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Steady-State and N-Stages Control for Isolated Controlled Intersections

Jack Haddad, Bart De Schutter, David Mahalel, and Per-Olof Gutman

Abstract— **In this paper a simplified isolated controlled intersection is introduced. Discrete-event piecewise affine (PWA) and discrete-event max-plus models are proposed to formulate the optimization problem for the switching sequences. Two control problems are considered: steady-state control and N-stages control. The formulated discrete-event PWA and max-plus problems are converted to be solved by linear programming (LP), mixed-integer programming (MIP), and mixed-integer linear programming (MILP). In the special case when the criterion is a strictly monotonous and linear function of the queue lengths, the steady-state control problem is solved analytically.**

I. Introduction

Transportation networks over the world are becoming more and more congested. Congestion has several effects on travelers, businesses, agencies, and cities. One significant element is the value of the additional time and wasted fuel. The congestion in USA's metropolitan areas is increasing continuously, e.g. in 2005 congestion (based on wasted time and fuel) cost about \$ 78.2 billion or an average of \$ 707 per traveler [24].

As traffic becomes more congested, utilization of the available infrastructure and increasing the capacity are an essential goal, which can be achieved by traffic control and management. The urban transportation network consists of number of signalized intersections. The congestion is not distributed equally between all the signalized intersections. There is usually a group of intersections that are more congested than others. These intersections are called critical intersections. Increasing the capacity of critical intersections will increase the total throughput flow of the network, and as a result the network capacity will be increased and the delays will decrease.

Different models, methods, and strategies have been proposed and applied for controlling urban isolated signalized intersections [1], [2], [14], [17], [18], [20], [21], [23], [26]. These researches aim to minimize delays or to maximize the intersection capacity. Some recent research considers the isolated intersection in the urban traffic network as a hybrid system [8], [9], [11], [19] and others propose the game theory approach [27] to model signalized intersections.

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In [6], the optimal acyclic (or N-stages) control was dealt with, where the Extended Linear Complementary Problem (ELCP), which is a mathematical programming problem, was used. In this paper, we introduce the steady-state and N-stages control problems and study the design of optimal traffic signal switching time sequences for a traffic signal controlled intersection through discrete-event models: max-plus and PWA (piecewise affine).

The paper is organized as follows. After describing the problem definition in Section II, the discrete-event models of an isolated intersection and the formulation of the optimal problems are given in Section III. The control problem for steady-state and N-stages control is dealt with in Sections IV and V respectively, which is followed by conclusions and topics for future research.

II. Problem Definition

In this paper, a typical simplified isolated intersection will be dealt with[1]. As shown in Fig. 1, there are two movements (m_1 and m_2), where each movement has a traffic signal that can be green or red. There is a traffic conflict in the intersection area between the two movements, therefore they cannot travel simultaneously and the traffic signal will be opposite, i.e. when movement m_1 has a green light movement m_2 has a red light and vice versa.
A given movement will encounter intertwined green and red periods. A cycle is defined as a pair of one green and one red period, and may change over time.

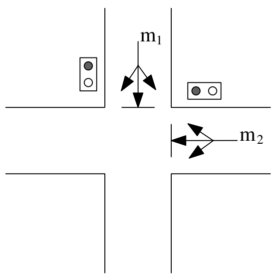

Fig. 1. Simplified isolated controlled intersection

In this case there are two movements m_1 and m_2, therefore the evolution of the queue lengths will be considered only for these two movements. The length of queue movement i at time t, which is the number of vehicles stopping behind the stop line in the intersection, is denoted by $q_i(t)$ [veh]. Let $f_{\mathrm{arr},i}(t)$, $f_{\mathrm{dep},i}(t)$ be, respectively, the arrival rate [veh/s] and the departure rate [veh/s] for queue i at time t. The

[1]Extension to more complex arrangements or setups is possible.

queue length growth rate $\alpha_i(t)$ [veh/s] for queue i at time t is given by $\alpha_i(t) = f_{\mathrm{arr},i}(t) - f_{\mathrm{dep},i}(t)$.

