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MPC for max-plus-linear systems with an output cost criterion: steady-state behavior and guaranteed stability

T.J.J. van den Boom and B. De Schutter

Abstract—In previous work we have extended the popular Model Predictive Control (MPC) design technique to a class of discrete event systems that can be described by a model that is “linear” in the max-plus algebra. In this paper we study the steady-state behavior of these so-called max-plus-linear systems in the case of an output cost criterion. We derive improved tuning rules for the controller parameters that guarantee us a feasible and stable operation of the controller in the unconstrained case. An example shows that violation of the tuning rules may destabilize the closed-loop system.

I. INTRODUCTION

Model predictive control (MPC) [8] is a proven technology for the control of multivariable systems in the presence of input, output and state constraints and it is capable of tracking pre-scheduled reference signals. These attractive features make MPC widely accepted in the process industry. Usually MPC uses linear or nonlinear discrete-time models. However, the attractive features mentioned above have led us to extend MPC to discrete event systems. In this paper we consider the class of discrete event systems with synchronization but no concurrency. Such systems can be described by models that are “linear” in the max-plus algebra [1], [2], [7], and therefore they are called max-plus-linear (MPL) systems. In [3] we have extended MPC to MPL systems. In [14], [15], [16] we have studied stability and tuning of MPC controllers for MPL systems using an output cost criterion. In [14] we observed that for MPL systems, stability is not an intrinsic feature of the system, but it also depends on the input and the due dates (i.e., the reference signal) of the system. In [15] we used an end-point constraint to guarantee stability. In [16] we derived tuning rules that guarantee stability for the case where the initial state is bounded. In [10], [11] the steady-state behavior of MPC for unconstrained MPL systems was considered in the case of a state cost criterion. Also simple tuning rules were derived for guaranteed stability of the closed loop. Unfortunately, these tuning rules could not be easily extended to the case of an output cost criterion.

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The reason to study the output cost criterion lies in the fact that in many applications the delay of the output event signal with respect to some due date signal is important rather than the delay of the state events. This motivates us to study MPC for unconstrained MPL systems like in [11], but now with an output cost criterion.

Related papers on MPC for MPL systems are [5], [6], [9]. The paper is organized as follows: We start with some preliminaries in Section II. In Section III we introduce a so-called dynamic steady-state sequence for MPL systems and we show that its length is always equal to 1. In Section IV we discuss the model predictive control algorithm for unconstrained MPL systems, and in Section V we discuss the issue of stability. We derive tuning rules that guarantee stability without conditions on the initial state (as was done in [16]) and without using an end-point constraint (as was done in [15]). Section VI presents a worked example to show that violation of the derived tuning rules may indeed lead to an unstable closed-loop system.

II. PRELIMINARIES

Max-Plus algebra: Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition (\oplus) and multiplication (\otimes) are defined as follows [1], [2]: $x \oplus y = \max(x, y)$, $x \otimes y = x + y$ for numbers $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij})$$

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^n A_{ik} \otimes C_{kj} = \max_{k=1, \dots, n} (A_{ik} + C_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$. A max-plus diagonal matrix $S = \text{diag}_{\oplus}(s_1, \dots, s_n)$ has elements $S_{ij} = \varepsilon$ for $i \neq j$ and diagonal elements $S_{ii} = s_i$ for $i = 1, \dots, n$. $E = \text{diag}_{\oplus}(0, \dots, 0)$ is the max-plus identity matrix. The matrix $\varepsilon_{m \times n}$ is the $m \times n$ max-plus-algebraic zero matrix: $(\varepsilon_{m \times n})_{ij} = \varepsilon$ for all i, j . The max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows: $A^{\otimes 0} = E$ and $A^{\otimes k} = A \otimes A^{\otimes k-1}$ for $k = 1, 2, \dots$. For any matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ we can define

$$A^* = E \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots$$

If for a max-plus diagonal matrix $S = \text{diag}_{\oplus}(s_1, \dots, s_n)$ all s_i are finite, the inverse of S is equal to $S^{\otimes -1} = \text{diag}_{\oplus}(-s_1, \dots, -s_n)$. Then it holds that $S \otimes S^{\otimes -1} = S^{\otimes -1} \otimes S = E$.

