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Max-plus algebra and max-plus linear discrete event systems: An introduction

Bart De Schutter and Ton van den Boom

Abstract— **We provide an introduction to the max-plus algebra and explain how it can be used to model a specific class of discrete event systems with synchronization but no concurrency. Such systems are called max-plus linear discrete event systems because they can be described by a model that is "linear" in the max-plus algebra. We discuss some key properties of the max-plus algebra and indicate how these properties can be used to analyze the behavior of max-plus linear discrete event systems. We also briefly present some control approaches for max-plus linear discrete event systems, including model predictive control. Finally, we discuss some extensions of the max-plus algebra and of max-plus linear systems.**

I. Introduction

In recent years both industry and the academic world have become more and more interested in techniques to model, to analyze, and to control complex discrete event systems (DES) such as flexible manufacturing systems, telecommunication networks, multiprocessor operating systems, railway networks, traffic control systems, logistic systems, intelligent transportation systems, computer networks, multi-level monitoring and control systems, and so on.

Although in general DES lead to a nonlinear description in conventional algebra, there exists a subclass of DES for which this model becomes "linear" when we formulate it in the max-plus algebra [1]–[3], which has maximization and addition as its basic operations. More specifically, DES in which only synchronization and no concurrency or choice occur can be modeled using the operations maximization (corresponding to synchronization: a new operation starts as soon as all preceding operations have been finished) and addition (corresponding to the duration of activities: the finishing time of an operation equals the starting time plus the duration). This leads to a description that is "linear" in the max-plus algebra. Therefore, DES with synchronization but no concurrency are called *max-plus linear DES*. Some examples of max-plus linear DES are production systems, railroad networks, urban traffic networks, queuing systems, and array processors [1]–[3].

In the early sixties the fact that certain classes of DES can be described by max-linear models has been discovered independently by a number of researchers, among whom Cuninghame-Green [4], [5] and Giffler [6]–[8]. An account of the pioneering work of Cuninghame-Green on max-algebraic system theory for DES has been given in [2]. Related work has been done by Gondran and Minoux [9]–[11]. In the late eighties and early the topic attracted new interest due to the research of Cohen, Dubois, Moller, Quadrat, Viot [12]–[14], Olsder [15]–[17], Gaubert [18]–[20], which resulted in the publication of [1]. Since then, several other researchers have entered the field.

The class of DES that can be described by a max-plus linear time-invariant model is only a small subclass of the class of all DES. However, for max-plus linear DES there are many efficient analytic methods available to assess the characteristics and the performance of the system since we can use the properties of the max-plus algebra to analyze max-plus linear models in a very efficient way (as opposed to, e.g., computer simulation where, before we can determine the steady-state behavior of a given DES, we may first have to simulate the transient behavior, which in some cases might require a rather large amount of computation time).

We will see that there exists a remarkable analogy between the basic operations of the max-plus algebra (maximization and addition) on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence, many concepts and properties of conventional algebra also have a max-plus analogue. This analogy also allows us to translate many concepts, properties, and techniques from conventional linear system theory to system theory for max-plus linear DES. However, there are also some major differences that prevent a straightforward translation of properties, concepts, and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-plus linear system theory for DES. Hence, there is a need for a dedicated theory and dedicated methods for max-plus linear DES.

In this paper we give an introduction to the max-plus algebra and to max-plus linear systems. We will highlight the most important properties and analysis methods of the max-plus algebra, discuss some important characteristics of max-plus linear DES, and give a brief overview of performance analysis and control methods for max-plus linear DES. More extensive overviews of the max-plus algebra and max-plus linear systems can be found in [1]–[3], [19].

II. Max-plus algebra

A. Basic operations of the max-plus algebra

The basic operations of the max-plus algebra [1]–[3] are maximization and addition, which will be represented by $\oplus$ and $\otimes$ respectively:

$$x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y$$

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for $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$. The reason for using these symbols is that there is a remarkable analogy between $\oplus$ and conventional addition, and between $\otimes$ and conventional multiplication: many concepts and properties from linear algebra (such as the Cayley-Hamilton theorem, eigenvectors and eigenvalues, Cramer's rule, ...) can be translated to the max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$ [1]–[3], [16], [19]. Therefore, we also call $\oplus$ the max-plus-algebraic addition, and $\otimes$ the max-plus-algebraic multiplication. Note however that one of the major differences between conventional algebra and max-plus algebra is that in general there do not exist inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$. The zero element for $\oplus$ is $\varepsilon \stackrel{\text{def}}{=} -\infty$: we have $a \oplus \varepsilon = a = \varepsilon \oplus a$ for all $a \in \mathbb{R}_\varepsilon$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra.

