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Model Predictive Traffic Control to Reduce Vehicular Emissions – An LPV-Based Approach

S. K. Zegeye, B. De Schutter, and J. Hellendoorn

***Abstract*— We propose a traffic control approach that can reduce both traffic emissions and travel times based on model predictive control (MPC). We approximate the traffic flow and emission models into a linear parameter varying (LPV) form, which leads to an LPV-MPC control approach. We consider two objective functions and formulate them as convex functions, so that convex optimization methods can be used to generate the optimal control sequences of the LPV-MPC control approach. The resulting LPV-MPC solutions can next be used as a good initial point in the nonlinear-nonconvex optimization of the nonlinear original MPC problem.**

I. Introduction

A potential control approach for traffic systems is Model Predictive Control (MPC) [12]. In this control approach models are used to predict the evolution of the traffic states and to provide a prediction of the total travel time, throughput, emissions, and fuel consumption. One of the typical traffic modeling categories is the class of the microscopic models. Microscopic traffic flow models are accurate, because they describe the dynamics of each vehicle in a traffic network. However, they require longer simulation time. Another traffic modeling categories is the class of macroscopic models. Such models treat the traffic flow as a compressible fluid and they require a low computation time [6]. However, emission models that use the output of such models are subject to larger estimation errors [1]. In order to improve the estimation error the macroscopic traffic flow models can be integrated with microscopic emission models [19].

The METANET [13] traffic flow model and the VT-macro [19] are employed in this paper. Since both the METANET traffic flow model and VT-macro emission model are nonlinear and nonconvex models, the resulting MPC problem boils down to a nonlinear-nonconvex optimization problem. Nonconvex optimization methods heavily depend on the initial points of the optimized variables, which are usually difficult to determine. But, there are nonlinear Model Predictive Control (MPC) approaches that are well developed for certain class of models [3], [7], [17]. One of them is Linear Parameter Varying (LPV) models.

In this paper, we use the LPV formulation of the METANET-like traffic flow model as described in [9], [10], [11]. We further extend it to explicitly describe metered on-ramp flow and node equations of the METANET [13] traffic flow model in an LPV framework. Moreover, we integrate the VT-macro [19] emission and fuel consumption model to the traffic flow model in an LPV form by adding new states, defining new outputs, and introducing approximations of the emission model. Finally, the resulting integrated traffic flow and emission LPV model is used in the design of an LPV-MPC controller.

Since the LPV-MPC control problem is formulated based on approximation of both the METANET [13] traffic flow model and the VT-macro [19] emission and fuel consumption model, the control solutions may not be optimal. Nevertheless, the control solutions from the LPV-MPC approach can be used as a good initial point for the nonlinear-nonconvex optimization of the original nonlinear MPC optimization problem.

II. Models

A. METANET

METANET [4], [8], [13] is a macroscopic second-order traffic flow model. The model describes the evolution of the traffic variables — the density, the flow, and the space-mean speed — as nonlinear difference equations. The METANET model is discrete in space and time. In this model, a node is placed at a point where there is a change in the geometry of a freeway (such as lane drop, on/off-ramp, and bifurcation). A homogeneous freeway that connects such nodes is called a link. Links are divided into equal segments of length 300-500 m. The equations that describe the traffic dynamics in a segment of a link are given by

$$q_{m,i}(k) = \lambda_m \rho_{m,i}(k) v_{m,i}(k) \quad (1)$$

$$\rho_{m,i}(k+1) = \rho_{m,i}(k) + \frac{T_s}{L_m \lambda_m}\left[q_{m,i-1}(k) - q_{m,i}(k)\right] \quad (2)$$

$$\begin{aligned} v_{m,i}(k+1) = {} & v_{m,i}(k) + \frac{T_s}{\tau}\left[V[\rho_{m,i}(k)] - v_{m,i}(k)\right] \\ & + \frac{T_s v_{m,i}(k)\left[v_{m,i-1}(k) - v_{m,i}(k)\right]}{L_m} \\ & - \frac{T_s \eta \left[\rho_{m,i+1}(k) - \rho_{m,i}(k)\right]}{\tau L_m \left(\rho_{m,i}(k) + \kappa\right)} \end{aligned} \quad (3)$$

$$V[\rho_{m,i}(k)] = v_{\text{free},m} \exp\left[-\frac{1}{b_m}\left(\frac{\rho_{m,i}(k)}{\rho_{\text{cr},m}}\right)^{b_m}\right] \quad (4)$$

where $q_{m,i}$, $\rho_{m,i}$, and $v_{m,i}$, denote respectively the flow, density, and space-mean speed of segment i of link m, L_m denotes the length of the segments of link m, λ_m denotes the number of lanes of link m, and T_s denotes the simulation time step. Furthermore, $\rho_{\text{cr},m}$ is the critical density, τ a time constant, η the anticipation constant, b_m the parameter of the fundamental diagram, and κ is a model parameter.