Finally we introduce the max-plus-algebraic (MPA) eigenvalue of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$. The scalar $\lambda \in \mathbb{R}_\varepsilon$ is an

MPA eigenvalue if there exists a $v \in \mathbb{R}_\varepsilon^n$ with at least one finite entry such that $A \otimes v = \lambda \otimes v$ [1]. The vector v is called an MPA eigenvector.

Max-Plus-Linear (MPL) systems: Discrete-event systems with only synchronization and no concurrency can be modeled by a max-plus-algebraic model of the following form [1]:

$$x_{\text{sys}}(k) = A_{\text{sys}} \otimes x_{\text{sys}}(k-1) \oplus B_{\text{sys}} \otimes u_{\text{sys}}(k) \quad (1)$$

$$y_{\text{sys}}(k) = C_{\text{sys}} \otimes x_{\text{sys}}(k) \quad (2)$$

with $A_{\text{sys}} \in \mathbb{R}_\varepsilon^{n \times n}$, $B_{\text{sys}} \in \mathbb{R}_\varepsilon^{n \times m}$, and $C_{\text{sys}} \in \mathbb{R}_\varepsilon^{p \times n}$, where n is the number of states, m is the number of inputs and p the number of outputs. The system (1)-(2) is called a max-plus-linear (MPL) system.

The MPL system (1)-(2) is controllable if all states are connected to some input [1]. It can be checked that the system is controllable iff the matrix

$$\Gamma_n = \begin{bmatrix} B & A \otimes B & \dots & A^{\otimes n-1} \otimes B \end{bmatrix}$$

is row-finite [4].

Model Predictive Control (MPC) for MPL systems: Consider an MPL system (1)-(2). The Model Predictive Control (MPC) problem for this system is formulated as follows [3]:

$$\min_{u_{\text{sys}}(k|k), \dots, u_{\text{sys}}(k+N_p-1|k)} \sum_{j=0}^{N_p-1} \quad (3)$$

$$\left(\sum_{i=1}^p \max(y_{\text{sys},i}(k+j|k) - r_{\text{sys},i}(k+j), 0) \right) - \beta \left(\sum_{\ell=1}^m u_{\text{sys},\ell}(k+j|k) \right) \quad (4)$$

subject to

$$u_{\text{sys},\ell}(k+j|k) - u_{\text{sys},\ell}(k+j-1|k) \geq 0, \quad \text{for } j = 0, \dots, N_p-1, \ell = 1 \dots, m, \quad (5)$$

where $r_{\text{sys}}(k)$ is the reference vector containing the due dates for the output events $y_{\text{sys}}(k)$, $\hat{y}_{\text{sys}}(k+j|k)$ denotes the prediction of $y_{\text{sys}}(k+j)$ based on knowledge at event step k , $u_{\text{sys}}(k+j|k)$ denotes the future input for step $k+j$, based on knowledge at event step k , and N_p is the prediction horizon. The first term in cost criterion (4) is also called the ‘‘tardiness’’. Note that it penalizes all delays with respect to the due dates. The second term in cost criterion (4) reflects that we have just-in-time control, which means that the input events take place as late as possible. Equation (5) guarantees a non-decreasing input signal. Note that the input sequences correspond to occurrence times of consecutive events, and so $u_{\text{sys}}(k)$ should be nondecreasing.

MPC uses a receding horizon strategy. So after computation of the optimal control sequence $u_{\text{sys}}(k|k), \dots, u_{\text{sys}}(k+N_p-1|k)$, only the first control sample $u_{\text{sys}}(k) = u_{\text{sys}}(k|k)$ will be implemented, subsequently the horizon is shifted and the model and the initial state estimate

can be updated if new measurements are available, then the new MPC problem is solved, etc. The above problem is called the MPL-MPC problem.

In this paper we consider a reference signal that the output should track:

$$r_{\text{sys}}(k) = r_{\text{sys},0} + \rho k \quad (6)$$

where $r_{\text{sys},0} \in \mathbb{R}$ is a vector of offsets and

$$\rho > \lambda_{\text{max}}, \quad (7)$$

in which λ_{max} denotes the largest MPA eigenvalue of A_{sys} . In [14] it has been shown that (7) is a necessary condition for stability.