Let $r \in \mathbb{R}$. The rth max-plus-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra. If $x \in \mathbb{R}$ then $x^{\otimes^0} = 0$ and the inverse element of x w.r.t. $\otimes$ is $x^{\otimes^{-1}} = -x$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r < 0$ then $\varepsilon^{\otimes^r}$ is not defined. In this paper we have $\varepsilon^{\otimes^0} = 0$ by definition.

The rules for the order of evaluation of the max-plus-algebraic operators correspond to those of conventional algebra. So max-plus-algebraic power has the highest priority, and max-plus-algebraic multiplication has a higher priority than max-plus-algebraic addition.

B. Max-plus-algebraic matrix operations

The basic max-plus-algebraic operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$ then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_k (a_{ik} + c_{kj})$$

for all i, j. Note the analogy with the definitions of matrix sum and product in conventional linear algebra.

The matrix $\mathcal{E}_{m \times n}$ is the $m \times n$ max-plus-algebraic zero matrix: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j; and the matrix E_n is the $n \times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. If the size of the max-plus-algebraic identity matrix or the max-plus-algebraic zero matrix is not specified, it should be clear from the context. The max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows: $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ for $k = 1, 2, \ldots$

C. Connection with graph theory

There exists a close relation between max-plus algebra (and related structures) and graphs [1], [9], [21].

Definition 2.1 (Precedence graph): Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. The precedence graph of A, denoted by $\mathcal{G}(A)$, is a weighted directed graph with vertices 1, 2, ..., n and an arc (j, i) with weight a_{ij} for each $a_{ij} \neq \varepsilon$.
It easy to verify that every weighted directed graph corresponds to the precedence graph of an appropriately defined matrix with entries in $\mathbb{R}_\varepsilon$.

Now we can give a graph-theoretic interpretation of the max-plus-algebraic matrix power. Let $A \in \mathbb{R}_\varepsilon^{n \times n}$. If $k \in \mathbb{N}_0$ then we have

$$(A^{\otimes^k})_{ij} = \max_{i_1, i_2, \ldots, i_{k-1}} (a_{ii_1} + a_{i_1 i_2} + \ldots + a_{i_{k-1} j})$$

for all i, j. Hence, $(A^{\otimes^k})_{ij}$ is the maximal weight of all paths of $\mathcal{G}(A)$ of length k that have j as their initial vertex and i as their final vertex — where we assume that if there does not exist a path of length k from j to i, then the maximal weight is equal to ε by definition.

A directed graph $\mathcal{G}$ is called strongly connected if for any two different vertices i, j of the graph, there exists a path from i to j.

Definition 2.2 (Irreducible matrix): A matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is called irreducible if its precedence graph $\mathcal{G}(A)$ is strongly connected.
If we reformulate this in the max-plus algebra then a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is irreducible if

$$(A \oplus A^{\otimes^2} \oplus \ldots \oplus A^{\otimes^{n-1}})_{ij} \neq \varepsilon \quad \text{for all } i, j \text{ with } i \neq j,$$

since this condition means that for two arbitrary vertices i and j of $\mathcal{G}(A)$ with $i \neq j$ there exists at least one path (of length 1, 2, ... or $n-1$) from j to i.

Example 2.3 Consider $A = \begin{bmatrix} 0 & \varepsilon & 2 \\ 2 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$. The precedence graph $\mathcal{G}(A)$ of A is given in Figure 1.

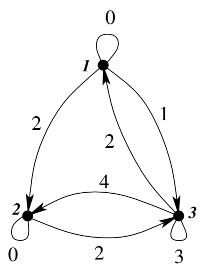

Fig. 1. Precedence graph of the matrix A of Example 2.3. The vertices are indicated in a bold italic font, and the weights are indicated next to the arcs in a regular font.

Clearly, $\mathcal{G}(A)$ is strongly connected, and hence A is irreducible. $\square$

III. Some basic problems in the max-plus algebra

In this section we present some basic max-plus-algebraic problems and some methods to solve them.

A. Systems of max-plus linear equations

Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and $b \in \mathbb{R}_\varepsilon^n$. In general, the system of max-plus linear equations $A \otimes x = b$ will not always have a solution, even if A is square or if it has more columns than rows. Therefore, the concept of subsolution is introduced [1], [2].

