

Technical report 08-008

Randomly switching max-plus linear systems and equivalent classes of discrete event systems*

T. van den Boom and B. De Schutter

If you want to cite this report, please use the following reference instead:

T. van den Boom and B. De Schutter, “Randomly switching max-plus linear systems and equivalent classes of discrete event systems,” *Proceedings of the 9th International Workshop on Discrete Event Systems (WODES’08)*, Göteborg, Sweden, pp. 242–247, May 2008.

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: <https://www.dcsc.tudelft.nl>

* This report can also be downloaded via https://pub.bartdeschutter.org/abs/08_008.html

Randomly switching max-plus linear systems and equivalent classes of discrete event systems

Ton van den Boom and Bart De Schutter

Abstract—In switching max-plus-linear discrete event systems we can switch between different modes of operation. In each mode the discrete event system is described by a max-plus-linear state space model with different system matrices for each mode. In randomly switching max-plus-linear systems, the switching between the modes can be both deterministic and stochastic. The switching changes the structure of the system, which allows the system to break synchronization and to change the order of events. In this paper two equivalent descriptions for randomly switching max-plus-linear systems will be discussed. Furthermore, we will show that a randomly switching max-plus-linear system can be written as a piecewise affine system or a max-min-plus-scaling system. The last transformation can be established under (rather mild) additional assumptions on the boundedness of state and input.

I. INTRODUCTION

The class of discrete event systems (DES) essentially consists of man-made systems that contain a finite number of resources (such as machines, communications channels, or processors) that are shared by several users (such as product types, information packets, or jobs) all of which contribute to the achievement of some common goal (the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation) [1].

In this paper we consider switching max-plus-linear (SMPL) systems, discrete event systems that can switch between different modes of operation, in which the mode switching depends on a stochastic sequence or on the input and previous state. In each mode the system is described by a max-plus-linear state equation and a max-plus-linear output equation, with different system matrices for each mode. In [8] we have discussed SMPL systems with *deterministic* switching, and in [9], [10] we have discussed SMPL systems with *random* switching.

The class of SMPL systems contains discrete event systems with synchronization but no concurrency, in which the order of synchronization of the event steps may vary randomly, or is determined by input signals or the previous state.

This work was partially funded by the Dutch Technology Foundation STW project “Multi-agent control of large-scale hybrid systems” (DWV.6188), by the European 6th Framework Network of Excellence “HYbrid CONtrol: Taming Heterogeneity and Complexity of Networked Embedded Systems (HYCON)” (FP6-IST-511368), by the Delft Research Center Next Generation Infrastructure, and by the Transport Research Center Delft.

Ton van den Boom and Bart De Schutter are with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands a.j.j.vandenboom@tudelft.nl, b@deschutter.info

Bart De Schutter is also with the Maritime & Transport Technology Group, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands

Typical examples of SMPL systems are flexible manufacturing systems, telecommunication networks, logistic networks, and signal controlled urban traffic networks.

Mode switching depending on input signals allows us to model a change in the structure of the system, such as breaking a synchronization or changing the order of events. Mode switching depending on the state may be due to concurrency between various events (see [8]). Random mode switching between may be due to e.g. (randomly) changing production recipes, varying customer demands or traffic demands, or failures in production units, transmission lines, or traffic links.

In this paper we review the type 1 RSMPL system description introduced in [9], [10] and we introduce a type 2 RSMPL system description. We show that these two RSMPL systems descriptions are in fact equivalent. Furthermore, we show that a type 2 RSMPL system can be rewritten as a piecewise affine (PWA) system or a max-min-plus-scaling (MMPS) system. Using the results of [5] we also implicitly prove that a RSMPL system can be rewritten as an (extended) linear complementarity (ELC/LC) system or a mixed logic dynamical (MLD) system, which are well-known system descriptions used in the field of hybrid systems. Using these results we can transfer properties of and methods for PWA systems and MMPS systems to RSMPL systems.

II. SWITCHING MAX-PLUS LINEAR SYSTEMS

Max-plus algebra

We start this section by giving the some basic definitions in the max-plus algebra [1], [4], [6].

Define $\mathcal{E} = -\infty$ and $\mathbb{R}_{\mathcal{E}} = \mathbb{R} \cup \{\mathcal{E}\}$. The max-plus-algebraic addition (\oplus) and multiplication (\otimes) are defined as follows:

$$x \oplus y = \max(x, y) \quad , \quad x \otimes y = x + y$$

for numbers $x, y \in \mathbb{R}_{\mathcal{E}}$ and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^n a_{ik} \otimes c_{kj} = \max_{k=1, \dots, n} (a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_{\mathcal{E}}^{m \times n}$ and $C \in \mathbb{R}_{\mathcal{E}}^{n \times p}$.