III. STEADY-STATE BEHAVIOR

Given system (1)-(2) and a reference signal (6) with $\rho > \lambda_{\text{max}}$, we can shift the system as follows. Define

$$\bar{A} = A_{\text{sys}} - \rho, \quad \bar{B} = B_{\text{sys}}, \quad \bar{C} = C_{\text{sys}} \quad (8)$$

and

$$\bar{x}(k) = x_{\text{sys}}(k) - \rho k, \quad (9)$$

$$\bar{u}(k) = u_{\text{sys}}(k) - \rho k, \quad \bar{y}(k) = y_{\text{sys}}(k) - \rho k, \quad (10)$$

then the MPL system (1)-(2) is equivalent to the shifted MPL system

$$\bar{x}(k) = \bar{A} \otimes \bar{x}(k-1) \oplus \bar{B} \otimes \bar{u}(k) \quad (11)$$

$$\bar{y}(k) = \bar{C} \otimes \bar{x}(k). \quad (12)$$

Note that the reference signal for the output $\bar{y}(k)$ of the shifted system will be a set point $\bar{r}(k) = \bar{r}_{\text{ss}} = r_{\text{sys},0}, \forall k$. We now consider the steady-state behavior of this shifted system.

Definition 1: The dynamic steady-state sequence (DSSS)

$$(\bar{x}_{\text{ss}}, \bar{u}_{\text{ss}}, \bar{r}_{\text{ss}}, c) = (\bar{x}_{\text{ss}}(1), \dots, \bar{x}_{\text{ss}}(c), \bar{u}_{\text{ss}}(1), \dots, \bar{u}_{\text{ss}}(c), \bar{r}_{\text{ss}}, c),$$

where $c \in \mathbb{N}_0$ is defined as the solution of

$$\max_{\bar{u}_{\text{ss}}, \bar{x}_{\text{ss}}, c} \frac{1}{c} \sum_{j=1}^c \sum_{i=1}^m [\bar{u}_{\text{ss}}(j)]_i \quad (13)$$

$$\bar{x}_{\text{ss}}(1) = \bar{A} \otimes \bar{x}_{\text{ss}}(c) \oplus \bar{B} \otimes \bar{u}_{\text{ss}}(1) \quad (14)$$

$$\bar{x}_{\text{ss}}(j) = \bar{A} \otimes \bar{x}_{\text{ss}}(j-1) \oplus \bar{B} \otimes \bar{u}_{\text{ss}}(j) \quad \text{for } j = 2, \dots, c \quad (15)$$

$$\bar{r}_{\text{ss}} \geq \bar{C} \otimes \bar{x}_{\text{ss}}(j) \quad \text{for } j = 1, \dots, c \quad (16)$$

If there are multiple solutions for c , we choose smallest positive integer c for which the criterion is maximized. The value c is called the DSSS length. \diamond

Lemma 2: For any MPL system (11)-(12) the DSSS length is equal to $c_{\text{ss}} = 1$. \diamond

The proof of Lemma 2 is in [13].

Note that Lemma 2 means that system (11)-(11) has a static equilibrium point $(\bar{x}_{ss}, \bar{u}_{ss}, \bar{r}_{ss}, 1)$. In that case, the values \bar{x}_{ss} and \bar{u}_{ss} can easily be computed for a given \bar{r}_{ss} as follows [11]:

$$\begin{aligned}\bar{u}_{ss} &= -((\bar{C} \otimes \bar{A}^* \otimes \bar{B})^T \otimes \bar{r}_{ss}), \\ \bar{x}_{ss} &= \bar{A}^* \otimes \bar{B} \otimes \bar{u}_{ss}.\end{aligned}\quad (17)$$

IV. THE MPC PROBLEM FOR MAX-PLUS-LINEAR SYSTEMS

Consider a system (1)-(2) and a reference signal (6). To simplify the derivations we will restrict ourselves to SISO systems, so $B \in \mathbb{R}_\varepsilon^{n \times 1}$, and $C \in \mathbb{R}_\varepsilon^{1 \times n}$. However, all derivations in this and the next section can easily be extended to MIMO systems. Let λ_{\max} be the largest eigenvalue of the matrix A_{sys} , and consider a tracking rate $\rho > \lambda_{\max}$. We will now normalize this system. There exists an MPA invertible diagonal matrix P such that the matrix