Definition 3.1 (Subsolution): Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and $b \in \mathbb{R}_\varepsilon^n$. We say that $x \in \mathbb{R}_\varepsilon^n$ is a *subsolution* of the system of max-plus linear equations $A \otimes x = b$ if $A \otimes x \leqslant b$.

Although the system $A \otimes x = b$ does not always have a solution, it always possible to determine the *largest subsolution* if we allow components that are equal to ∞ in the solution and if we assume that $\varepsilon \otimes \infty = \infty \otimes \varepsilon = \varepsilon$ by definition. For the sake of simplicity and to avoid expressions like $\varepsilon - \varepsilon$, we assume from now on that all the components of b are finite. The largest subsolution $\hat{x}$ of $Ax = b$ is then given by

$$\hat{x}_j = \min_i (b_i - a_{ij}) \qquad \text{for } j = 1,2,\ldots,n.$$

Example 3.2 Consider the matrix A of Example 2.3 and $b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^{\mathrm{T}}$. The system of equations $A \otimes x = b$ does not have a solution. However, the largest subsolution is given by $\hat{x} = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^{\mathrm{T}}$, and we have $A \otimes \hat{x} = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}^{\mathrm{T}} \leqslant b$. □

Note that for the largest subsolution $\hat{x}$ we have $A \otimes \hat{x} \leqslant b$. In some cases, we want to minimize the difference between $A \otimes x$ and b, i.e., find x such that $\max_i |b_i - (A \otimes x)_i|$ is minimized. A solution $\tilde{x}$ of this problem is given by

$$\tilde{x} = \hat{x} \otimes \frac{\delta}{2} \qquad \text{with } \delta = \max_i (b_i - (A \otimes \hat{x})_i) \quad . \tag{1}$$

We then have $\max_i |b_i - (A \otimes \tilde{x})_i| = \frac{\delta}{2}$.

B. Max-plus-algebraic eigenvalue problem

Definition 3.3 (Max-plus-algebraic eigenvalue): Let $A \in \mathbb{R}_\varepsilon^{n \times n}$. If there exist $\lambda \in \mathbb{R}_\varepsilon$ and $v \in \mathbb{R}_\varepsilon^n$ with $v \neq \boldsymbol{\varepsilon}_{n\times 1}$ such that $A \otimes v = \lambda \otimes v$ then we say that λ is a max-plus-algebraic eigenvalue of A and that v is a corresponding max-plus-algebraic eigenvector of A.
It can be shown that every square matrix with entries in $\mathbb{R}_\varepsilon$ has at least one eigenvalue [1]. However, in contrast to linear algebra, the number of max-plus-algebraic eigenvalues of an n by n matrix is in general less than n. Moreover, if a matrix is irreducible, it has is only one eigenvalue (see e.g., [13]).

Example 3.4 Consider the (irreducible) matrix A of Example 2.3. This matrix has one max-plus-algebraic eigenvalue $\lambda = 3$ and a corresponding max-plus-algebraic eigenvector is $v = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}^{\mathrm{T}}$. We have $A \otimes v = \begin{bmatrix} 3 & 5 & 4 \end{bmatrix}^{\mathrm{T}} = 3 \otimes v$. □

There exist several efficient algorithms to determine max-plus-algebraic eigenvalues such as the power algorithm of [22] or the policy iteration algorithm of [23].

We also have the following property [1], [13], [24]:

Theorem 3.5: If $A \in \mathbb{R}_\varepsilon$ is irreducible, then

$$\exists k_0 \in \mathbb{N}, \exists c \in \mathbb{N}_0 \text{ such that } \forall k \geqslant k_0 : A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$$

where λ is the (unique) max-plus-algebraic eigenvalue of A. In the case where A is not irreducible the behavior of $A^{\otimes^k}$ for k is more complex (see, e.g., [1], [3], [25]).