Randomly switching Max-Plus-Linear systems

Switching Max-Plus-Linear (SMPL) systems are discrete event systems that can switch between different modes of operation [8]. In each mode $\ell \in \{1, \dots, L\}$, the system is

described by a max-plus-linear state equation and a max-plus-linear output equation:

$$x(k) = A^{(\ell(k))} \otimes x(k-1) \oplus B^{(\ell(k))} \otimes u(k) \quad (1)$$

$$y(k) = C^{(\ell(k))} \otimes x(k) \quad (2)$$

in which the matrices $A^{(\ell)} \in \mathbb{R}_{\mathcal{E}}^{n_x \times n_x}$, $B^{(\ell)} \in \mathbb{R}_{\mathcal{E}}^{n_x \times n_u}$, $C^{(\ell)} \in \mathbb{R}_{\mathcal{E}}^{n_y \times n_x}$ are the system matrices for the ℓ -th mode. The index k is called the event counter. For discrete event systems the state $x(k)$ typically contains the time instants at which the internal events occur for the k th time, the input $u(k)$ contains the time instants at which the input events occur for the k th time, and the output $y(k)$ contains the time instants at which the output events occur for the k th time.

In [8] we have considered deterministic switching, that was a function of the previous state or an input signal. In [9] we have introduced random switching, i.e. the mode switching depended on a stochastic sequence. In [10] we have combined both switching types. For the SMPL system (1)-(2), the mode switching variable $\ell(k)$ then depends on both stochastic variables as well as deterministic variables (state and inputs). The switching times are determined by a switching mechanism. The mode switching variable $\ell(k)$ is a stochastic process, and the probability of switching depends on the previous mode $\ell(k-1)$, the previous state $x(k-1)$, the input variable $u(k)$, and an (additional) control vector $v(k) \in \mathbb{R}^{n_v}$.

Definition 1 ([9], [10]): A type 1 Randomly Switching Max-Plus-Linear (RSMPL) system is defined as follows: Consider system (1)-(2) with L possible modes. The probability of switching to mode $\ell(k)$ given $\ell(k-1)$, $x(k-1)$, $u(k)$, $v(k)$ is denoted by $P(\ell(k)|\ell(k-1), x(k-1), u(k), v(k))$. For any given $\ell(k) \in \{1, \dots, L\}$, the probability P is piecewise affine on polyhedral sets in the variables $\ell(k-1)$, $x(k-1)$, $u(k)$, $v(k)$. \diamond

Define $w(k) = [\ell(k-1) \ x^T(k-1) \ u^T(k) \ v^T(k)]^T$, then for any $m \in \{1, \dots, L\}$ there exist matrices $S_{i,m}$, vectors $\alpha_{i,m}$, $s_{i,m}$ and scalars $\beta_{i,m}$ such that the probability¹ P can be written as

$$P(\ell(k)|w(k)) = \alpha_{i,\ell(k)}^T w(k) + \beta_{i,\ell(k)},$$

if $w(k) \in \Gamma_{i,\ell(k)}$ for $i = 1, \dots, n_{\ell(k)}$

where $\Gamma_{i,\ell(k)} = \{w(k) \mid S_{i,\ell(k)} w(k) \leq s_{i,\ell(k)}\}$, and the sets $\Gamma_{i,\ell(k)}$ are such that

$$\bigcup_{i=1}^{n_m} \Gamma_{i,m} = \mathbb{R}^{n_w} \quad \text{and} \quad \text{int}(\Gamma_{i,m}) \cap \text{int}(\Gamma_{j,m}) = \emptyset \quad \text{for } i \neq j.$$

where $\text{int}(\cdot)$ denotes the interior.

Definition 2: A type 2 Randomly Switching Max-Plus-Linear (RSMPL) system is defined as follows: Consider system (1)-(2) with L possible modes. The mode $\ell(k) = m$ if

$$z(k) = [\ell(k-1) \ x^T(k-1) \ u^T(k) \ v^T(k) \ d(k)]^T \in \Omega_m$$

¹Note that P is a probability, so obviously for any $w(k)$ we find $0 \leq P(m|w(k)) \leq 1$, $m = 1, \dots, L$ and $\sum_{m=1}^L P(m|w(k)) = 1$.

where $d(k) \in [0, 1]$ is a uniformly distributed scalar signal, and where $\Omega_m = \bigcup_{j=1}^{n_m} \Omega_{m,j}$ in which $\Omega_{m,j}$ are closed convex polyhedra (i.e. given by a finite number of linear inequalities) with non-overlapping interiors in the variable $z(k)$:

$$\Omega_{m,j} = \{z(k) \mid R_{m,j} z(k) \leq r_{m,j}\}, \quad \text{for } j = 1, \dots, n_m \quad \diamond$$

In the next section we will show that the two types of RSMPL systems are equivalent and therefore model the same class of discrete event systems. A type 1 RSMPL model is easy for modeling where we consider the probabilities of switching from one mode to another. The model gives a lot of physical insight into the system and is usually more intuitive for the user. A type 2 RSMPL model is signal based and the properties of probabilities of switching are translated into the properties of a stochastic signal. The fact that type 2 RSMPL models are signal based makes that this type of RSMPL system can easily be translated into another model in one of the well-known classes of hybrid systems (see Section IV).