$$A = (P^{\otimes -1} \otimes A_{\text{sys}} \otimes P) - \rho \quad (18)$$

satisfies $A_{ij} < 0$, $\forall i, j$ [11]. Now define

$$B = (P^{\otimes -1} \otimes B_{\text{sys}}) + \bar{u}_{ss}, \quad (19)$$

$$C = (C_{\text{sys}} \otimes P) - r_{\text{sys},0}, \quad (20)$$

where \bar{u}_{ss} is given by (17). Define the normalized signals

$$x(k) = (P^{\otimes -1} \otimes x_{\text{sys}}(k)) - \rho k, \quad (21)$$

$$u(k) = u_{\text{sys}}(k) - \rho k - \bar{u}_{ss}, \quad (22)$$

$$y(k) = y_{\text{sys}}(k) - \rho k - r_{\text{sys},0}. \quad (23)$$

then the MPL system (1)-(2) is equivalent to the normalized MPL system

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \quad (24)$$

$$y(k) = C \otimes x(k) \quad (25)$$

Corollary 3: For the normalized system (24)-(25) we have

$$r_{ss} = 0 \quad (26)$$

$$u_{ss} = -((C \otimes A^* \otimes B)^T \otimes r_{ss}) = 0 \quad (27)$$

$$y_{ss} = C \otimes x_{ss} = C \otimes A^* \otimes B \otimes u_{ss} = 0 \quad (28)$$

$$C \otimes A^{\otimes i} \otimes B \leq 0, \quad \forall i \geq 0 \quad (29)$$

Note that (26)-(28) can be found by substitution of the system matrices of the normalized system in (17). From (28) we find $C \otimes A^* \otimes B = 0$ and so (29).

Consider the normalized MPL system (24)-(25). The Model Predictive Control (MPC) problem for this system is reformulated as follows:

$$\begin{aligned}\min_{u(k|k), \dots, u(k+N_p-1|k)} & \sum_{j=0}^{N_p-1} \\ & (\max(y(k+j|k), 0) - \beta u(k+j|k))\end{aligned}\quad (30)$$

subject to

$$u(k+j|k) - u(k+j-1|k) \geq -\rho,$$

$$\text{for } j = 0, \dots, N_p-1, \quad (31)$$

Define the signals u^b and z :

$$u^b(k+j|k) = (u(k-1) - \rho(j+1)) \oplus 0$$

$$z(k+j|k) = 0 \oplus C \otimes A^{\otimes j+1} \otimes x(k-1)$$

$$\oplus \bigoplus_{i=0}^j C \otimes A^{\otimes j-i} \otimes B \otimes u^b(k+i|k)$$

for $j = 0, \dots, N_p-1$.

Proposition 4: Assume $\beta < 1/N_p$ and define the scalars $\tilde{D}_\ell = C \otimes A^{\otimes \ell} \otimes B$ for $\ell = 0, \dots, N_p-1$. Consider the maximization problem

$$\max_{u(k|k), \dots, u(k+N_p-1|k)} \sum_{j=0}^{N_p-1} u(k+j|k) \quad (32)$$

subject to

$$\begin{aligned}\tilde{D}_{i-j} + u(k+j|k) &\leq z(k+i|k), \\ \forall i \geq j, i, j &\in \{0, \dots, N_p-1\}\end{aligned}\quad (33)$$

$$\begin{aligned}u(k+j|k) - u(k+j-1|k) &\geq -\rho \\ \forall j &\in \{0, \dots, N_p-1\}\end{aligned}\quad (34)$$

Then, $u(k|k), \dots, u(k+N_p-1|k)$ is the optimal input sequence for the MPL-MPC problem (30)-(31) at event step k . The output for this optimal input sequence is given by

$$y(k+j|k) = z(k+j|k)$$

◇

The proof is similar to the proof of Lemma 1 of [16] and the proof of Proposition 8 of [11].

The maximization problem of (32)-(34) can be solved using linear programming algorithms. Note that because $z(k+j|k) \geq 0$ and $\tilde{D}_\ell = C \otimes A^{\otimes \ell} \otimes B \leq 0$ by (29), we find that due to the maximization of $u(k+j|k)$ in (32) we have that $u(k+j|k) \geq 0$ for all j .