Example 3.6 For the matrix A of Example 2.3 we have

$$A = \begin{bmatrix} 0 & \varepsilon & 2 \\ 2 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad A^{\otimes^2} = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 4 & 5 & 6 \end{bmatrix},$$

$$A^{\otimes^3} = \begin{bmatrix} 6 & 7 & 8 \\ 8 & 9 & 10 \\ 7 & 8 & 9 \end{bmatrix}, \quad A^{\otimes^4} = \begin{bmatrix} 9 & 10 & 11 \\ 11 & 12 & 13 \\ 10 & 11 & 12 \end{bmatrix}, \ldots$$

So $A^{\otimes^{k+1}} = 3 \otimes A^{\otimes^k}$ for $k = 2,3,\ldots$ □

C. Systems of max-plus-algebraic multivariate polynomial equalities and inequalities

A *system of multivariate polynomial equalities and inequalities in the max-plus algebra* is defined as follows:

> Given a set of integers $\{m_k\}$ and sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1,\ldots,m_k\}$, $j \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,p_1,p_1+1,\ldots,p_1+p_2\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \quad \text{for } k = 1,2,\ldots,p_1,$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \quad \text{for } k = p_1+1,\ldots,p_1+p_2.$$

Note that the exponents can be negative or real. In conventional algebra the equations can be written as

$$\max_{i=1,\ldots,m_k} \Big(a_{ki} + \sum_{j=1}^{n} c_{kij} x_j\Big) = b_k \quad \text{for } k = 1,2,\ldots,p_1,$$

$$\max_{i=1,\ldots,m_k} \Big(a_{ki} + \sum_{j=1}^{n} c_{kij} x_j\Big) \leqslant b_k \quad \text{for } k = p_1+1,\ldots,p_1+p_2.$$

In [26]–[28] it has been shown that the above problem and related max-plus problems such as computing max-plus matrix decompositions, transformation of max-plus linear state space models, state space realization of max-plus linear systems, construction of matrices with a given max-plus characteristic polynomial, and solving systems of max-min-plus equations can be recast as a so-called extended linear complementarity problem (ELCP), which is defined as follows:

> Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1,2,\ldots,p\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{2}$$
>
> subject to $Ax \geqslant c$ and $Bx = d$.

Algorithms for solving ELCPs can be found in [29] (to compute the entire solution set) and in [30] (to find one solution only).

IV. Max-plus linear systems

A. Max-plus linear state space models

DES with only synchronization and no concurrency can be modeled by a max-plus-algebraic model of the following form [1]–[3]:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \tag{3}$$

$$y(k) = C \otimes x(k) \tag{4}$$

with $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times n}$ where m is the number of inputs and l the number of outputs. The vector x represents the state, u is the input vector, and y is the output vector of the system. It is important to note that in (3)–(4) the components of the input, the output, and the state are event times, and that the counter k in (3)–(4) is an event counter. For a manufacturing system, $u(k)$ would typically represent the time instants at which raw material is fed to the system for the kth time, $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products, and $y(k)$ the time instants at which the kth batch of finished products leaves the system.

Due to the analogy with conventional linear time-invariant systems, a DES that can be modeled by (3)–(4) will be called a max-plus linear time-invariant DES system.

Typical examples of systems that can be modeled as max-plus linear DES are production systems, railroad networks, urban traffic networks, and queuing systems. We will now illustrate in detail how the behavior of a simple manufacturing system can be described by a max-plus linear model of the form (3)–(4).

B. Example: A simple production system

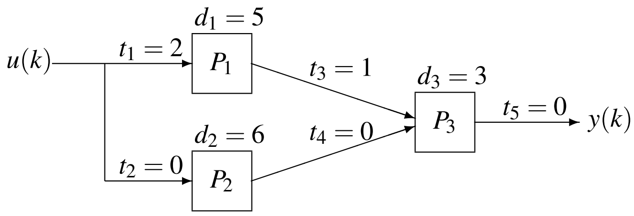

Fig. 2. A simple production system.

Consider the system of Figure 2. This production system consists of 3 processing units: P_1, P_2, and P_3. Raw material is fed to P_1 and P_2, processed, and sent to P_3 where assembly takes place. The processing times for P_1, P_2, and P_3 are respectively $d_1 = 5$, $d_2 = 6$, and $d_3 = 3$ time units. We assume that it takes $t_1 = 2$ time units for the raw material to get from the input source to P_1 and that it takes $t_3 = 1$ time unit for the finished products of processing unit P_1 to reach P_3. The other transportation times (t_2, t_4, and t_5) are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Initially all buffers are empty and none of the processing units contains raw material or intermediate products.

A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available. Define

- $u(k)$: time instant at which raw material is fed to the system for the kth time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y(k)$: time instant at which the kth finished product leaves the system.