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For origins (such as on-ramps and mainstream entry points) a queue model is used. The dynamics of the queue length w_o at the origin o are modeled as

$$w_o(k+1) = w_o(k) + T_s(d_o(k) - q_o(k)) \tag{5}$$

where d_o and q_o denote respectively the demand and outflow of the origin o. The outflow q_o is given by

$$q_o(k) = \min\left[d_o(k) + \frac{w_o(k)}{T_s},\ r_o(k)C_o,\ C_o\left(\frac{\rho_{\text{jam},m} - \rho_{m,1}(k)}{\rho_{\text{jam},m} - \rho_{\text{cr},m}}\right)\right] \tag{6}$$

where $r_o(k) \in [0\ 1]$ for a metered on-ramp and $r_o(k) = 1$ for an unmetered on-ramp or mainstream origin, $\rho_{\text{jam},m}$ and $\rho_{\text{cr},m}$ are respectively the maximum and critical densities of link m, and C_o denotes the capacity of on-ramp or mainstream origin o.

If m is the link out of a node to which an on-ramp o is connected, then for the first segment of link m the term

$$-\frac{\delta T_s q_o(k) v_{m,1}(k)}{L_m \lambda_m (\rho_{m,1}(k) + \kappa)} \tag{7}$$

is added to (3) in order to account for the speed drop caused by the merging phenomena, where δ is model parameter. METANET can also include lane drops, merging lanes, off-ramps, and so on [5], [8], [13].

A node provides a downstream density to incoming links, and an upstream speed to leaving links. The flow that enters node n is distributed among the leaving links according to

$$Q_n(k) = \sum_{\mu \in I_n} q_{\mu,N_\mu}(k) \tag{8}$$

$$q_{m,0}(k) = \beta_{n,m}(k) Q_n(k) \tag{9}$$

where $Q_n(k)$ is the total flow that enters the node at simulation step k, I_n is the set of links that enter node n, $\beta_{n,m}(k)$ are the turning rates (i.e., the fraction of the total flow through node n that leaves via link m), and $q_{m,0}(k)$ is the flow that leaves node n via link m.

When node n has more than one leaving link, the virtual downstream density $\rho_{m,N_m+1}(k)$ of entering link m is given by

$$\rho_{m,N_m+1}(k) = \frac{\sum_{\mu \in O_n} \rho_{\mu,1}^2(k)}{\sum_{\mu \in O_n} \rho_{\mu,1}(k)} \tag{10}$$

where O_n is the set of links leaving node n.

When node n has more than one entering link, the virtual upstream speed $v_{m,0}(k)$ of leaving link m is given by

$$v_{m,0}(k) = \frac{\sum_{\mu \in I_n} v_{\mu,N_\mu}(k) q_{\mu,N_\mu}(k)}{\sum_{\mu \in I_n} q_{\mu,N_\mu}(k)}. \tag{11}$$

B. VT-macro

VT-macro [19] is a dynamic macroscopic emission and fuel consumption model that we have developed for integration with the METANET model. It provides estimates of the traffic emissions and fuel consumption based on the space-mean speed, the average acceleration, and the corresponding number of vehicles subject to the space-mean speed and average acceleration.

In this model a distinction is made between the temporal and spatial-temporal accelerations and number of vehicles subject to it [19]. The temporal acceleration and the corresponding number of vehicles are given by

$$a_{m,i}(k) = \frac{v_{m,i}(k) - v_{m,i}(k-1)}{T_s} \tag{12}$$

$$n_{m,i}(k) = L_m \lambda_m \rho_{m,i}(k) - T_s \lambda_m v_{m,i-1}(k-1) \rho_{m,i-1}(k-1). \tag{13}$$

On the other hand, the spatial-temporal acceleration is different for different freeway geometries [19]. For brevity, we only consider spatial-temporal accelerations of links and on-ramps here. The spatial-temporal acceleration and the number of vehicles corresponding to segment i of a link m are given by

$$a_{m,i,i+1}(k) = \frac{v_{m,i+1}(k) - v_{m,i}(k-1)}{T_s} \tag{14}$$

$$n_{m,i,i+1}(k) = T_s q_{m,i}(k-1), \tag{15}$$

while the spatial-temporal acceleration and number of vehicles for an on-ramp o are given by

$$a_o(k) = \frac{v_{m,i}(k) - v_o}{T_s} \tag{16}$$

$$n_o(k) = T_s q_o(k-1). \tag{17}$$

where v_o is the on-ramp speed.

