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Stability bounds for fuzzy estimation and control – Part I: State estimation

Zs. Lendek, R. Babuška, and B. De Schutter

***Abstract*— Analysis and observer design for nonlinear systems have long been investigated, but no generally applicable methods exist as yet. A large class of nonlinear systems can be well approximated by Takagi-Sugeno fuzzy models, for which methods and algorithms have been developed to analyze their stability and to design observers. However, results obtained for Takagi-Sugeno fuzzy models are in general not directly applicable to the original nonlinear system. In this paper, we investigate what conclusions can be drawn and what guarantees can be expected when an observer is designed based on an approximate fuzzy model and applied to the original nonlinear system. It is shown that in general, exponential stability of the estimation error dynamics cannot be obtained. However, the estimation error is bounded. This bound is computed based on the approximation error and the Lyapunov function used. The results are illustrated using simulation examples.**

I. Introduction

A large class of nonlinear functions can be exactly represented or accurately approximated by Takagi-Sugeno (TS) fuzzy models [16]. A well-known method to obtain an exact fuzzy representation of a nonlinear system is the sector nonlinearity approach [13]. However, when using this method, the observability of the local models is not guaranteed, even when the nonlinear system is observable. Although for fuzzy models well-established methods exist to analyze their stability or to design observers, these cannot be used if the local models are not stable or observable, respectively.

Therefore, in this paper we consider fuzzy models that retain observability in their local models, even though they only approximate the nonlinear system. Several methods exist to construct TS models such that they approximate a given nonlinear model to an arbitrary degree of accuracy [5], [10]. In this case, since the fuzzy model only approximates the original nonlinear system, when the analysis or design concerns the fuzzy model, the results may not directly hold true for the nonlinear system. For instance, the observers designed for the fuzzy model are in general not guaranteed to perform as expected for the nonlinear system.

In fuzzy control, the problem of model-plant mismatch has been addressed for specific types of fuzzy systems by the use of robust controllers [1]–[4], [6]–[8]. However, observer design and the contribution of the estimation error to stabilization using output-feedback is rarely discussed. In particular, the performance of the observer designed for the approximate model and then applied to the original nonlinear system has not been studied for the case when the scheduling variables themselves have to be estimated. Therefore, in this paper, we investigate whether and when conclusions can be drawn on the performance of a fuzzy observer designed based on an approximate fuzzy model and applied to the original nonlinear system that is approximated by the fuzzy model. To simplify the computations, a common quadratic Lyapunov function is used. Similar, although considerably more complex conditions can be derived if other Lyapunov functions or relaxed conditions are used.

The structure of the paper is as follows. Section II presents the models used and reviews some classic results for the stability of autonomous fuzzy systems. Section III investigates when the stability of a TS system implies the stability of the nonlinear system. These results serve as the basis for investigating the expected performance the observer designed for the fuzzy model and applied to the nonlinear system, which is studied and illustrated using examples in Section IV. The companion paper [11] continues this analysis for control design and output feedback control. Conclusions on the stability of the observers are drawn in Section V.

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II. Preliminaries

In this paper, we consider the following nonlinear system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})\end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the vector of the state variables, $\boldsymbol{u}$ is the input vector, $\boldsymbol{y}$ is the measurement vector. We assume that the variables are defined on a compact set $\mathcal{C}_{\boldsymbol{xuy}}$, i.e., $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}$. A TS fuzzy approximation of this system can be obtained (e.g., by linearization) as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\ \boldsymbol{y} &= \boldsymbol{h}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i)\end{aligned} \tag{2}$$

so that the approximation errors $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^\diamond$ and $\overline{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}^\diamond$ satisfy

$$\begin{aligned}\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x},\, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}} \\ \|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_h + \delta_h\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x},\, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}\end{aligned} \tag{3}$$

where σ_f, σ_h, δ_f, and δ_h are known nonnegative finite constants, and $\mathcal{C}_{\boldsymbol{xu}} = \{(\boldsymbol{x},\, \boldsymbol{u}) | \exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x},\, \boldsymbol{u},\, \boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}\}$. A bound similar to (3) is frequently used in controller design (see the companion paper [11]). In (2), A_i, B_i, C_i, D_i, a_i, and d_i, $i = 1, 2, \ldots, m$ represent the matrices and biases of the ith local linear model and w_i, $i = 1, 2, \ldots, m$ are the corresponding normalized membership functions, that depend on the scheduling variables $\boldsymbol{x}$, $\boldsymbol{u}$.

Throughout the paper it is assumed that the membership functions are normalized, i.e., $w_i(\boldsymbol{x}, \boldsymbol{u}) \geq 0$, $\sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u}) = 1$, $\forall(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}$. I and 0, respectively, denote the identity and the zero matrices of the appropriate dimensions, $\mathcal{H}(A)$ represents the Hermitian of the matrix A, i.e., $\mathcal{H}(A) = A + A^T$, and $\|.\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.

The nonlinear system (1) is now expressed as an uncertain TS system, given as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) + \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i) + \overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{4}$$

where the uncertainties $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ satisfy (3).