III. EQUIVALENCE IN CLASSES OF RSMPL SYSTEMS

Proposition 3: Every type 1 RSMPL system can be written as a type 2 RSMPL system. \diamond

Proof: Consider a type 1 RSMPL system, let $w(k)$ be fixed and let (i_1, i_2, \dots, i_L) be such that $w(k) \in \Gamma_{i_j, j}$ for $j = 1, \dots, L$. Now define the function

$$\eta(m, w(k)) = \begin{cases} \sum_{j=1}^m \alpha_{i_j, j}^T w(k) + \beta_{i_j, j} & \text{for } m = 1, \dots, L \\ 0 & \text{for } m = 0 \end{cases}$$

Note that for varying $w(k)$ the function $\eta(m, w(k))$ is a sum of piecewise affine functions in the variable $w(k)$, and therefore is piecewise affine in $w(k)$ itself. Now define a system (1)-(2) and a uniformly distributed scalar signal $d(k) \in [0, 1]$, and let

$$\ell(k) = m \quad \text{if} \quad \eta(m-1, w(k)) \leq d(k) \leq \eta(m, w(k))$$

This means that the probability that $\ell(k) = m$ is equal to:

$$\begin{aligned} P(m|w(k)) &= P(\eta(m-1, w(k)) \leq d(k) \leq \eta(m, w(k))) \\ &= \eta(m, w(k)) - \eta(m-1, w(k)) \\ &= \alpha_{i_m, m}^T w(k) + \beta_{i_m, m}. \end{aligned}$$

like in a type 1 RSMPL system. So we obtain

$$\ell(k) = m \quad \text{if} \quad \begin{bmatrix} -d(k) + \eta(m-1, w(k)) \\ d(k) - \eta(m, w(k)) \end{bmatrix} \leq 0$$

together with the constraints $S_{i_j, j} w(k) \leq s_{i_j, j}$ for $j = 1, \dots, L$. Define $z(k) = [w^T(k) \ d(k)]^T$ and the function

$$\xi(m, z(k)) = \begin{bmatrix} -d(k) + \eta(m-1, w(k)) \\ d(k) - \eta(m, w(k)) \\ S_{i_1, 1} w(k) - s_{i_1, 1} \\ \vdots \\ S_{i_L, L} w(k) - s_{i_L, L} \end{bmatrix},$$

Define the set $\Phi = \{\phi_1, \dots, \phi_{n_t}\} = \{\phi_t = (i_1, i_2, \dots, i_L) | S_{i_j, j} w(k) \leq s_{i_j, j} \text{ for } j = 1, \dots, L\}$ of all possible permutations, such that $S_{i_j, j} w(k) \leq s_{i_j, j}$ for $j = 1, \dots, L$. Now there exist matrices $R_{m,t}$ and scalars $r_{m,t}$ such that the function $\xi(m, z(k))$ for the corresponding ϕ_t can be written as $\xi(m, z(k)) = R_{m,t} z(k) - r_{m,t}$

This means that

$$\ell(k) = m \text{ if } R_{m,t} z(k) \leq r_{m,t}, t = 1, \dots, n_t,$$

which is equal to a type 2 RSMPL system. ■

Proposition 4: Every type 2 RSMPL system can be written as a type 1 RSMPL system. ◇

Proof: Consider a type 2 RSMPL system, so

$$\ell(k) = m \text{ if } R_{m,t} z(k) \leq r_{m,t}$$

Define $R_{m,t,1}$ and $R_{m,t,2}$ such that

$$R_{m,t} z(k) = R_{m,t,1} w(k) + R_{m,t,2} d(k) \leq r_{m,t} \quad (3)$$

Note that $d(k) \in [0, 1]$ is a scalar. Let $d_{m,t,\max}(w(k))$ be the maximum value d such that (3) is satisfied, and let $d_{m,t,\min}(w(k))$ be the minimum value d such that (3) is satisfied. If for some $w(k)$ there exists no $d(k) \in [0, 1]$ such that (3) is satisfied, we define $d_{m,t,\max}(w(k)) = d_{m,t,\min}(w(k)) = 0$. Finding $d_{m,t,\max}(w(k))$ and $d_{m,t,\min}(w(k))$ can be done using a linear programming algorithm, which means that $d_{m,t,\max}(w(k))$ and $d_{m,t,\min}(w(k))$ are piecewise affine in $w(k)$ ([3]). So there exist vectors $p_{m,t,i,\max}$ and $p_{m,t,i,\min}$ and scalars $q_{m,t,i,\max}$ and $q_{m,t,i,\min}$, such that

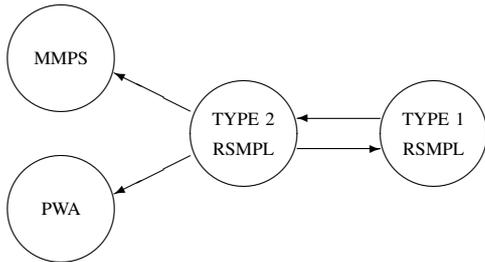
$$\begin{aligned} d_{m,t,\max}(w(k)) &= p_{m,t,i,\max}^T w(k) + q_{m,t,i,\max} \\ d_{m,t,\min}(w(k)) &= p_{m,t,i,\min}^T w(k) + q_{m,t,i,\min} \end{aligned}$$

if $S_{i,m,t} w(k) \leq s_{i,m,t}$ for $i = 1, \dots, M_m$, $t = 1, \dots, n_t$.