Let us now determine the time instant at which processing unit P_1 starts working for the kth time. If we feed raw material to the system for the kth time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + 2$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the previous, i.e., the $(k-1)$st, batch. Since the processing time on P_1 is $d_1 = 5$ time units, the $(k-1)$st intermediate product will leave P_1 at time $t = x_1(k-1) + 5$. Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$x_1(k) = \max(x_1(k-1) + 5, u(k) + 2) \tag{5}$$

for $k = 1, 2, \ldots$ The condition that initially processing unit P_1 is empty and idle corresponds to the initial condition $x_1(0) = \varepsilon$ since then it follows from (5) that $x_1(1) = u(1) + 2$, i.e., the first batch of raw material that is fed to the system will be processed immediately (after a delay of 2 time units needed to transport the raw material from the input to P_1).

Using a similar reasoning we find the following expressions for the time instants at which P_2 and P_3 start working for the $(k+1)$st time and for the time instant at which the kth finished product leaves the system:

$$\begin{aligned}
x_2(k) &= \max(x_2(k-1) + 6, u(k) + 0) && (6)\\
x_3(k) &= \max(x_1(k) + 5 + 1, x_2(k) + 6 + 0, x_3(k-1) + 3)\\
&= \max(x_1(k-1) + 11, x_2(k-1) + 12,\\
&\qquad x_3(k-1) + 3, u(k) + 8) && (7)\\
y(k) &= x_3(k) + 3 + 0 && (8)
\end{aligned}$$

for $k = 1, 2, \ldots$ The condition that initially all buffers are empty corresponds to the initial condition $x_1(0) = x_2(0) = x_3(0) = \varepsilon$.

Let us now rewrite the evolution equations of the production system using the symbols $\oplus$ and $\otimes$. It is easy to verify that (5) can be rewritten as

$$x_1(k) = 5 \otimes x_1(k-1) \oplus 2 \otimes u(k) \ .$$

If we also do this for (6)–(8) and if we rewrite the resulting equations in max-plus-algebraic matrix notation, we obtain

$$\begin{aligned}
x(k) &= \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix} \otimes x(k-1) \oplus \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \otimes u(k)\\
y(k) &= \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} \otimes x(k)
\end{aligned}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^{\mathrm{T}}$. Note that this is a model of the form (3)–(4).

In the next section we shall use this production system to illustrate some of the max-plus-algebraic techniques that can be used to analyze max-plus linear time-invariant DES.

V. Performance analysis and control of max-plus linear systems

A. Analysis of max-plus linear systems

Now we present some analysis techniques for DES that can be described by a model of the form (3)–(4).

First we determine the input-output behavior of the DES. We have

$$\begin{aligned} x(1) &= A \otimes x(0) \oplus B \otimes u(1) \\ x(2) &= A \otimes x(1) \oplus B \otimes u(2) \\ &= A^{\otimes^2} \otimes x(0) \oplus A \otimes B \otimes u(1) \oplus B \otimes u(2) \end{aligned}$$

etc., which yields $x(k) = A^{\otimes^k} \otimes x(0) \oplus \bigoplus_{i=1}^{k} A^{\otimes^{k-i}} \otimes B \otimes u(i)$ for $k = 1, 2, \ldots$ Hence,

$$y(k) = C \otimes A^{\otimes^k} \otimes x(0) \oplus \bigoplus_{i=1}^{k} C \otimes A^{\otimes^{k-i}} \otimes B \otimes u(i) \quad (9)$$

for $k = 1, 2, \ldots$

Consider two input sequences $u_1 = \{u_1(k)\}_{k=1}^{\infty}$ and $u_2 = \{u_2(k)\}_{k=1}^{\infty}$. Let $y_1 = \{y_1(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence u_1 (with initial condition $x_1(0) = x_{1,0}$) and let $y_2 = \{y_2(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence u_2 (with initial condition $x_2(0) = x_{2,0}$). Let $\alpha, \beta \in \mathbb{R}_{\varepsilon}$. From (9) it follows that the output sequence that corresponds to the input sequence $\alpha \otimes u_1 \oplus \beta \otimes u_2 = \{\alpha \otimes u_1(k) \oplus \beta \otimes u_2(k)\}_{k=1}^{\infty}$ (with initial condition $\alpha \otimes x_{1,0} \oplus \beta \otimes x_{2,0}$) is given by $\alpha \otimes y_1 \oplus \beta \otimes y_2$. This explains why DES that can be described by a model of the form (3)–(4) are called *max-plus linear*.