The following assumptions are made:

- A1: The arrival and departure rates in the isolated intersection are known and constant within each cycle[2].
- A2: When the traffic signal is green, the departure rate is bigger than the arrival rate, i.e. $f_{\mathrm{dep},i}(t) > f_{\mathrm{arr},i}(t)$, and when the traffic signal is red, the departure rate is equal to zero, $f_{\mathrm{dep},i}(t) = 0$, and the arrival rate is $f_{\mathrm{arr},i}(t) \geq 0$.
- A3: The queue lengths (number of vehicles) are approximated by real numbers.
- A4: Each movement will have only one green signal per cycle.

For the isolated controlled intersection with a constant traffic arrival and departure rates, we determine the control traffic signals that optimize the given control objective or criterion. We also formulate the stability conditions for the optimal traffic signal solution in the two control cases: steady-state and N-stages control.

III. Discrete Models for Isolated Controlled Intersections

A variety of models [10], [22] are based on the store-and-forward approach of modeling traffic networks that was first suggested by [15], [16]. This approach enables the simplification of the mathematical description of the traffic flow process without the use of switched variables. In this paper we consider the isolated controlled intersection as switching systems, as was done in [5], [6], [7].

Optimizing traffic signal switching sequences will be done through two models: discrete-event max-plus and discrete-event PWA.

A. Basic model

Let k be the index of the cycle. By assumption A4, in the cycle sequence each movement (m_1 or m_2) will have only one green signal per cycle. For each cycle of the cycle sequence we want to determine two decision variables: the cycle time, $T_{\mathrm{cyc},k}$ [s], and g_k [%] the proportion of the green time of movement m_1 in cycle k. The cycle time is expressed as number of seconds and the proportion of the green time is expressed as a percentage of the cycle time.

The evolution of the system begins at time t_0. This implies that the state of the queue length i at time t is given by

$$q_i(t) = q_i(t_0) + \int_{t_0}^{t} \alpha_i(t)dt \tag{1}$$

There are two switching times for cycle k: t_{2k+1} and t_{2k+2} (see Fig. 2). Without loss of generality, let the green light for movement m_1 start at t_{2k}, which coincides with the start of the cycle time. Hence, t_{2k+1} is the end of the green light for movement m_1 (or the start of the green light for movement m_2) and t_{2k+2} is the end of the green light for movement m_2 (or the start of the green light for movement m_1 in the next cycle time). The cycle time duration $T_{\mathrm{cyc},k}$ is equal to $t_{2k+2} - t_{2k}$. By assumption A1 the arrival rate

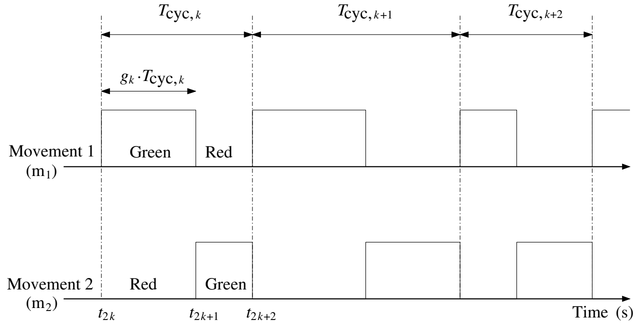

Fig. 2. Traffic signal switching sequences for movements m_1 and m_2

of queue i in phase $2k$ (i.e. the time period between t_{2k} and t_{2k+1}), $f_{\mathrm{arr},i}(t_{2k})$, and the departure rate of queue i in phase $2k$, $f_{\mathrm{dep},i}(t_{2k})$, are known and constant. The same holds for phase $2k+1$: the arrival rate of queue i, $f_{\mathrm{arr},i}(t_{2k+1})$, and the departure rate of queue i, $f_{\mathrm{dep},i}(t_{2k+1})$, are known and constant. This means, for example, that the growth rate $\alpha_i(t_{2k}) = f_{\mathrm{arr},i}(t_{2k}) - f_{\mathrm{dep},i}(t_{2k})$ has a constant rate value between the two discrete event time t_{2k} and t_{2k+1}.
The relations between the time sequences are the following,

$$t_{2k+1} = t_{2k} + g_k \cdot T_{\mathrm{cyc},k} \tag{2}$$

$$t_{2k+2} = t_{2k} + T_{\mathrm{cyc},k} \tag{3}$$

B. Formulation of an optimal discrete-event max-plus problem

The value of the queue length for movement m_1 in cycle k at the switching time instant t_{2k+1} is given by