Using the temporal and spatial-temporal components of the space-mean speed, acceleration, and number of vehicles, the VT-macro model is expressed as

$$\bar{J}_\gamma(k) = n_{\text{temp}}(k) \exp\left(\check{v}_{\text{temp}}^\top(k) P_\gamma \check{a}_{\text{temp}}(k)\right) + n_{\text{spat}}(k) \exp\left(\check{v}_{\text{spat}}^\top(k) P_\gamma \check{a}_{\text{spat}}(k)\right) \tag{18}$$

where $\bar{J}_\gamma(k)$ is the emission or fuel consumption $\gamma \in \mathscr{S} = \{\text{CO, HC, CO}_2\text{, NO}_x\text{, Fuel Consumption}\}$ during the time period $[kT_s, (k+1)T_s]$, the subscripts 'temp' and 'spat' respectively are the shorthand representation of 'temporal' and 'spatial-temporal', $n_{(\cdot)}$ denotes the number of vehicles that are subject to the space-mean speed $v_{(\cdot)}$ and the average acceleration $a_{(\cdot)}$ with the speed vector $\check{v}_{(\cdot)}^\top(k) = [1\ v_{(\cdot)}(k)\ v_{(\cdot)}^2(k)\ v_{(\cdot)}^3(k)]$ and the acceleration vector $\check{a}_{(\cdot)}^\top(k) = [1\ a_{(\cdot)}(k)\ a_{(\cdot)}^2(k)\ a_{(\cdot)}^3(k)]$, and P_γ denotes the model parameter. The values of the parameter matrices P_γ can be found in [2].

C. LPV formulation of METANET

An LPV model of a system can be considered as a weighted sum of a set of Linear Time Invariant (LTI) models, where the weighting is determined by scheduling parameters that are known a priori or that can be estimated [16], [18].

In general a discrete-time LPV system can be written

$$x(k+1) = \sum_{i=0}^{n_p} p_i(k)\left(A_i x(k) + B_i u(k)\right) \quad (19)$$

$$y(k) = \sum_{i=0}^{n_p} p_i(k)\left(C_i x(k) + D_i u(k)\right) \quad (20)$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ are respectively the state vector, the input vector, and the output vector with $n_x, n_y, n_u, n_p \in \mathbb{Z}^+$, p_i denotes the scheduling parameter, and A_i, B_i, C_i, and D_i are the system matrices.

In order to design a traffic controller based on an LPV-MPC approach, the METANET model has to be transformed into an LPV form. The exact and approximate LPV transformation of the METANET model have been developed in [11]. The exact LPV model in [11] has four scheduling variables for every segment i of a link m. The number of the scheduling variables increases with the number of segments. Hence, in order to avoid the computational problems in the LMI solvers, we use the approximate LPV model proposed in [11]. The approximate LPV model reduces the number of scheduling variables to two for every segment of i of link m. These are:

$$p_{m,i}^{(1)}(k) = \tilde{v}_{m,i}(k) \quad (21)$$

$$p_{m,i}^{(2)}(k) = \frac{T_s v_{\text{free},m}}{\tau \tilde{\rho}_{m,i}(k)} \exp\left(-\frac{1}{b_m}\left(\frac{\tilde{\rho}_{m,i}(k) + \rho_{m,i}^*}{\rho_{\text{cr},m}}\right)^{b_m}\right) \quad (22)$$

where $\tilde{v}_{m,i} = v_{m,i} - v_{m,i}^*$ and $\tilde{\rho}_{m,i} = \rho_{m,i} - \rho_{m,i}^*$ with $v_{m,i}^*$ and $\rho_{m,i}^*$ respectively are steady-state[1] values of the space-mean speed and the density.

By using the relation in (1), linear state transformation described by the $\tilde{x} = x - x^*$, and the scheduling variables in (21) and (22), one can transform (1)-(4) into

$$\tilde{\rho}_{m,i}(k+1) = \tilde{\rho}_{m,i}(k) + \frac{T_s}{L_m}\tilde{q}_{m,i-1} - \frac{T_s}{L_m}\Big[p_{m,i}^{(1)}(k)\tilde{\rho}_{m,i}(k) + \tilde{v}_{m,i}(k)\rho_{m,i}^* + v_{m,i}^*\tilde{\rho}_{m,i}(k) + v_{m,i}^*\rho_{m,i}^*\Big] \quad (23)$$

$$\begin{aligned}\tilde{v}_{m,i}(k+1) = &\tilde{v}_{m,i}(k) - \frac{T_s}{L_m}\tilde{v}_{m,i}(k) + p_{m,i}^{(2)}(k)\tilde{\rho}_{m,i}(k)\\ &+ \frac{T_s}{L_m} v_{m,i}^*\left(\tilde{v}_{m,i-1}(k) - \tilde{v}_{m,i}(k)\right)\\ &+ \frac{T_s}{L_m} p_{m,i}^{(1)}(k)\left(\tilde{v}_{m,i-1}(k) - \tilde{v}_{m,i}(k)\right)\\ &+ \frac{\nu T_s\left(\tilde{\rho}_{m,i+1}(k) - \tilde{\rho}_{m,i}(k)\right)}{\rho_{\text{cr},m} + \kappa}.\end{aligned} \quad (24)$$