The probability that $\ell(k) = m$, given $w(k)$ can be written as

$$\begin{aligned} P(m|w(k)) &= P(d_{m,t,\min} \leq d(k) \leq d_{m,t,\max}) \\ &= d_{m,t,\max} - d_{m,t,\min} \\ &= (p_{m,t,i,\max}^T - p_{m,t,i,\min}^T) w(k) \\ &\quad + (q_{m,t,i,\max} - q_{m,t,i,\min}) \end{aligned}$$

for $S_{i,m,t} w(k) \leq s_{i,m,t}$ for $i = 1, \dots, M_m$, $t = 1, \dots, n_t$, which is a type 1 RSMPL system. ■



Proposition 5: A type 2 RSMPL system can always be rewritten in the compact form:

$$x(k) = \bar{A}^{(\kappa(k))} \otimes x(k-1) \oplus \bar{B}^{(\kappa(k))} \otimes u(k) \quad (4)$$

$$y(k) = \bar{C}^{(\kappa(k))} \otimes x(k) \quad (5)$$

The mode $\kappa(k) = m$ if

$$z(k) = [\kappa(k-1) \quad x^T(k-1) \quad u^T(k) \quad v^T(k) \quad d(k)]^T \in \bar{\Omega}_m$$

where $d(k) \in [0, 1]$ is a uniformly distributed scalar signal, and where $\bar{\Omega}_m$ are polyhedra (i.e. given by a finite number of linear inequalities) with non-overlapping interior in the variable $z(k)$:

$$\bar{\Omega}_m = \{ z(k) | R_m z(k) \leq r_m \}$$

Proof: Consider the type 2 RSMPL system of definition 2, and introduce a new numbering

$$\kappa = j + \sum_{i=1}^{\ell-1} n_i, \text{ for } \ell = 1, \dots, L \text{ and } j = 1, \dots, n_\ell$$

then we find

$$\bar{\Omega}_\kappa = \Omega_{\ell,j}$$

and

$$[\bar{A}^\kappa \quad \bar{B}^\kappa \quad \bar{C}^\kappa] = [A^{(\ell(k))} \quad B^{(\ell(k))} \quad C^{(\ell(k))}]$$

Definition 6: An RSMPL system is bounded if for any bounded $(x(k-1), u(k))$ we have that $x(k)$ and $y(k)$ are bounded for all $\ell(k-1) \in \{1, \dots, L\}$ and $d(k) \in [0, 1]$. ◇

Corollary 7: For a bounded RSMPL system the matrix

$$M^{(\ell)} = \begin{bmatrix} A^{(\ell)} & B^{(\ell)} & \mathcal{E} \\ \mathcal{E} & \mathcal{E} & C^{(\ell)} \end{bmatrix}$$

is row-finite for all $\ell = 1, \dots, L$ (i.e. for all ℓ every row of the matrix $M^{(\ell)}$ has at least one finite entry).

IV. RSMPL SYSTEMS AND EQUIVALENT SYSTEM DESCRIPTIONS

The following definition is an extension of [7]:

Definition 8: Piecewise Affine (PWA) systems are described by

$$\begin{aligned} x(k) &= A_i x(k-1) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \text{ for } \begin{bmatrix} x(k-1) \\ u(k) \\ d(k) \end{bmatrix} \in \Omega_i, \quad (6)$$

for $i = 1, \dots, N$ where $\Omega_1, \dots, \Omega_N$ are closed polyhedra (i.e. given by a finite number of linear inequalities) with non-overlapping interiors in the variables $x(k-1)$, $u(k)$, and $d(k)$. The signal $d(k) \in [0, 1]$ is a uniformly distributed scalar signal. ◇

Proposition 9: Every bounded RSMPL system of type 2 can be written as a piecewise affine (PWA) system. \diamond

Proof: Consider the bounded RSMPL system of type 2 with state and output equations (1) and (2) where $\ell(k) = i$ if $z(k)$ satisfies $R_i z(k) \leq r_i$. Define

$$\omega(k) = \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}, \quad \eta(k) = \begin{bmatrix} x(k-1) \\ u(k) \\ x(k) \end{bmatrix} \quad (7)$$

$$M^{(\ell)} = \begin{bmatrix} A^{(\ell)} & B^{(\ell)} & \mathcal{E} \\ \mathcal{E} & \mathcal{E} & C^{(\ell)} \end{bmatrix} \quad (8)$$

then (1) and (2) can be written as

$$\omega(k) = M^{(\ell)}(k) \otimes \eta(k) \quad \text{for } R_\ell z(k) \leq r_\ell$$

The RSMPL system is well-defined, which means that if η is bounded, then ω will be bounded.