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot g_k \cdot T_{\mathrm{cyc},k}, 0) \tag{4}$$

and at the switching time instant t_{2k+2} is given by

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\mathrm{cyc},k} \tag{5}$$

Recall that the signal light for movement m_2 is opposite to m_1, therefore the value of the queue lengths for movement m_2 in cycle k are given by

$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot g_k \cdot T_{\mathrm{cyc},k} \tag{6}$$

$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\mathrm{cyc},k}, 0) \tag{7}$$

We now consider the following problem: for a given number of cycles N and starting time t_0 (recall that the starting time should also coincide with the start of green signal for m_1), we compute an optimal switching time sequence $t_1, t_2, \ldots, t_{2N}$ that minimizes a given performance criterion J. There are variety of criteria that can be chosen, e.g. average queue length, maximal queue length, and delay over all queues.
Two new variables are defined $T_1(k)$ [s] and $T_2(k)$ [s], where $T_1(k) = g_k \cdot T_{\mathrm{cyc},k}$ and $T_2(k) = (1 - g_k) \cdot T_{\mathrm{cyc},k}$. Substituting

[2]Also averaged values can be considered.

these variables into (4) - (7) leads to the following Discrete-event Max-Plus (DMP) problem:

$$\min_{\substack{T_1(0),\, T_2(0),\\ T_1(1),\, T_2(1),\, \cdots,\\ T_1(N-1),\, T_2(N-1)}} J \tag{8}$$

subject to

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k), 0) \tag{9}$$
$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot T_2(k) \tag{10}$$
$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot T_1(k) \tag{11}$$
$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k), 0) \tag{12}$$

for $k = 0, 1, 2, \ldots, N-1$.
The optimization problem is formulated by minimization of the criterion J over N cycles. Hence, the number of variables to be determined is $2N$.

C. Formulation of an optimal discrete-event PWA problem

The $\max$ operator in (9) and (12) can be rewritten in such a way that PWA equations are obtained. This leads to the Discrete-event PWA (DPWA) problem:

$$\min_{\substack{T_1(0),\, T_2(0),\\ T_1(1),\, T_2(1),\, \cdots,\\ T_1(N-1),\, T_2(N-1)}} J \tag{13}$$

subject to

$$q_1(t_{2k+1}) = \begin{cases} q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k) \\ \qquad \text{if } q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k) \geq 0, \\ 0 \qquad \text{if } q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k) < 0, \end{cases} \tag{14}$$

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot T_2(k) \tag{15}$$
$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot T_1(k) \tag{16}$$

$$q_2(t_{2k+2}) = \begin{cases} q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k) \\ \qquad \text{if } q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k) \geq 0, \\ 0 \qquad \text{if } q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k) < 0, \end{cases} \tag{17}$$

for $k = 0, 1, 2, \ldots, N-1$.

IV. STEADY-STATE CONTROL

In order to develop and to implement dynamic models for controlling the traffic system, which can be helpful in decreasing congestion, first the steady-state control problem will be solved. The optimal solution and the feasibility condition for the steady-state control are useful in the control theory for the N-stages control problem, e.g., the steady-state solution can be the initial solution for the optimization process for the N-stages control problem.

In the steady-state control problem it is assumed that after some time, denoted by τ_0 ($k = 0$), the system will be in a steady-state mode. In the steady-state mode the cycle time and the green time will be constant, i.e. the traffic flows at the intersection and the evolution of the queues at the stop lines will be cyclic. Hence, only one cycle and two switching times (τ_1 and τ_2) are required to calculate the optimal cyclic switching sequences in the steady-state mode. The queue length for movement i at the start of the cycle will be equal to the queue length at the start of the next cycle:

$$q_1(\tau_0) = q_1(\tau_2) \tag{18}$$
$$q_2(\tau_0) = q_2(\tau_2) \tag{19}$$

In the following, two cases will be distinguished depending on the properties of the criterion J. The first case is when the criterion J is a strictly monotonous[3] function of the queue lengths (i.e. of $q_1(\tau_1)$, $q_2(\tau_1)$, $q_1(\tau_2)$ and $q_2(\tau_2)$), such as average queue length, positively weighted sum of queue lengths, or average travel time. The second case is when the criterion J is not a strictly monotonous function of the queue lengths, such as maximum queue length, weighted sum of queue lengths with some weights equal to zero, or maximal travel time.