These transformed equations can be rewritten into LPV state-space representation as

$$x_{m,i}(k+1) = \sum_{\ell=0}^{2} p_{m,i}^{(\ell)}(k)\left(A_{m,i}^{(\ell)} x_{m,i}(k) + G_{m,i}^{(\ell)} u_G(k)\right) + H_0 \quad (25)$$

[1]The values of $v_{m,i}^*$ and $\rho_{m,i}^*$ are determined by setting $\rho_{m,i}(k+1) = \rho_{m,i}(k) = \rho_{m,i}^*$, $v_{m,i}(k+1) = v_{m,i}(k) = v_{m,i}^*$, $w_o(k+1) = w_o(k) = w_o^*$, etc. in the equations of the METANET model for each segment and each node as well as in the boundary conditions and solving the resulting system of equations.

where $x_{m,i} = [\tilde{\rho}_{m,i}\ \tilde{v}_{m,i}]^\top$ is the state vector, $u_G = [\tilde{\rho}_{m,i+1}\ \tilde{q}_{m,i-1}\ \tilde{v}_{m,i-1}]^\top$ is an exogenous input vector, $p_{m,i}^{(0)} = 1$, $p_{m,i}^{(1)} =$(21), $p_{m,i}^{(2)} =$(22), with appropriately defined system matrices $A_{m,i}^{(\ell)}$, $G_{m,i}^{(\ell)}$, and H_0.

III. LPV EXTENSION AND INTEGRATION

A. *Metered on-ramp LPV modeling*

The LPV formulation in Section II-C does not explicitly model an on-ramp flow and the queue model in the LPV model. In this section we consider these separately and show how they can be rewritten into LPV from.

Due to the speed drop in (7) and the change in the inflow of (2) for $i = 1$, the LPV description of a segment with an on-ramp is slightly different from (25). To reformulate the on-ramp flow into an LPV form, we first convert the min function of the on-ramp flow in (6) into a linear description. Next we define new states and then formulate the on-ramp flow as an LPV form.

The on-ramp flow in (6) is the minimum of three quantities: the flow $q_{\text{want}}(k)$ of vehicles that want to enter the freeway, the maximum flow $q_{\text{control}}(k)$ determined by the controller (which is equal to C_o if there is no ramp metering present), and the maximal flow $q_{\text{space}}(k)$ determined by the available space in the freeway. These three quantities are analyzed and reformulated into linear inequalities as follows.

The flow of the on-ramp in (6) can be equivalently recast to

$$q_o(k) = \min\Bigg[\underbrace{d_o(k) + \frac{w_o(k)}{T_s}}_{q_{\text{want}}(k)},\ C_o \min\Bigg(\underbrace{r_o(k)}_{\frac{q_{\text{control}}(k)}{C_o}},\ \underbrace{\frac{\rho_{\text{jam},m} - \rho_{m,1}(k)}{\rho_{\text{jam},m} - \rho_{\text{cr},m}}}_{\frac{q_{\text{space}}(k)}{C_o}}\Bigg)\Bigg]. \quad (26)$$

For simplicity of notation, let us rewrite the above on-ramp flow equation as

$$q_o(k) = \min\left\{q_{\text{want}}(k),\ \bar{q}_{r_o}(k)\right\} \quad (27)$$

where $\bar{q}_{r_o}(k) = \min\left\{q_{\text{control}}(k),\ q_{\text{space}}(k)\right\}$. According to (26) a ramp controller can generate $q_{\text{control}}(k) > q_{\text{space}}(k)$, but in the end the flow that can enter to the link is determined by the available space in the segment. Hence, we can simply restrict the controller to only generate flows that do not exceed the available space. In other words, the ramp metering rate $r_o(k)$ has to be designed in such a way that $q_{\text{control}}(k) \leq q_{\text{space}}(k)$ is satisfied. Since $r_o(k) \in [0\ 1]$, this is equivalent to stating that

$$\begin{bmatrix}-1\\1\\1\end{bmatrix} r_o(k) \leq \begin{bmatrix}0\\1\\\frac{\rho_{\text{jam},m}}{\rho_{\text{jam},m}-\rho_{\text{cr},m}}\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{-1}{\rho_{\text{jam},m}-\rho_{\text{cr},m}}\end{bmatrix}\rho_{m,1}(k). \quad (28)$$

Then, we can reformulate $\bar{q}_{r_o}(k)$ in (27) as $\bar{q}_{r_o}(k) = C_o r_o(k)$ with $r_o(k)$ is subject to the linear inequality (28). Hence, we have $q_o(k) = \min\{q_{\text{want}}(k),\ C_o r_o(k)\}$ subject to (28).