Define the set Φ_ℓ of all elements $\phi_t = (\ell_t, p_{1,t}, p_{2,t}, \dots, p_{n,t})$, $t = 1, \dots, n_t$, where $\ell_t \in \{0, \dots, L\}$ and $p_{i,t} \in \{1, \dots, n\}$ such that there exists an $\eta(k)$ satisfying

$$\omega_j(k) = \max_p (m_{j,p}^{(\ell_t)} + \eta_p(k)) = m_{j,p_{j,t}}^{(\ell_t)} + \eta_{p_{j,t}}(k), \quad j = 1, \dots, n$$

or equivalently

$$m_{j,p_{j,t}}^{(\ell_t)} + \eta_{p_{j,t}}(k) \geq m_{j,i}^{(\ell_t)} + \eta_i(k) \quad \text{for } i, j = 1, \dots, n$$

Now by collecting all entries for $i, j = 1, \dots, n$ we obtain that

$$\omega(k) = H^{(t)} \eta(k) + h^{(t)} \quad (9)$$

if $E^{(t)} \eta(k) \leq e^{(t)}$ and $R_{\ell_t} z(k) \leq r_{\ell_t}$

where, for $i, j = 1, \dots, n$, and for $t = 1, \dots, n_t$, we have

$$[H^{(t)}]_{i,j} = \begin{cases} 1 & \text{for } j = p_{i,t} \\ 0 & \text{otherwise} \end{cases}, \quad [h^{(t)}]_i = m_{i,p_{i,t}}^{(\ell_t)}$$

$$[E^{(t)}]_{i(n-1)+j,l} = \begin{cases} 1 & \text{for } l = j \text{ and } j \neq p_{i,t} \\ -1 & \text{for } l = p_{i,t} \text{ and } j \neq p_{i,t} \\ 0 & \text{otherwise} \end{cases}$$

$$[e^{(t)}]_{i(n-1)+j,l} = m_{i,p_{i,t}}^{(\ell_t)} - m_{i,j}^{(\ell_t)}$$

It is clear that from equation (9) we can derive the matrices A_i, B_i, C_i, D_i , and vectors f_i and g_i . Note that the polyhedra are described by the inequalities $E^{(t,\ell)} \eta(k) \leq e^{(t,\ell)}$ and $R_\ell z(k) \leq r_\ell$.

The number of polyhedra is less than or equal to $N = Ln^n$. \blacksquare

Definition 10: An MMPS expression f of the variables x_1, \dots, x_n is defined by the grammar²

$$f := x_i | \alpha | \max(f_k, f_l) | \min(f_k, f_l) | f_k + f_l | \beta f_k \quad (10)$$

with $i \in \{1, 2, \dots, n\}$, $\alpha, \beta \in \mathbb{R}$, and where f_k, f_l are again MMPS expressions. \diamond

²The symbol $|$ stands for OR and the definition is recursive.

Definition 11: Consider systems that can be described by

$$x(k+1) = f_x(x(k), u(k), d(k)) \quad (11)$$

$$y(k) = f_y(x(k), u(k), d(k)), \quad (12)$$

where f_x, f_y are MMPS expressions in terms of the components of $x(k), u(k)$, and the auxiliary variables $d(k)$, which are all real-valued. Such systems will be called MMPS systems. If in addition, we have a condition of the form

$$f_c(x(k), u(k), d(k)) \leq c(k),$$

with f_c an MMPS expression, we speak about constrained MMPS systems. \diamond

Proposition 12: Every bounded RSMPL system of type 2 can be written as a constrained max-min-plus-scaling (MMPS) system provided that the variables x and u are bounded. \diamond

Note that this Proposition is a direct consequence of the equivalence between the class of PWA systems and constrained MMPS system (see [5]). However we will provide a direct proof here which transfers RSMPL systems directly into MMPS systems.

Proof: Consider a bounded RSMPL system of type 2 with state and output equations (1) and (2) where $\ell(k) = i$ if $z(k)$ satisfies $R_i z(k) \leq r_i$. Define ω, η , and $M^{(\ell)}$ as (7) and (8). Then

$$\omega(k) = M^{(\ell(k))}(k) \otimes \eta(k) \quad \text{for } R_{\ell(k)} z(k) \leq r_{\ell(k)}$$

so

$$\omega_i(k) = \max_j \left(m_{i,j}^{(\ell(k))} + \eta_j(k) \right) \quad \text{for } R_{\ell(k)} z(k) \leq r_{\ell(k)}$$

where $[M^{(\ell)}]_{i,j} = m_{i,j}^{(\ell)}$. Introduce the binary variables $\delta_i \in \{0, 1\}$, for all $i = 1, \dots, L$ such that

