered in time.

VII. DISCUSSION

In this paper we have considered the control of randomly switching max-plus-linear systems, a subclass of the discrete event systems, in which we can switch between different modes of operation. In each mode the system is described by max-plus-linear equations with different system matrices for each mode. The moments of switching are determined by a stochastic variable.

We have derived a stabilizing model predictive controller for switching max-plus-linear systems. The resulting optimization problem can be solved using linear programming algorithms.

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