The term $q_{\text{want}}(k) = d_o(k) + \frac{w_o(k)}{T_s}$ determines the flow of available vehicles that want to enter the network if there is enough capacity. This limits the maximum amount of vehicles the controller can "ask" for, i.e., it imposes a constraint such that $q_{\text{control}}(k) \le q_{\text{want}}(k)$. In fact, this prevents the on-ramp queue $w_o(k+1)$ in (5) from becoming negative. But, we can also impose non-negativity constraints on the queue length to avoid having to use the min function. This leaves us with an on-ramp flow equation that is equal to

$$q_o(k) = C_o r_o(k) \tag{29}$$

where $r_o(k)$ satisfies (28) and $w_o(k+1) \ge 0$.

Now using the relations given in (7) and (29), applying the linear state transformation on q_o and $v_{m,1}$, setting $i = 1$ for the LPV model of a segment without an on-ramp in (25), and increasing the $\tilde{\rho}_{m,1}(k+1)$ by $\frac{T_s(\tilde{q}_o(k)+q_o^*)}{L_m \lambda_m}$ due to the on-ramp flow, an extra term

$$A_r^{(0)} x_r(k) + (B_r^{(0)} + p_{m,1}^{(1)}(k) B_r^{(1)}) \tilde{r}_o(k) + H_r + G_r^{(0)} u_r(k) \tag{30}$$

is added to (25) for $i = 1$, with $x_r(k) = [\tilde{\rho}_{m,1}(k) \;\; \tilde{v}_{m,1}(k) \;\; \tilde{w}_o(k)]^\top$, $u_r(k) = [d_o(k) \; q_o(k)]^\top$ and with appropriate matrices $A_r^{(0)}$, $B_r^{(0)}$, $B_r^{(1)}$, H_r, and $G_r^{(0)}$, and $p_{m,1}^{(1)}(k) = \tilde{v}_{m,1}(k)$.

B. LPV formulation of node equations

When the turning rate $\beta_{n,m}(k)$ in (9) is approximated by a constant[2] $\beta_{n,m}$, which is the average over a certain time window, one can model the flow that leaves a node n in an LPV form as

$$\tilde{q}_{m,0}(k) = \beta_{n,m} \sum_{\mu \in I_n} \lambda_\mu p_{\mu,N_\mu}^{(1)}(k) \tilde{\rho}_{\mu,N_\mu}(k) \tag{31}$$

where $p_{\mu,N_\mu}^{(1)}(k) = \tilde{v}_{\mu,N_\mu}(k)$.

The node equations corresponding to the virtual density in (10) and (11), are also nonlinear in the state variables. Therefore, we approximate (10) as

$$\tilde{\rho}_{m,N_m+1}(k) = \frac{\sum_{\mu \in O_n} \rho_{\mu,1}^* \tilde{\rho}_{\mu,1}(k)}{\sum_{\mu \in O_n} \rho_{\mu,1}^*} \tag{32}$$

by introducing steady-state values $\rho_{\mu,1}^*$. Moreover, we approximate (11) as

$$\tilde{v}_{m,0}(k) = \frac{\sum_{\mu \in I_n} \lambda_\mu v_{\mu,N_\mu}^* \rho_{\mu,N_\mu}^* \tilde{v}_{\mu,N_\mu}(k)}{\sum_{\mu \in I_n} \lambda_\mu v_{\mu,N_\mu}^* \rho_{\mu,,N_\mu}^*} \tag{33}$$

by introducing additional steady-state values v_{μ,N_μ}^* and ρ_{μ,N_μ}^*. These linear formulations, i.e., (31), (32), and (33) describe a way of combining the LPV models of different links.

[2]The time dependent $\beta_{n,m}(k)$ can also be considered as a scheduling variable. But, to reduce the computation time (which increases exponentially with the number of scheduling variables) we take the constant approximation of the $\beta_{n,m}(k)$.

C. State augmentation

The VT-macro model is a nonlinear function of the states of the LPV formulation of the METANET model described in (25) and (30). In general, an increased number of scheduling variables increases the computation time exponentially. So, in order to avoid an increase in the number of the scheduling variables of the LPV formulation of the METANET, we first define new memory states. Then, we define new output variables of the LPV model. Finally, we define an objective function that can be optimized using LMIs.