$$[\delta_i = 1] \Leftrightarrow [R_i z(k) \leq r_i] \quad (13)$$

Now assume that x and u are bounded. Then with $\ell \in \{1, \dots, L\}$ and $d \in [0, 1]$ we find that there exist bounded sets \mathcal{E} and \mathcal{Z} such that $\eta \in \mathcal{E}$ and $z \in \mathcal{Z}$. Let

$$\rho_i^* = \max_{z \in \mathcal{Z}} R_i z(k) - r_i$$

then according to [2], using the fact that the interiors of the sets $\{R_i z(k) \leq r_i\}$ are non-overlapping, we can rewrite (13) as

$$R_i z(k) - r_i \leq \rho_i^* (1 - \delta_i(k)) \quad (14)$$

$$\sum_{i=1}^L \delta_i(k) = 1 \quad (15)$$

Note that constraints (14) and (15) are linear constraints. Define the lower bound

$$\sigma_{\min}^* = \min_l \min_i \min_{\eta \in \mathcal{E}} \max_j (m_{i,j}^{(l)} + \eta_j(k))$$

(note that σ_{\min}^* is bounded because both $m_{i,j}^{(l)}$ and $\eta_j(k)$ are bounded), then the system is described by the following equations:

$$\omega_i(k) = \max_{l,j} \left(m_{i,j}^{(l)} + \eta_j(k) + (1 - \delta_l(k)) \sigma_{\min} \right)$$

Note that this is an MMPS function in the variables η and δ_i . Together with the linear constraints (14) and (15) (which are also MMPS constraints) we have proven that a type 2 RSMPL system, with bounded x and u , can be rewritten as a constrained MMPS system. ■

V. EXAMPLE

Consider a type RSMPL system

$$\begin{aligned} x(k) &= \max \left(x(k-1) + 1, u(k) + 0.2 \right) & \text{for } \ell(k) = 1 \\ x(k) &= \max \left(x(k-1) - 0.1, u(k) + 0.1 \right) & \text{for } \ell(k) = 2 \\ y(k) &= x(k) \end{aligned}$$

with

$$\begin{aligned} P(1|1, x(k-1)) &= 0 \\ P(2|1, x(k-1)) &= 1 \\ P(1|2, x(k-1)) &= \text{sat}(-x(k-1)) \\ P(2|2, x(k-1)) &= \text{sat}(1 + x(k-1)) \end{aligned}$$

$$\text{where } \text{sat}(\alpha) = \begin{cases} 0 & \text{for } \alpha < 0 \\ \alpha & \text{for } 0 \leq \alpha \leq 1. \\ 1 & \text{for } \alpha > 1 \end{cases}$$

Define the vector $w(k) = [\ell(k-1) \quad x(k-1) \quad u(k)]^T$. The probability $P(\ell(k)|w(k))$ can be written as a piecewise function:

$$\begin{aligned} P(\ell(k)|w(k)) &= \alpha_{i,\ell(k)}^T w(k) + \beta_{i,\ell(k)}, \\ \text{if } S_{i,\ell(k)} w(k) &\leq s_{i,\ell(k)}, \quad i = 1, \dots, 4 \end{aligned}$$

where

$$\begin{aligned} i=1 \quad S_{1,\ell(k)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & S_{1,\ell(k)} &= \begin{bmatrix} 1.5 \\ 0 \\ 0 \end{bmatrix} \\ \alpha_{1,1}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{1,1} &= 0 \\ \alpha_{1,2}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{1,2} &= 1 \\ i=2 \quad S_{2,\ell(k)} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} & S_{2,\ell(k)} &= \begin{bmatrix} -1.5 \\ 0 \end{bmatrix} \\ \alpha_{2,1}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{2,1} &= 0 \\ \alpha_{2,2}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{2,2} &= 1 \\ i=3 \quad S_{3,\ell(k)} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} & S_{3,\ell(k)} &= \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix} \\ \alpha_{3,1}^T &= \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} & \beta_{3,1} &= 0 \\ \alpha_{3,2}^T &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} & \beta_{3,2} &= 1 \\ i=4 \quad S_{4,\ell(k)} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & S_{4,\ell(k)} &= \begin{bmatrix} -1.5 \\ -1 \end{bmatrix} \\ \alpha_{4,1}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{4,1} &= 1 \\ \alpha_{4,2}^T &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \beta_{4,2} &= 0 \end{aligned}$$

This is a type 1 RSMPL system.