V. STABILITY

Stability in conventional system theory is concerned with boundedness of the states. In MPL systems however, k is an event counter and $x_i(k)$ refers to the occurrence time of an event. So the sequence $x_i(k), x_i(k+1), \dots$ is always non-decreasing, and for $k \rightarrow \infty$ the event time $x_i(k)$ will usually grow unbounded. We therefore adopt the notion of stability for discrete event systems from [12], in which a discrete event systems is called stable if all its buffer levels remain bounded. This implies that for an observable max-plus linear systems with $\rho > \lambda_{\max}$ the closed-loop stability is achieved if there exist finite constants $u_{\max}, x_{\max}, y_{\max}, K$ such that for the output, state, and input of the corresponding normalized system we have

$$|y(k)| \leq y_{\max}, \quad \forall k \geq K \quad (35)$$

$$|x(k)| \leq x_{\max}, \quad \forall k \geq K \quad (36)$$

$$|u(k)| \leq u_{\max} \quad , \quad \forall k \geq K \quad (37)$$

Condition (35) means that the delay $y(k) = y_{\text{sys}} - r_{\text{sys}}(k)$ remains bounded. Condition (36) implies that the number of parts in the output buffer will remain bounded. Finally, condition (37) together with (35) means that the time between the starting date $u_{\text{sys}}(k)$ and the output date $y_{\text{sys}}(k)$ (i.e., the throughput time) is bounded.

Lemma 5: Let $x \in \mathbb{R}_{\varepsilon}^n$ and $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ with $A_{ij} \leq 0$ for all i, j . Then we have

$$\begin{aligned} A^{\otimes l+n} \otimes x &\leq A^{\otimes l+n-1} \otimes x \oplus A^{\otimes l+n-2} \otimes x \oplus \\ &\dots \oplus A^{\otimes l} \otimes x \end{aligned} \quad (38)$$

for any integer $l \geq 1$. \diamond

Proof: Note that if (38) holds for $l = 1$, it will hold for any integer $l \geq 1$ due to the monotonicity of max-plus-algebraic multiplication [1], [2].

We will first show that (38) holds for $l = 1$ and for the max-plus-algebraic unit vectors e_1, \dots, e_n where e_j is defined as follows:

$$(e_j)_i = \begin{cases} 0 & \text{if } i = j \\ \varepsilon & \text{otherwise} \end{cases}$$

for $i = 1, \dots, n$, i.e. we prove

$$\begin{aligned} (A^{\otimes n+1} \otimes e_j)_i &\leq (A^{\otimes n} \otimes e_j)_i \oplus (A^{\otimes n-1} \otimes e_j)_i \oplus \\ &\dots \oplus (A \otimes e_j)_i \end{aligned} \quad (39)$$

Note that for e_j and any integer $\ell \geq 0$, we have $(A^{\otimes \ell} \otimes e_j)_i = (A^{\otimes \ell})_{ij}$ for all i .

Now we use the fact that the max-plus-algebraic matrix power has the following graph-theoretic interpretation [1]: the value of $(A^{\otimes \ell})_{ij}$ with ℓ a positive integer corresponds the maximum weight of a path of length ℓ from vertex j to vertex i in the precedence graph $\mathcal{G}(A)$ of A , which is defined as follows: $\mathcal{G}(A)$ has n vertices and an arc with weight A_{ij} from vertex j to vertex i for every pair (i, j) such that $A_{ij} \neq \varepsilon$ (So $A_{ij} \neq \varepsilon$ indicates that there is no arc from vertex j to vertex i).

Let $i, j \in \{1, \dots, n\}$. Now we consider two cases: if there is no path of any length from vertex j to vertex i , then we have $(A^{\otimes \ell})_{ij} = \varepsilon$ for all ℓ and thus

$$\varepsilon = (A^{\otimes n+1} \otimes e_j)_i \leq (A^{\otimes n} \otimes e_j)_i \oplus \dots \oplus (A \otimes e_j)_i = \varepsilon \quad (40)$$

So in this case (39) holds. Now we consider the case that there is at least one path from vertex j to vertex i in $\mathcal{G}(A)$. Since $\mathcal{G}(A)$ has n vertices, we obtain — after the removal of any loops in the path, if present — a path of length ℓ with $1 \leq \ell \leq n$. So the right-hand side of (39) is different from ε . Let us denote the value of the right-hand side of (39) in this case by w_{\max} . If we now consider a path P of length $n+1$ from vertex j to vertex i , then this path has to contain at least one loop, as well as loop-free path from vertex j to

vertex i with a length between 1 and n . The maximal weight of the loop-free path will be less than or equal to w_{\max} , and due to the fact that the entries of A are less than or equal to zero, the weight of the loops is also less than or equal to zero, which implies that the weight of P is also less than or equal to w_{\max} . So (39) also holds in this case.