1) Temporal variables: The LPV formulation of the METANET model cannot capture the $\tilde{v}_{m,i-1}(k-1)$ and $\tilde{\rho}_{m,i-1}(k-1)$ in (12) and (13). Moreover, an additional scheduling variable should be introduced due to (13). In order to solve these two issues we add two new states to the LPV model as

$$\tilde{q}_{\text{ms},m,i}(k+1) = \lambda_m \tilde{v}_{m,i}(k) \tilde{\rho}_{m,i}(k), \tag{34}$$

$$\tilde{v}_{\text{ms},m,i}(k+1) = \tilde{v}_{m,i}(k). \tag{35}$$

With the newly added states, (12) and (13) become

$$\tilde{a}_{m,i}(k) = \frac{\tilde{v}_{m,i}(k) - \tilde{v}_{\text{ms},m,i}(k)}{T_s} \tag{36}$$

$$\tilde{n}_{m,i}(k) = L_m \lambda_m \tilde{\rho}_{m,i}(k) - T_s \tilde{q}_{\text{ms},m,i-1}(k) \tag{37}$$

where $\tilde{q}_{\text{ms},m,i-1}(k) = \tilde{q}_{m,i-1}(k)$ for $i = 1$, and (34) for $i > 1$.

Now (36) and (37) are linear with respect to the states or external inputs of the LPV model.

2) Spatial variables: Using the newly introduced states in (34) and (35) the spatial acceleration from one segment to another segment of a link and the number of vehicles subject to it given in (14) and (15) are formulated as

$$\tilde{a}_{m,i,i+1}(k) = \frac{\tilde{v}_{m,i+1}(k) - \tilde{v}_{\text{ms},m,i}(k)}{T_s} \tag{38}$$

$$\tilde{n}_{m,i,i+1}(k) = T_s \tilde{q}_{\text{ms},m,i}(k). \tag{39}$$

In order to recast the on-ramp spatial variables in (16) and (17), we once more define two extra memory states $\tilde{v}_{\text{ms},o}(k+1) = \tilde{v}_o$ and $\tilde{q}_{\text{ms},o}(k+1) = \tilde{q}_o(k)$. Thus, the spatial acceleration in (16) and the number of vehicles in (17) are formulated as linear function of the states as

$$\tilde{a}_o(k) = \frac{\tilde{v}_{m,i}(k) - \tilde{v}_{\text{ms},o}(k)}{T_s} \tag{40}$$

$$\tilde{n}_o(k) = T_s \tilde{q}_{\text{ms},o}(k). \tag{41}$$

Therefore, the state vector of the LPV model for a segment i of a link is[3] $\tilde{x}_{m,i} = [\tilde{\rho}_{m,i} \;\; \tilde{v}_{m,i} \;\; \tilde{q}_{\text{ms},m,i} \;\; \tilde{v}_{\text{ms},m,i}]^\top$ and the state vector for a segment $i = 1$ with an on-ramp is $\tilde{x}_{m,1} = [\tilde{\rho}_{m,1} \;\; \tilde{v}_{m,1} \;\; \tilde{q}_{\text{ms},m,1} \;\; \tilde{v}_{\text{ms},m,1} \;\; \tilde{w}_o \;\; \tilde{q}_{\text{ms},o} \;\; \tilde{v}_{\text{ms},o}]^\top$.

Accordingly, the system matrices in (25) and (30) are changed appropriately to adopt these new states. Therefore, the LPV model of a freeway can be formulated by combining the states depending on the nature of the segments (i.e., depending on whether a segment has an on-ramp or not).

[3]The independent simulation step k is removed for the sake of brevity.

D. Output formulation

Since in the LPV-MPC formulation we want the objective to be a function of the states and the output; since we need the space-mean speed, density, queue length, acceleration, and number of vehicles in the expression for the total time spent and the emission; and since space-mean speed, density, and queue length are already the state variables, the output of the LPV model should only contain the acceleration and the number of vehicles. Thus, we have

$$\tilde{y}_{m,i}(k) = C_{m,i}\tilde{x}_{m,i}(k) + D_{m,i}\tilde{u}_{m,i}(k) \tag{42}$$

where $\tilde{y}_{m,i} = [\tilde{a}_{m,i} \;\; \tilde{n}_{m,i} \;\; \tilde{a}_{m,i,i+1} \;\; \tilde{n}_{m,i,i+1}]^\top$ and $\tilde{u}_{m,i} = [\tilde{q}_{\mathrm{ms},i-1} \;\; \tilde{v}_o]^\top$ for a segment i without an on-ramp, and $\tilde{y}_{m,i} = [\tilde{a}_{m,1} \;\; \tilde{n}_{m,1} \;\; \tilde{a}_{m,1,2} \;\; \tilde{n}_{m,1,2} \;\; \tilde{a}_o \;\; \tilde{n}_o]^\top$ and $\tilde{u}_{m,i} = \tilde{q}_{\mathrm{ms},0}$ for a segment $i = 1$ with an on-ramp, and with appropriately defined matrices $C_{m,i}$ and $D_{m,i}$.