Now we will rewrite this system as a type 2 RSMPL system. First define

$$\begin{aligned} \eta(0, w(k)) &= 0 \\ \eta(1, w(k)) &= \alpha_{i,1}^T w(k) + \beta_{i,1} \\ &\quad \text{for } S_{i,1} w(k) \leq s_{i,1}, \quad i = 1, \dots, 4 \\ \eta(2, w(k)) &= 1 \end{aligned}$$

Now define a uniformly distributed scalar signal $d(k) \in [0, 1]$, define $z(k) = [w(k) \quad d(k)]^T$, and the functions

$$\begin{aligned} \xi(1, z(k)) &= \begin{bmatrix} -d(k) + 0 \\ d(k) - \alpha_{i,1}^T w(k) - \beta_{i,1} \\ S_{i,1} w(k) - s_{i,1} \end{bmatrix}, \\ \xi(2, z(k)) &= \begin{bmatrix} -d(k) + \alpha_{i,1}^T w(k) + \beta_{i,1} \\ d(k) - 1 \\ S_{i,1} w(k) - s_{i,1} \end{bmatrix} \end{aligned}$$

for $i = 1, \dots, 4$. Note that the first entry of $\xi(1, z(k))$ is always smaller than zero, and the same holds for the second entry of $\xi(2, z(k))$, so we can remove these entries. Now we can compute the matrices for $i = 1, \dots, 4$:

$$\begin{aligned} R_{1,i} &= \begin{bmatrix} 1 & -\alpha_{i,1}^T \\ 0 & S_{i,1} \end{bmatrix} & r_{1,i} &= \begin{bmatrix} \beta_{i,1} \\ s_{i,1} \end{bmatrix} \\ R_{2,i} &= \begin{bmatrix} -1 & \alpha_{i,1}^T \\ 0 & S_{i,2} \end{bmatrix} & r_{2,i} &= \begin{bmatrix} -\beta_{i,2} \\ s_{i,2} \end{bmatrix} \end{aligned}$$

which results in

$$\begin{aligned} R_{1,1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & r_{1,1} &= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} \\ R_{2,1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & r_{2,1} &= \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} \\ R_{1,2} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & r_{1,2} &= \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} \\ R_{2,2} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & r_{2,2} &= \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix} \\ R_{1,3} &= \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & r_{1,3} &= \begin{bmatrix} -1.5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ R_{2,3} &= \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & r_{2,3} &= \begin{bmatrix} -1.5 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ R_{1,4} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & r_{1,4} &= \begin{bmatrix} -1.5 \\ 1 \\ -1 \end{bmatrix} \\ R_{2,4} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & r_{2,4} &= \begin{bmatrix} -1.5 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

To rewrite this type 2 RSMPL system in a compact form we have to introduce 8 modes, where we have

$$\begin{aligned} x(k) &= \max(x(k-1) + 1, u(k) + 0.2) & \text{for } \kappa(k) = 1, 2, 3, 4 \\ x(k) &= \max(x(k-1) - 0.1, u(k) + 0.1) & \text{for } \kappa(k) = 5, 6, 7, 8 \\ y(k) &= x(k) \end{aligned} \quad (16)$$

We redefine $\bar{R}_\kappa = \bar{R}_{(i-1)*4+j} = R_{i,j}$ and we define

$$\begin{aligned} \bar{r}_1 &= \begin{bmatrix} 0 \\ 4.5 \end{bmatrix}, \bar{r}_2 = \begin{bmatrix} 0 \\ -4.5 \\ 0 \end{bmatrix}, \bar{r}_3 = \begin{bmatrix} 0 \\ -4.5 \\ 1 \\ 0 \end{bmatrix}, \bar{r}_4 = \begin{bmatrix} 1 \\ -4.5 \\ -1 \end{bmatrix}, \\ \bar{r}_5 &= \begin{bmatrix} 1 \\ 4.5 \end{bmatrix}, \bar{r}_6 = \begin{bmatrix} 1 \\ -4.5 \\ 0 \end{bmatrix}, \bar{r}_7 = \begin{bmatrix} 1 \\ -4.5 \\ 1 \\ 0 \end{bmatrix}, \bar{r}_8 = \begin{bmatrix} 0 \\ -4.5 \\ 1 \end{bmatrix}, \end{aligned}$$

and we obtain that the now mode $\kappa(k) = m$ if

$$\bar{z}(k) = [\kappa(k-1) \quad x^T(k-1) \quad u^T(k) \quad v^T(k) \quad d(k)]^T \in \bar{\Omega}_m$$

where

$$\bar{\Omega}_m = \{ z(k) \mid \bar{R}_m \bar{z}(k) \leq \bar{r}_m \}$$

Now it is straightforward to derive the PWA system: Note that

$$\begin{aligned} x(k) &= x(k-1) + 1 & \text{if } \kappa(k) < 4.5, x(k-1) + 1 \leq u(k) + 0.2 \\ x(k) &= u(k) + 0.2 & \text{if } \kappa(k) < 4.5, x(k-1) + 1 < u(k) + 0.2 \\ x(k) &= x(k-1) - 0.1 & \text{if } \kappa(k) > 4.5, x(k-1) - 0.1 \leq u(k) + 0.1 \\ x(k) &= u(k) + 0.1 & \text{if } \kappa(k) > 4.5, x(k-1) - 0.1 < u(k) + 0.1 \end{aligned}$$