So now we have proven that (38) holds for the max-plus-algebraic unit vectors e_1, \dots, e_n . Since any vector $x \in \mathbb{R}_{\varepsilon}^n$ can be written as

$$x = \bigoplus_{i=1}^n x_i \otimes e_i$$

and since max-plus-algebraic addition and multiplication are monotonous [1], (38) also holds. \blacksquare

Lemma 6: Let $x \in \mathbb{R}_{\varepsilon}^n$ and $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ with $A_{ij} \leq 0$ for all i, j . Then we have

$$A^{\otimes l} \otimes x \leq A^{\otimes l-1} \otimes x \oplus A^{\otimes l-2} \otimes x \oplus \dots \oplus A \otimes x \quad (41)$$

for any integer $l \geq n$.

Proof: From Lemma 5 it follows that

$$A^{\otimes l} \otimes x \leq A^{\otimes l-1} \otimes x \oplus A^{\otimes l-2} \otimes x \oplus \dots \oplus A^{\otimes l-n} \otimes x \quad (42)$$

Using the implication $w \leq v \implies w \leq v \oplus z$ it immediate follows that

$$\begin{aligned} A^{\otimes l} \otimes x &\leq \left(A^{\otimes l-1} \otimes x \oplus A^{\otimes l-2} \otimes x \oplus \dots \oplus A^{\otimes l-n} \otimes x \right) \\ &\oplus \left(A^{\otimes l-n-1} \otimes x \oplus A^{\otimes l-n-2} \otimes x \oplus \dots \oplus A \otimes x \right) \end{aligned} \quad (43)$$

Theorem 7: Let a normalized MPL system (24)-(25) be controllable. For every event step k we compute the optimal input sequence by solving (30)-(31) and we apply only $u(k) = u(k|k)$. Let $N_p \geq n$ and $0 < \beta < 1/N_p$. Define the function

$$V(k) = N_p \max_{j \in \{0, 1, \dots, N_p-1\}} (y(k+j|k), 0) \quad (44)$$

There holds:

$$V(k) \geq 0 \quad , \quad V(k+1) \leq V(k) \quad (45)$$

Furthermore, we have

$$V(k) \geq J(k) \quad (46)$$

Together with the fact that $J(k) \geq 0$, this means that the closed-loop system is stable. \diamond

Proof: First we prove (46):

$$\begin{aligned} V(k) &= N_p \max_{j \in \{0, 1, \dots, N_p-1\}} \max(y(k+j|k), 0) \\ &\geq \sum_{j=0}^{N_p-1} \max(y(k+j|k), 0) \end{aligned}$$

$$\begin{aligned} &\geq \sum_{j=0}^{N_p-1} \left(\max(y(k+j|k), 0) - \beta u(k+j|k) \right) \\ &= J(k) \end{aligned}$$

where we have used the fact that $u(k+j|k) \geq 0$ for all $j = 0, 1, \dots, N_p - 1$.

Next we prove (45). First of all, note that by definition (44) we have that $V(k) \geq 0$. The next step is to prove $V(k+1) \leq V(k)$:

Consider

$$V(k+1) = N_p \max_{j \in \{1, \dots, N_p\}} (y(k+j|k+1), 0)$$

We will first prove that $y(k+j|k+1) = y(k+j|k)$ for $j = 1, \dots, N_p - 1$. With $x(k|k+1) = x(k|k)$ we can easily observe that $z(k+j|k+1) = z(k+j|k)$ and so according to Proposition 4 we have $y(k+j|k+1) = y(k+j|k)$. To prove that $V(k+1) \leq V(k)$ we have to prove:

$$y(k+N_p|k+1) \leq \max_{j \in \{0, \dots, N_p-1\}} y(k+j|k)$$

First note that at event step $k+1$ the signals u^b and z for $j \geq 1$ are given by

$$\begin{aligned} u^b(k+j|k+1) &= (u(k) - \rho j) \oplus 0 \\ z(k+j|k+1) &= 0 \oplus C \otimes A^{\otimes j} \otimes x(k) \\ &\oplus \bigoplus_{i=1}^j C \otimes A^{\otimes j-i} \otimes B \otimes u^b(k+i|k). \end{aligned}$$