Now we assume that $x(0) = \varepsilon_{n \times 1}$. For the simple production system of Section IV-B this would mean that all the buffers are empty at the beginning. Let $p \in \mathbb{N}_0$. If we define $Y = \begin{bmatrix} y(1) & y(2) & \ldots & y(p) \end{bmatrix}^{\mathrm{T}}$ and $U = \begin{bmatrix} u(1) & u(2) & \ldots & u(p) \end{bmatrix}^{\mathrm{T}}$, then (9) results in

$$Y = H \otimes U \quad (10)$$

with

$$H = \begin{bmatrix} C \otimes B & \varepsilon & \ldots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \ldots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{p-1}} \otimes B & C \otimes A^{\otimes^{p-2}} \otimes B & \ldots & C \otimes B \end{bmatrix}.$$

For the production system of Section IV-B this means that if we know the time instants at which raw material is fed to the system, we can compute the time instants at which the finished products will leave the system.

Example 5.1 Consider the production system of Section IV-B. Define $Y = \begin{bmatrix} y(1) & y(2) & y(3) & y(4) \end{bmatrix}^{\mathrm{T}}$ and $U = \begin{bmatrix} u(1) & u(2) & u(3) & u(4) \end{bmatrix}^{\mathrm{T}}$. If $x(0) = \varepsilon_{3 \times 1}$ then we have $Y = H \otimes U$ with

$$H = \begin{bmatrix} 11 & \varepsilon & \varepsilon & \varepsilon \\ 16 & 11 & \varepsilon & \varepsilon \\ 21 & 16 & 11 & \varepsilon \\ 27 & 21 & 16 & 11 \end{bmatrix}.$$

If we feed raw material to the system at time instants $u(1) = 0$, $u(2) = 9$, $u(3) = 12$, $u(4) = 15$, the finished products will leave the system at time instants $y(1) = 11$, $y(2) = 20$, $y(3) = 25$, and $y(4) = 30$ since

$$H \otimes \begin{bmatrix} 0 \\ 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 11 \\ 20 \\ 25 \\ 30 \end{bmatrix}.$$

□

Now we consider the autonomous DES described by

$$\begin{aligned} x(k+1) &= A \otimes x(k) \\ y(k) &= C \otimes x(k) \end{aligned}$$

with $x(0) = x_0$. For the production system of Section IV-B this would mean that we start from a situation in which some internal buffers and all the input buffer are not empty at the beginning (if $x_0 \neq \varepsilon_{n \times 1}$) and that afterwards the raw material is fed to the system at such a rate that the input buffers never become empty.

If the system matrix A of the autonomous DES is irreducible, then we can compute the "ultimate" behavior of the autonomous DES by solving the max-plus-algebraic eigenvalue problem $A \otimes v = \lambda \otimes v$. By Theorem 3.5 there exist integers $k_0 \in \mathbb{N}$ and $c \in \mathbb{N}_0$ such that $x(k+c) = \lambda^{\otimes^c} \otimes x(k)$ for all $k \geqslant k_0$. This means that

$$x_i(k+c) - x_i(k) = c\lambda \quad (11)$$

for $i = 1, 2, \ldots, n$ and for all $k \geqslant k_0$. From this relation it follows that for a production system the average duration of a cycle of the production process when the system has reached its cyclic behavior will be given by λ. The average production rate will then be equal to $\frac{1}{\lambda}$. This also enables us to calculate the utilization levels of the various machines in the production process. Furthermore, some algorithms to compute the eigenvalue also yield the critical paths of the production process and the bottleneck machines [13].

Example 5.2 The system matrix A of the production system of Section IV-B is not irreducible and it does not lead to a behavior of the form (11). In fact, it can be verified that A has three eigenvalues $\lambda_1 = 3$, $\lambda_2 = 5$, and $\lambda_3 = 6$, with corresponding eigenvectors

$$v_1 = \begin{bmatrix} \varepsilon \\ \varepsilon \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ \varepsilon \\ 6 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} \varepsilon \\ 0 \\ 6 \end{bmatrix}.$$

□

B. *Control of max-plus linear DES*

1) Residuation-based control: The basic control problem for max-plus linear DES consists in determining the optimal feeding times of raw material to the system and/or the optimal starting times of the (internal) processes.