A. The criterion is a strictly monotonous function of the queue lengths

Now we show that for a criterion that is a strictly monotonous function of the queue lengths the optimal cyclic switching sequences problem and the feasibility condition for the steady-state control can be formulated through a discrete-event max-plus model, and solved analytically for a strictly monotonous and linear criterion.

1) Formulation of an optimal cyclic discrete-event max-plus problem: The formulation is based on the DMP problem (8) - (12). The cyclic queue lengths equations (18) - (19) are added to the DMP problem and then we optimize it over only one cycle time ($N = 1$ and $k = 0$). Therefore, the number of decision variables will decrease to two: $T_1(0)$ and $T_2(0)$. For simplicity we write $T_1(0)$ and $T_2(0)$ as T_1 and T_2, respectively. We also assume that a lower bound T (with $T > 0$) for the sum of T_1 and T_2 is given, i.e. $T_1 + T_2 \geq T$. The Cyclic Discrete-event Max-Plus (CDMP) problem is then defined as follows:

$$\min_{T_1, T_2} J \tag{20}$$

subject to

$$q_1(\tau_1) = \max(q_1(\tau_0) + \alpha_1(\tau_0) \cdot T_1, 0) \tag{21}$$
$$q_1(\tau_2) = q_1(\tau_1) + \alpha_1(\tau_1) \cdot T_2 \tag{22}$$
$$q_2(\tau_1) = q_2(\tau_0) + \alpha_2(\tau_0) \cdot T_1 \tag{23}$$
$$q_2(\tau_2) = \max(q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2, 0) \tag{24}$$
$$T_1 + T_2 \geq T \tag{25}$$
$$\text{and } (18), (19)$$

Note that for scalars $a, b, c \in \mathbb{R}$ we have $a = \max(b, c)$ implies $a \geq b$ and $a \geq c$. In a similar way the CDMP problem can be rewritten in such a way that the $\max$ equations are "relaxed" to linear inequality equations. But

[3]The function J is strictly monotonous if for all queue length vectors with $\hat{q} \leq \tilde{q}$ and with $\hat{q}_i < \tilde{q}_i$ for at least one index i, we have $J(\hat{q}) < J(\tilde{q})$.

first, the cyclic queue lengths equations (18) and (19) are substituted into (21) and (23) respectively,

$$q_1(\tau_1) = \max(q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1, 0) \tag{26}$$

$$q_2(\tau_1) = q_2(\tau_2) + \alpha_2(\tau_0) \cdot T_1 \tag{27}$$

The $\max$ equations (24) and (26) can be relaxed into linear inequality equations as follows,

$$q_1(\tau_1) \geq q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1 \tag{28}$$

$$q_1(\tau_1) \geq 0 \tag{29}$$

$$q_2(\tau_2) \geq q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2 \tag{30}$$

$$q_2(\tau_2) \geq 0 \tag{31}$$

This leads to the "Relaxed" Cyclic Discrete-event Max-Plus (R-CDMP) problem:

$$\min_{T_1, T_2} J \tag{32}$$

subject to

$$(22), (25), (27), (28), (29), (30), (31)$$

Proposition 1: If the criterion J is a strictly monotonous function of the queue lengths, then any optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem.

Proof: The proof is done by contradiction.
Let $\tilde{q} = \big(\tilde{q}_1(\tau_1), \tilde{q}_1(\tau_2), \tilde{q}_2(\tau_1), \tilde{q}_2(\tau_2)\big)$ and $\tilde{T} = \big(\tilde{T}_1, \tilde{T}_2\big)$ be an optimal solution of the R-CDMP problem such that (21) is not satisfied, i.e.