Now define

$$\begin{aligned} y_1(k+N_p|k+1) &= C \otimes A^{\otimes N_p+1} \otimes x(k-1) \\ y_2(k+N_p|k+1) &= \bigoplus_{j=0}^{N_p} C \otimes A^{\otimes N_p+1} \otimes B \otimes u(k|k) \\ &\oplus \bigoplus_{j=1}^{N_p} C \otimes A^{\otimes N_p-j} \\ &\otimes B \otimes u^b(k+j|k+1) \end{aligned}$$

Using this we derive

$$\begin{aligned} 0 \oplus y_1(k+N_p|k+1) \oplus y_2(k+N_p|k+1) &= \\ &= 0 \oplus C \otimes A^{\otimes N_p+1} \otimes x(k-1) \\ &\oplus C \otimes A^{\otimes N_p+1} \otimes B \otimes u(k|k) \\ &\oplus \bigoplus_{j=1}^{N_p} C \otimes A^{\otimes N_p-j} \\ &\otimes B \otimes u^b(k+j|k+1) \\ &= 0 \oplus C \otimes A^{\otimes N_p} \otimes x(k) \\ &\oplus \bigoplus_{j=1}^{N_p} C \otimes A^{\otimes N_p-j} \otimes B \otimes u^b(k+j|k+1) \\ &= z(k+N_p|k+1) \\ &= y(k+N_p|k+1) \end{aligned}$$

From Lemma 6 we know that for all $x \in \mathbb{R}_\varepsilon^n$ we have

$$A^{\otimes N_p+1} \otimes x \leq A^{\otimes N_p} \otimes x \oplus A^{\otimes N_p-1} \otimes x \oplus \dots \oplus A \otimes x$$

since $N_p \geq n$, and so

$$\begin{aligned} C \otimes A^{\otimes N_p+1} \otimes x(k-1) &\leq C \otimes A^{\otimes N_p} \otimes x(k-1) \\ &\oplus C \otimes A^{\otimes N_p-1} \otimes x(k-1) \oplus \dots \oplus C \otimes A \otimes x(k-1) \end{aligned}$$

This results in

$$y_1(k+N_p|k+1) \leq \max_{j \in \{0, \dots, N_p-1\}} y_1(k+j|k+1)$$

for $j = 0, \dots, N_p$. With $y(k+j|k) \leq C \otimes A^{\otimes j+1} \otimes x(k-1)$ for $j = 0, \dots, N_p$ we derive

$$\begin{aligned} y_1(k+N_p|k+1) &\leq \max_{j \in \{0, \dots, N_p-1\}} y_1(k+j|k+1) \\ &\leq \max_{j \in \{0, \dots, N_p-1\}} y(k+j|k+1) \end{aligned}$$

for $j = 0, \dots, N_p$. Using $y(k+j|k) = z(k+j|k) \geq 0$ we find

$$0 \oplus y_1(k+N_p|k+1) \leq \max_{j \in \{0, \dots, N_p-1\}} y(k+j|k)$$

Further we know that for $j = 0, \dots, N_p$.

$$u^b(k+j|k+1) = (u(k) - \rho j) \oplus 0 \leq u(k+j|k)$$

This means that

$$\begin{aligned} C \otimes A^{\otimes N_p-j} \otimes B \otimes u^b(k+j|k+1) &\leq C \otimes A^{\otimes N_p-j} \otimes B \otimes u(k+j|k) \\ &\leq y(k+j|k) \end{aligned}$$

for $j = 0, \dots, N_p$. This results in

$$y_2(k+N_p|k+1) \leq \max_{j \in \{0, \dots, N_p-1\}} y(k+j|k)$$

and so it follows:

$$\begin{aligned} y(k+N_p|k+1) &= 0 \oplus y_1(k+N_p|k+1) \oplus y_2(k+N_p|k+1) \\ &\leq \max_{j \in \{0, \dots, N_p-1\}} y(k+j|k) \end{aligned}$$