Consider (10). If we know the vector Y of latest times at which the finished products have to leave the system, we can compute the vector U of (latest) time instants at which raw material has to be fed to the system by solving the system of max-plus linear equations $H \otimes U = Y$, if this system has a solution, or by determining the largest subsolution of $H \otimes U = Y$, i.e., determining the largest U such that $H \otimes U \leqslant Y$. This approach is also called residuation [31].

The residuation-based approach for computing the optimal feeding times is used in [32], [33]. Note that the sequence $u(1), u(2), \ldots, u(p)$ should be non-decreasing as it corresponds to a sequence of consecutive feeding times. However, a residuation-based solution does not always satisfy this property. This problem can be overcome by projection on the set of non-increasing sequences [34].

Note that the method above corresponds to just-in-time production. However, if we have perishable goods it is sometimes better to minimize the maximal deviation between the desired and the actual finishing times. In this case we have to solve the problem $\min_U \max_i |(Y - H \otimes U)_i|$. This problem can be solved using formula (1).

Example 5.3 Let us again consider the production system of Section IV-B and the matrix H and the vectors U and Y as defined in Example 5.1. If the finished parts should leave the system before time instants 17, 19, 24, and 27 and if we want to feed the raw material to the system as late as possible, then we should feed raw material to the system at time instants 0, 6, 11, 16 since the largest subsolution of $H \otimes U = \begin{bmatrix} 17 & 19 & 24 & 27 \end{bmatrix}^{\mathrm{T}}$ is $\hat{U} = \begin{bmatrix} 0 & 6 & 11 & 16 \end{bmatrix}^{\mathrm{T}}$. The actual output times $\hat{Y}$ are given by $\hat{Y} = H \otimes \hat{U} = \begin{bmatrix} 11 & 17 & 22 & 27 \end{bmatrix}^{\mathrm{T}}$. Note that the largest deviation δ between the desired and the actual output times is equal to 6. The input times that minimize this deviation are given by $\tilde{U} = \hat{U} \otimes \dfrac{\delta}{2} = \hat{U} \otimes 3 = \begin{bmatrix} 3 & 9 & 14 & 19 \end{bmatrix}^{\mathrm{T}}$. The corresponding output times are given by $\tilde{Y} = \begin{bmatrix} 14 & 20 & 25 & 30 \end{bmatrix}^{\mathrm{T}}$. Note that the largest deviation between the desired finishing and the actual finishing times is now equal to $\frac{\delta}{2} = 3$. □

2) Model predictive control: A somewhat more advanced control approach for max-plus linear DES has been developed in [35]. This approach is an extension to max-plus linear DES of the model-based predictive control approach called Model Predictive Control (MPC) [36], [37] that has originally been developed for time-driven systems.

In MPC for max-plus linear DES at each event step k the controller computes the input sequence that optimizes a performance criterion J over the next N_{p} event steps, where N_{p} is called the prediction horizon, subject to various constraints on the inputs, states, and outputs of the system. Typically, the performance criterion aims at minimizing the difference or the tardiness with respect to a due date signal, while at the same time making the inputs as large as possible (just-in-time production). This results in an optimization problem that has to be solved at each event step k. In order to reduce the computational complexity, often a control horizon N_{c} is introduced with $N_{\mathrm{c}} < N_{\mathrm{p}}$ and it is assumed that the input rate is constant after event step $k + N_{\mathrm{c}}$.

MPC uses a receding horizon approach. This means that once the optimal input sequence has been determined only the input for the first event step is applied to the system. Next, at event step $k+1$ the new state of the system is determined or measured[1], the horizon is shifted, and the whole process is repeated again. This receding horizon approach introduces a feedback mechanism, which allows to reduce the effects of possible disturbances and model mismatch errors.

In [35] it has been shown that for a broad range of performance criteria and constraints, max-plus linear MPC results in a linear programming problem, which can be solved very efficiently. Worked examples of MPC for max-plus linear systems and related results can be found in [35], [38]–[41].

VI. Summary

We have presented an overview of the basic notions of the max-plus algebra and max-plus linear discrete event systems (DES). We have introduced the basic operations of the max-plus algebra and stated some of the main definitions, theorems, and properties of the max-plus algebra. Next, we have given an introduction to max-plus linear DES, and presented some elementary analysis and control methods for max-plus linear DES.

For more information on the analysis of max-plus linear time-invariant DES such as production systems, timetable dependent transportation networks, queuing systems, array processors, and so on the interested reader is referred to [1]–[3], [13], [14], [16], [17], [19], [32], [34], [40], [42]–[49] and the references therein.

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