$$\tilde{q}_1(\tau_1) > \max(\tilde{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \tilde{T}_1, 0) \tag{33}$$

or equivalently

$$\tilde{q}_1(\tau_1) > \tilde{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \tilde{T}_1 \tag{34}$$

$$\tilde{q}_1(\tau_1) > 0 \tag{35}$$

and such that $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$ satisfy (24) (note that we consider the case (34) and (35), but the proof for other cases is similar).
Now we replace $\tilde{q}_1(\tau_1)$ and $\tilde{q}_1(\tau_2)$ by

$$\hat{q}_1(\tau_1) = \tilde{q}_1(\tau_1) - \varepsilon \tag{36}$$

$$\hat{q}_1(\tau_2) = \tilde{q}_1(\tau_2) - \varepsilon \tag{37}$$

where $\varepsilon > 0$. The other variables stay the same, i.e. $\hat{q}_2(\tau_1)$, $\hat{q}_2(\tau_2)$, and $\hat{T}$ are equal to $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$, and $\tilde{T}$, respectively. In the following we verify that $(\hat{q}, \hat{T})$ is also a feasible solution of the R-CDMP problem as long as $\hat{q}_1(\tau_1) \geq 0$ (i.e. (29)) is satisfied. We fill out $\hat{q}_1(\tau_1)$ and $\hat{q}_1(\tau_2)$ into (34) and obtain $\tilde{q}_1(\tau_1) - \varepsilon > \tilde{q}_1(\tau_2) - \varepsilon + \alpha_1(\tau_0) \cdot \tilde{T}_1$ for any ε, which implies $\hat{q}_1(\tau_1) \geq \hat{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \hat{T}_1$ (i.e. (28) holds). Since the variables $\hat{q}_2(\tau_1)$, $\hat{q}_2(\tau_2)$, and $\hat{T}$ are assumed to be unchanged, they imply (25), (27), (30), and (31).
Equation (22) implies $\tilde{q}_1(\tau_2) - \varepsilon = \tilde{q}_1(\tau_1) - \varepsilon + \alpha_1(\tau_1) \cdot \tilde{T}_2$ or equivalently $\hat{q}_1(\tau_2) = \hat{q}_1(\tau_1) + \alpha_1(\tau_1) \cdot \hat{T}_2$.
Now we select ε such that $\hat{q}_1(\tau_1) = 0$. Then $(\hat{q}, \hat{T})$ is a feasible solution of the R-CDMP problem. Recall that the criterion J is a strictly monotonous function of the queue lengths. Since $\hat{q} \leq \tilde{q}$ and $\hat{q}_i < \tilde{q}_i$ for some i due to (36) and (37), this implies $J(\hat{q}, \hat{T}) < J(\tilde{q}, \tilde{T})$ which is in contradiction with the fact that $(\tilde{q}, \tilde{T})$ is an optimal solution of the R-CDMP problem.

Hence, the optimal solution of the R-CDMP problem should satisfy (21) and as a consequence the optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem. ■
So in the sequel we consider the R-CDMP problem instead of the CDMP problem.

2) Feasibility condition: In this section, the existence condition for the steady-state control is derived based on the R-CDMP problem.
We can eliminate $q_1(\tau_2)$ and $q_2(\tau_1)$ from the constraints of the R-CDMP problem by substituting (22) and (27) into (28) and (30) respectively, resulting in

$$-\alpha_1(\tau_0) \cdot T_1 \geq \alpha_1(\tau_1) \cdot T_2 \tag{38}$$

$$-\alpha_2(\tau_1) \cdot T_2 \geq \alpha_2(\tau_0) \cdot T_1 \tag{39}$$

If $T_1 = 0$, then it is implied from (38) and (39) that $T_2 = 0$, and vice versa. But this is a contradiction with (25) and the fact that $T > 0$. Hence, we will have $T_1 \neq 0$ and $T_2 \neq 0$. However, it follows from assumptions A1 and A2 that $\alpha_1(\tau_0) < 0$ and $\alpha_2(\tau_1) < 0$ where $\alpha_1(\tau_1) > 0$ and $\alpha_2(\tau_0) > 0$. Hence, from (38) and (39) we obtain the following feasibility condition:

$$\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} \leq \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)} \tag{40}$$

3) Analytic solution for linear criterion: Now we show that if the criterion J is a strictly monotonous *linear* function of the queue lengths, then the R-CDMP problem can be solved analytically.