We now have that $V(k)$ will be non-increasing, and so the function $J(k)$ will be bounded. This implies that there exists an upper bound for $y(k)$, and that $u(k)$ will have both an upper and lower bound. With the property that $y(k) - u(k) \geq C \otimes B$ we also prove that $y(k)$ has a lower bound. The system is controllable, which means that if $u(k)$ has a lower bound, then $x(k)$ will have a lower bound. Due to the fact that $\lambda_{\max}(A) \leq 0$, we find that if the initial state $x(0)$ has an upper bound and $u(k)$ has an upper bound, then $x(k)$ will have an upper bound. This proves that the closed-loop system is stable. \blacksquare

VI. WORKED EXAMPLE

Consider an MPL system (1)-(2) with

$$A_{\text{sys}} = \begin{bmatrix} \varepsilon & 0 & \varepsilon & 9 \\ 4 & 3 & 4 & 5 \\ 8 & \varepsilon & 1 & 8 \\ 0 & 0 & \varepsilon & \varepsilon \end{bmatrix}, \quad B_{\text{sys}} = \begin{bmatrix} 0 \\ 5 \\ 2 \\ 8 \end{bmatrix},$$

$$C_{\text{sys}} = [7 \quad 5 \quad 8 \quad \varepsilon].$$

The matrix A_{sys} has an eigenvalue $\lambda = 5.25$. We choose $\rho = 6.3 > \lambda$. The normalized system is now given by:

$$A = \begin{bmatrix} \varepsilon & -4.6 & \varepsilon & 0 \\ -4 & -3.3 & -2.3 & -5.7 \\ 0 & \varepsilon & -5.3 & -2.7 \\ -3.6 & -1.9 & \varepsilon & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} -18.7 \\ -15.4 \\ -18.4 \\ -8 \end{bmatrix},$$

$$C = [5.3 \quad 5 \quad 8 \quad \varepsilon].$$

We have a lower bound $N_p \geq n = 4$, and an upper bound for $\beta < 1/N_p$ for stability (cf. Theorem 7). In a first simulation we choose $N_p = 4$ and $\beta = 0.2$. We obtain a stable operation and all buffers remain bounded. In a second simulation we choose $N_p = 2$ and $\beta = 0.2$, which means that the first condition is violated. In a third simulation we choose $N_p = 4$ and $\beta = 0.8$, which means that the second condition is violated. In both cases the closed loop will become unstable and the output signal $y_{\text{sys}}(k)$ will grow unboundedly. The evolution of output signal $y_{\text{sys}}(k)$ for the three simulations is given in Figure 1.

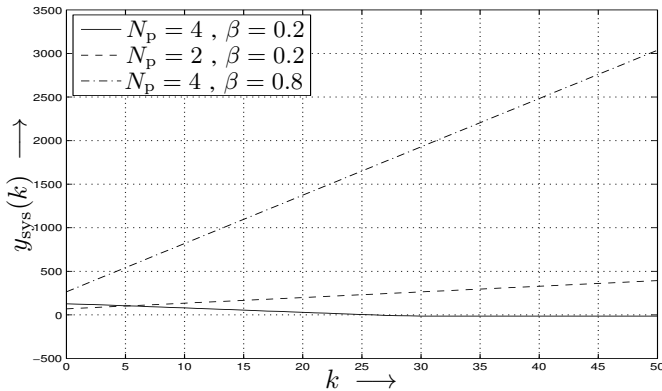


Fig. 1. $y_{\text{sys}}(k)$ for various values of λ and N_p

VII. DISCUSSION

Model predictive control (MPC) for max-plus-linear (MPL) systems is a practical approach to design optimal input sequences for a specific class of discrete event systems without concurrency or choice and in which only synchronization plays a role. In this paper we have studied the steady-state behavior and stability of unconstrained MPL-MPC in the case of an output cost criterion.

A discrete event system is called stable if all its buffer levels remain bounded. Therefore the steady-state properties of MPL systems have been considered in the case of a due

date sequence with a constant slope. We have shown that we can derive a Lyapunov function for the system and we have provided necessary conditions for stability.

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