IV. Optimizing Total Time Spent

One performance function that we can consider is the total time spent of all the vehicles in a traffic network. The total time spent (TTS) is given by [5]

$$J_{\mathrm{TTS}} = T_{\mathrm{s}} \sum_{k=0}^{N_{\mathrm{sim}}} \left(\sum_{(m,i)\in I_{\mathrm{all}}} L_m \lambda_m \rho_{m,i}(k) + \sum_{o\in O_{\mathrm{all}}} w_o(k) \right) \tag{43}$$

where N_{sim} is the simulation time, I_{all} is the set of pairs of indexes (m,i) of all links and segments in the network, and O_{all} is the set of indexes of all origins.

The TTS is linear and convex in the original state variables $\rho_{m,i}(k)$ and $w_{o,m}(k)$. Since the $\rho_{m,i}(k) = \tilde{\rho}_{m,i}(k) + \rho^*_{m,i}$ and $w_o(k) = \tilde{w}_o(k) + w^*_o$, the TTS remains linear and convex in the linearly transformed variables $\tilde{\rho}_{m,i}(k)$ and $\tilde{w}_o(k)$ of the LPV model. Now we discuss how the minimization of the TTS for the METANET LPV model can be transformed into an LMI problem.

The objective of the MPC controller is to reduce the TTS over the prediction horizon N_{p}, i.e.

$$J^{\mathrm{MPC}}_{\mathrm{TTS}}(k) = \sum_{j=0}^{N_{\mathrm{p}}-1} \left(\sum_{(m,i)\in I_{\mathrm{all}}} L_m \lambda_m \rho_{m,i}(k+j) + \sum_{o\in O_{\mathrm{all}}} w_o(k+j) \right). \tag{44}$$

The MPC problem subject to the METANET model is a nonlinear-nonconvex optimization problem [5]. But, by using the approximate LPV model the minimization of the TTS can be turned into convex optimization problem subject to linear matrix inequalities (LMIs).

We propose a state feedback control law

$$u(k+i) = Kx(k+i), \text{ for } i = 0,1,2,\ldots,N_{\mathrm{p}}-1 \tag{45}$$

where K is the state feedback gain.

Moreover, naturally the state variables of the LPV traffic flow model are bounded due to physical limitations of the system. This implies that the system is stable. Then we can find matrices P and K such that

$$\begin{gathered} \Phi_K(p(k))^\top P \Phi_K(p(k)) \prec P,\ \forall k \in \mathbb{Z}^+, \\ K^\top K \preceq \frac{\tilde{r}^2_{\mathrm{o,max}}}{4} P \end{gathered} \tag{46}$$

where $P = P^\top \succ 0$, $P \in \mathbb{R}^{n\times n}$, $\Phi(p(k)) = A(p(k)) + B(p(k))K$, $\tilde{r}_{\mathrm{o,max}} = 1 + r^*_{\mathrm{o}}$, and $A(\cdot)$ and $B(\cdot)$ are appropriate matrices that describe (30).

Thus, the resulting LPV-MPC problem is to determine a state feedback gain K such that the TTS in (44) is minimized. This can be done by formulating the state feedback control law (45) and (46), the minimization of the objective function $J^{\mathrm{MPC}}_{\mathrm{TTS}}$, the LPV model, and the state and input constraints into a minimization of a scalar variable subjected to LMIs (this reformulation can be performed using the methods of [7], [17]).

The solution of the resulting LPV-MPC problem, i.e. $u(0)$, $u(1)$, $\ldots$, $u(N_{\mathrm{p}}-1)$ obtained using the approximated LPV model can be used as a good initial point for the original nonlinear-nonconvex MPC problem.

V. Optimizing Total Time Spent and Vehicular Emissions

Another objective function we can consider is reduction of vehicular emission while improving the traffic flow. This requires the inclusion of the total emission (TE) in the objective function of the LPV-MPC problem. So, one way to define the control objective is as follows

$$J_{\mathrm{con}} = \zeta_1 \frac{\mathrm{TTS}}{\mathrm{TTS}_{\mathrm{nom}}} + \zeta_2 \frac{\mathrm{TE}}{\mathrm{TE}_{\mathrm{nom}}} \tag{47}$$

where TE is the total emissions computed using (18), ζ_j for $j = 1,2$ denotes the weighting factor, and $\mathrm{TTS}_{\mathrm{nom}}$ and $\mathrm{TE}_{\mathrm{nom}}$ are respectively the nominal values of the TTS and TE.