As an example, let the weighted average queue length over all the queues be our criterion function J,

$$J = \sum_{i=1}^{2} w_i \frac{1}{\tau_2 - \tau_0} \int_{\tau_0}^{\tau_2} q_i(t) dt \tag{41}$$

where $w_i > 0$ for all i.
Based on assumption A1 we write the following integral as:

$$\int_{t_k}^{t_{k+1}} q_i(t) dt = \frac{(t_{k+1} - t_k)}{2} (q_i(t_k) + q_i(t_{k+1})) \tag{42}$$

Recall that $\tau_2 - \tau_0 = T_1 + T_2$ and $q_i(\tau_0) = q_i(\tau_2)$. Substituting these equations and (42) into (41), we obtain the following criterion J which is strictly monotonous linear function of the queue lengths

$$J = \sum_{i=1}^{2} \frac{w_i}{2} (q_i(\tau_1) + q_i(\tau_2)) \tag{43}$$

Note that J is strictly monotonous in the queue lengths due to the fact that $w_i > 0$ for all i.

We eliminate $q_1(\tau_2)$ and $q_2(\tau_1)$ in the objective function by substituting (22) and (27) into (43), which leads to

$$J = \frac{1}{2}(2w_1 q_1(\tau_1) + 2w_2 q_2(\tau_2) + w_1 \alpha_1(\tau_1) \cdot T_2 + w_2 \alpha_2(\tau_0) \cdot T_1) \tag{44}$$

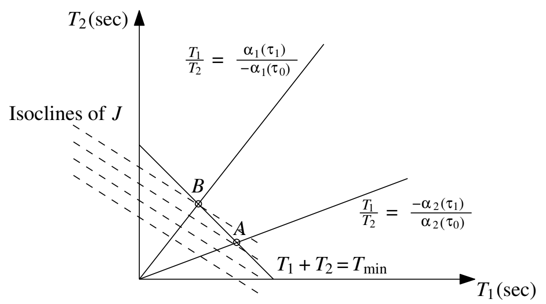

Fig. 3. Analytic solution for the linear programming problem

Looking at the R-CDMP problem with the linear criterion (44), the optimal solutions of the queue lengths $q_1(\tau_1)$ and $q_2(\tau_2)$ must be equal to zero in order to minimize J, i.e.

$$q_1^*(\tau_1) = 0 \tag{45}$$
$$q_2^*(\tau_2) = 0 \tag{46}$$

Therefore, we obtain the following linear programming (LP) problem

$$\min_{T_1,T_2} J = \frac{1}{2}\left(w_2\alpha_2(\tau_0)\cdot T_1 + w_1\alpha_1(\tau_1)\cdot T_2\right) \tag{47}$$

subject to

$$-\alpha_1(\tau_0)\cdot T_1 \geq \alpha_1(\tau_1)\cdot T_2 \tag{48}$$
$$-\alpha_2(\tau_1)\cdot T_2 \geq \alpha_2(\tau_0)\cdot T_1 \tag{49}$$
$$T_1 + T_2 \geq T \tag{50}$$

The solution of this problem depends on the slope of the linear objective function (see Fig. 3). If $w_2\alpha_2(\tau_0) < w_1\alpha_1(\tau_1)$ the optimal solution will be point A, and when $w_2\alpha_2(\tau_0) > w_1\alpha_1(\tau_1)$ the optimal solution will be point B. In the case when $w_2\alpha_2(\tau_0) = w_1\alpha_1(\tau_1)$ all points between A and B are optimal solutions for the problem. Points A and B are equal to

$$(T_1,T_2)_A = \left(\frac{-T\alpha_2(\tau_1)}{\alpha_2(\tau_0)-\alpha_2(\tau_1)}, \frac{T\alpha_2(\tau_0)}{\alpha_2(\tau_0)-\alpha_2(\tau_1)}\right) \tag{51}$$
$$(T_1,T_2)_B = \left(\frac{-T\alpha_1(\tau_1)}{\alpha_1(\tau_0)-\alpha_1(\tau_1)}, \frac{T\alpha_1(\tau_0)}{\alpha_1(\tau_0)-\alpha_1(\tau_1)}\right) \tag{52}$$

B. The criterion is not a strictly monotonous function of the queue lengths

When the criterion J is not a strictly monotonous function of the queue lengths, the Cyclic Discrete-event PWA (CDPWA) problem can be formulated based on the DPWA problem (13) - (17). The cyclic queue lengths equations (18) and (19) are then added to the DPWA problem and then it is optimized over one cycle time (i.e. $N = 1$ and $k = 0$).