This translates to the following PWA system:

for $\kappa = 1, \dots, 4$:

$$\begin{aligned} x(k) &= x(k-1) + 1 & \text{if } \bar{R}_\kappa \bar{z}(k) \leq \bar{r}_\kappa, x(k-1) + 0.8 \geq u(k) \\ x(k) &= u(k) + 0.2 & \text{if } \bar{R}_\kappa \bar{z}(k) \leq \bar{r}_\kappa, x(k-1) + 0.8 \leq u(k) \end{aligned}$$

for $\kappa = 5, \dots, 8$:

$$\begin{aligned} x(k) &= x(k-1) + 1 & \text{if } \bar{R}_\kappa \bar{z}(k) \leq \bar{r}_\kappa, x(k-1) - 0.2 \geq u(k) \\ x(k) &= u(k) + 0.1 & \text{if } \bar{R}_\kappa \bar{z}(k) \leq \bar{r}_\kappa, x(k-1) - 0.2 \leq u(k) \end{aligned}$$

We conclude that this system is a PWA system with 16 polyhedral regions.

Finally we rewrite the system as a constrained MMPS system. Assume the input is in the bounded set $u_{\min} \leq u(k) \leq u_{\max}$ and that the initial state $x(0)$ satisfies $x_{\min} \leq x(0) \leq x_{\max}$. It is easy to derive that in that case $x(k)$, $k \leq 0$ will be bounded:

$$\max(x_{\min} - 0.1, u_{\min} + 0.1) \leq x(k) \leq \max(1, x_{\max} + 1, u_{\max} + 0.2).$$

We compute

$$\rho_i^* = \max_{\bar{z} \in \mathcal{Z}} \bar{R}_i \bar{z}(k) - \bar{r}_i = \min(-3.5, x_{\min} - 0.1)$$

and

$$\sigma_{\min}^* = \min_l \min_i \min_{\eta \in \mathcal{E}} \max_j (m_{i,j}^{(l)} + \eta_j(k)) = u_{\min} + 0.1$$

Introduce the binary variables $\delta_i \in \{0, 1\}$, for all $i = 1, \dots, 8$. Now we obtain the following model:

$$\begin{aligned} x(k) &= \max \left[\left(\max(x(k-1) + 1, u(k) + 0.2) + \right. \right. \\ &\quad \left. \left. (1 - \delta_1 - \delta_2 - \delta_3 - \delta_4) \sigma_{\min}^* \right), \right. \\ &\quad \left. \left(\max(x(k-1) - 0.1, u(k) + 0.1) + \right. \right. \\ &\quad \left. \left. (1 - \delta_5 - \delta_6 - \delta_7 - \delta_8) \sigma_{\min}^* \right) \right] \end{aligned}$$

for $\delta_i(k) \in [0, 1]$ and subject to

$$\bar{R}_i \bar{z}(k) \leq \bar{r}_i + \rho_i^* (1 - \delta_i(k)) \text{ for all } i = 1, \dots, 8$$

$$\sum_{i=1}^8 \delta_i(k) = 1$$

This is a constrained MMPS system.

VI. DISCUSSION

In randomly switching max-plus-linear (RSMPL) discrete event systems we can switch between different modes of operation. The switching between the modes can be both deterministic and stochastic, and in each mode the discrete event system is described by a max-plus-linear state space model with different system matrices for each mode.

In this paper we have revisited the type 1 randomly switching max-plus-linear (RSMPL) systems, introduced a new type 2 RSMPL system, and showed that these two classes of RSMPL systems are equivalent.

Furthermore we have proven that every bounded RSMPL system of type 1 or 2 can be written as a piecewise affine (PWA) system or a constrained max-min-plus-scaling (MMPS) system (provided that the variables x and u are bounded).

REFERENCES

- [1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. John Wiley & Sons, New York, 1992.
- [2] A. Bemporad and M. Morari. Control of integrated logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.
- [3] F. Borelli. *Constrained Optimal Control of Linear and Hybrid Systems*. Springer-Verlag, Berlin, Germany, 2003.
- [4] R.A. Cuninghame-Green. *Minimax Algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1979.
- [5] W. Heemels, B. De Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, July 2001.
- [6] B. Heidergott, G.J. Olsder, and J. van der Woude. *Max Plus at Work*. Princeton University Press, Princeton, 2006.
- [7] E. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Transactions in Automatic Control*, 26(2):346–358, 1981.
- [8] T.J.J. van den Boom and B. De Schutter. Modelling and control of discrete event systems using switching max-plus-linear systems. *Control Engineering Practice*, 14(10):1199–1211, October 2006.
- [9] T.J.J. van den Boom and B. De Schutter. Stabilizing model predictive controllers for randomly switching max-plus-linear systems. In *Proceedings of the ECC 2007*, Kos, Greece, July 2007.
- [10] T.J.J. van den Boom and B. De Schutter. Model predictive control for switching max-plus-linear systems with random and deterministic switching. In *IFAC World Congress 2008*, Seoul, South Korea, July 2008.