In general, however, due to the nonlinearity of (18), J_{con} in (47) is a nonlinear and nonconvex function, implying the advantages of the LPV modeling effort cannot be exploited. In other words, we cannot use LMI solvers to compute the optimal control inputs nor other convex optimization tools. But, the VT-macro emission model for CO, HC, and CO_2 emissions and fuel consumption can be approximated by a convex function in the operating region of the model [14].

We therefore propose an approximation of the emission plots by $\exp(f(v,a))$, where

$$f(v,a) = c_0 + c_1 v + c_2 a + c_3 v^2 + c_4 a^2 + c_5 av \tag{48}$$

with $c_i \in \mathbb{R}$ for $i \in \{0,1,\ldots 5\}$, and

$$Q = \begin{bmatrix} c_3 & \frac{c_5}{2} \\ \frac{c_5}{2} & c_4 \end{bmatrix} \succeq 0$$

i.e., $c_3 \geq 0$ and $c_3 c_4 - \frac{c_5^2}{4} \geq 0$. Thus, if Q is positive semidefinite, then $c_3 v^2 + c_4 a^2 + c_5 av$ is convex, as well as $f(v,a)$, and thus also $\exp(f(v,a))$ [15]. Since $\rho_{m,i}(k) = \tilde{\rho}_{m,i}(k) + \rho^*_{m,i}$, $v_{m,i}(k) = \tilde{v}_{m,i}(k) + v^*_{m,i}$, and $q_{m,i}(k) = \tilde{q}_{m,i}(k) + q^*_{m,i}$ are linear in $\tilde{\rho}_{m,i}(k)$, $\tilde{v}_{m,i}(k)$, and $\tilde{q}_{m,i}(k)$, the convexity of the function $f(v,a)$ will be retained [15].

So we approximate the VT-macro emission and fuel consumption model in (18) by

$$\begin{aligned} \bar{J}_\gamma(k) \approx\, & n_{\mathrm{temp}}(k) \exp\left(f(v_{\mathrm{temp}}(k), a_{\mathrm{temp}}(k))\right) \\ & + n_{\mathrm{spat}}(k) \exp\left(f(v_{\mathrm{spat}}(k), a_{\mathrm{spat}})(k)\right) \end{aligned} \tag{49}$$

where $f(v_{\text{temp}}, a_{\text{spat}})$ is as defined in (48).

The function in (49) is a nonconvex function due to the multiplication of the $\exp(\cdot)$ part and other states ($n_{\text{temp}}(k)$ and $n_{\text{spat}}(k)$). But, for a short prediction horizon N_{p} of an LPV-MPC control approach, one can approximate both $n_{\text{temp},i}(k)$ and $n_{\text{spat},i}(k)$ by constants $\bar{n}_{\text{temp},i}$ and $\bar{n}_{\text{spat},i}$ respectively. In this way the function in (49) can be approximated by a convex function J_{convex}.

The actual fit of (49) could be performed using a nonlinear constrained least squares optimization approach. Note that as this only has to be done once (for HC, CO, CO_2, and fuel consumption), we can argue that it is worthwhile to invest computation time in this. We could even use the original data and fit the proposed function on it [2].

Once again, using the approximate convex objective function J_{convex}, the LMI constraints, and the LPV model, one can formulate a LPV-MPC problem as described in the previous section.

VI. LPV-MPC

The LPV system is subject to linear inequality constraints (input and state). Moreover, we have defined two convex objective functions. The TTS is convex and holds for any predictions horizon. The objective of reducing emissions along with the TTS can be approximated by a convex function (at least for short prediction horizons). Hence, the MPC control problem based on the LPV formulation boils down to a convex optimization problem, which can be solved either using LMI solvers or other convex optimization tools.

However, since numerous approximations are introduced in the process of LPV formulation, the control solutions may not be optimal. Therefore, we propose to use the LPV-based MPC to quickly and efficiently get a good initial point for the original nonlinear-nonconvex MPC problem.

VII. Conclusions and Future Work

We have presented an LPV formulation of the traffic flow model METANET. In doing so, we have analyzed and simplified the nonlinear (metered) on-ramp flow and the node equations of the model. Finally, we have arrived at an LPV formulation of the model with linear constraints on the system states and control input. Moreover, by introducing new memory states we formulated the output of the LPV traffic flow model in such a way that they can be used as inputs to the emission and fuel consumption model. In this way we kept the number of the scheduling variables unchanged. Further, we have defined two objective functions that describe the total time spent only and total time spent and emissions together. Moreover, the objective functions describing the total time spent and the emissions have been analyzed and simplified to a convex formulation. Finally, we have briefly discussed on how an LPV-MPC approach can be used to quickly and efficiently provide good initial points for the original nonlinear-nonconvex MPC problem.

In our future work, we will consider extensive case studies, extensive comparison of LPV-MPC and other control methods (either applied directly or for providing good initial points), and extend the approach in [17].

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