The PWA equations can be transformed into mixed integer equations by introducing additional auxiliary variables as follows (cf. [4]). To perform these transformations we use the following equivalences, where δ represents a binary valued scalar variable, y a real valued scalar variable, and f a function defined on a bounded set X with upper and lower bounds M and m for the function values:

$$[f(x) \leq 0] \Leftrightarrow [\delta = 1] \text{ is true iff } \begin{cases} f(x) \leq M(1-\delta), \\ f(x) \geq \varepsilon + (m-\varepsilon)\delta, \end{cases}$$

where ε is a small tolerance (typically the machine precision),

$$y = \delta f(x) \text{ is equivalent to } \begin{cases} y \leq M\delta, \\ y \geq m\delta, \\ y \leq f(x) - m(1-\delta), \\ y \geq f(x) - M(1-\delta). \end{cases}$$

Now the CDPWA problem can be solved by mixed-integer programming (MIP) algorithms [3], [13] for a nonlinear criterion J and by a mixed-integer linear programming (MILP) algorithms [12], [25] for a criterion J that is a linear (but not strictly monotonous) function of the queue lengths.

V. N-stages Control

In the N-stages control problem we consider a finite number of switchings in the optimization procedure. Now we specifically consider the following problem: for a given integer N and a given starting time t_0 we want to compute an optimal switching sequence consisting of N cycles.
For the simplified isolated controlled intersection we formulate the problem for the following two cases.

A. The criterion is a strictly monotonous function of the queue lengths

We use the DMP problem (8) - (12) to solve the optimal problem for N-stages control when the criterion J is a strictly monotonous function of the queue lengths. In this case, each $\max$ equation can be relaxed to two inequality equations, which leads to the "Relaxed" Discrete-event Max-Plus (R-DMP) problem

$$\min_{\substack{T_1(0),\,T_2(0),\\ T_1(1),\,T_2(1),\,\cdots,\\ T_1(N-1),\,T_2(N-1)}} J \tag{53}$$

subject to

$$q_1(t_{2k+1}) \geq q_1(t_{2k}) + \alpha_1(t_{2k})\cdot T_1(k) \tag{54}$$
$$q_1(t_{2k+1}) \geq 0 \tag{55}$$
$$q_2(t_{2k+2}) \geq q_2(t_{2k+1}) + \alpha_2(t_{2k+1})\cdot T_2(k) \tag{56}$$
$$q_2(t_{2k+2}) \geq 0 \tag{57}$$
$$\text{and } (10), (11) \tag{58}$$

for $k = 0, 1, 2, \ldots, N-1$.

Proposition 2: If the criterion J is a strictly monotonous function of the queue lengths, then any optimal solution of the R-DMP problem is also an optimal solution of the DMP problem.

Proof: See the proof of Proposition 3.3 of [6] which also applies here. ∎

So the R-DMP problem can be solved by linear programming when the criterion J is a strictly monotonous linear function of the queue lengths.

B. The criterion is not a strictly monotonous function of the queue lengths

When the criterion J is not a strictly monotonous function of the queue lengths, the DPWA problem can be used to solve the optimal problem for N-stages. In this case the PWA equations can also be transformed into mixed integer inequalities (see Section IV-B) and can then be solved by using MIP approaches.

VI. Conclusions and Future Research

For the simplified isolated controlled intersection we can compute the optimal switching sequences for the steady-state and N-stages control problems by solving a linear programming problem, a mixed-integer programming problem, or a mixed-integer linear programming problem.

A feasibility condition for the steady-state control has been derived when the criterion J is a strictly monotonous function of the queue lengths. It is shown that if in addition the criterion is linear the problem can be solved analytically. When the criterion J is not strictly monotonous the problem can be recast as a mixed-integer linear or nonlinear programming problem.

The N-stages control problem can be solved by linear programming if the criterion J is linear and strictly monotonous. When the criterion is not strictly monotonous the control problem can be solved by mixed-integer programming.

As shown in this paper, the model of a signalized intersection can be rewritten as a piecewise affine system. Conducting stability analysis of such a system using Lyapunov functions is a topic for future research. Another topic for future research is the formulation of the control problem for signalized isolated intersections by optimal periodic control.

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