Note that the approximation error on a compact set of variables always satisfies

$$\begin{aligned} \|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma'_f \\ \|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma'_h \end{aligned} \tag{5}$$

for some σ'_f and σ'_h. However, as will be shown in the sequel, by using (3) whenever possible, less conservative conditions can be obtained.

Our results are based on the following conditions [17] for the stability of autonomous fuzzy systems:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z})A_i\boldsymbol{x} \tag{6}$$

where A_i, $i = 1, 2, \ldots, m$ represent the ith local linear model, w_i is the corresponding normalized membership function, and $\boldsymbol{z}$ the vector of the scheduling variables, which may depend on the states, input, output, or other measured exogenous variables.

Theorem 1: [17] System (6) *is exponentially stable if there exists* $P = P^T > 0$ *so that*

$$\mathcal{H}(PA_i) < 0 \tag{7}$$

for $i = 1, 2, \ldots, m$. □

Controller and observer design for fuzzy systems of the form (2) often leads to establishing the negative definiteness of double summations of the form $\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij}$, with Υ_{ij}, $i, j = 1, 2, \ldots m$ matrices of appropriate dimensions. In this paper we use the following relaxations for such sums [17]:

Theorem 2: Let Υ_{ij} *be matrices of proper dimensions. Then,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij} < 0 \tag{8}$$

holds, if

$$\begin{aligned} &\Upsilon_{ii} < 0 \qquad \text{for } i = 1, 2, \ldots, m, \\ &\frac{1}{2}(\Upsilon_{ij} + \Upsilon_{ji}) < 0, \qquad \text{for } i, j = 1, 2, \ldots, m, \ i \neq j \end{aligned} \tag{9}$$

Note that similar, although more complex results can also be derived using other types of Lyapunov functions, as long as using the derived conditions ensure the exponential stability of the TS system.

III. Stability analysis

Stability analysis of uncertain or perturbed nonlinear systems is in general investigated by using the Lyapunov function that establishes exponential stability of the nominal model for the uncertain system [9]. In this paper, we use a similar approach, i.e., the Lyapunov function that establishes stability of the fuzzy model is further used for the original nonlinear system.

For stability analysis, consider the *autonomous* nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{10}$$

that is approximated by the TS system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}^{\diamond}(\boldsymbol{x}) = \sum_{i=1}^{m} w_i(\boldsymbol{x})A_i\boldsymbol{x} \tag{11}$$

such that each local matrix A_i, $i = 1, 2, \ldots, m$ is stable and the approximation error $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^{\diamond}$ satisfies

$$\|\overline{\boldsymbol{f}}(\boldsymbol{x})\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall\boldsymbol{x} \tag{12}$$

where σ_f and δ_f are nonnegative finite constants. Consider the Lyapunov function $V = \boldsymbol{x}^T P\boldsymbol{x}$. If there exist $P = P^T > 0$, $Q = Q^T > 0$ so that the linear matrix inequality (LMI)

$$\mathcal{H}(PA_i) < -2Q, \quad i = 1, 2, \ldots, m \tag{13}$$

is satisfied, then, by applying the same Lyapunov function to the original nonlinear system (10), we obtain:

$$\begin{aligned} \dot{V} &= \boldsymbol{x}^T\mathcal{H}(P(\sum_{i=1}^{m} w_i(\boldsymbol{x})A_i\boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x}))) \\ &= \boldsymbol{x}^T\sum_{i=1}^{m} w_i(\boldsymbol{x})\mathcal{H}(PA_i)\boldsymbol{x} + 2\boldsymbol{x}^T P\overline{\boldsymbol{f}}(\boldsymbol{x}) \\ &\leq -2\lambda_{\min}(Q)\|\boldsymbol{x}\|^2 + 2\lambda_{\max}(P)\delta_f\|\boldsymbol{x}\|^2 + 2\lambda_{\max}(P)\sigma_f\|\boldsymbol{x}\| \\ &\leq -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)(1-\theta)\|\boldsymbol{x}\|^2 \\ &\quad - 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma_f) \end{aligned}$$

with $\theta \in (0, 1)$ arbitrarily chosen, and where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the eigenvalues with the smallest and largest absolute magnitude.

By analyzing the expression of $\dot{V}$, the following cases can be distinguished:

1) $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f < 0)$ or $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f > 0)$: no conclusion can be drawn;
2) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f = 0$: if the membership functions are sufficiently smooth, and $\boldsymbol{x} = 0$ is the only equilibrium point, based on LaSalle's

invariance principle and Barbalat's lemma [9], $\boldsymbol{x} = 0$ is a globally asymptotically stable equilibrium point of the nonlinear system (10). This result is in general obtained when adaptive fuzzy controllers are designed. In stability analysis of TS systems, this case is rarely encountered.

3) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f = 0$: the nonlinear system (10) has a globally exponentially stable equilibrium point in $\boldsymbol{x} = 0$. This result is found only if the approximation error is Lipschitz continuous in the states.
4) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f > 0$: the states of the nonlinear system (10) are uniformly ultimately bounded by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f} \frac{\sigma_f}{\theta}. \quad (14)$$

or

$$\gamma < \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f} \sigma_f. \quad (15)$$

As soon as the nonlinear system is approximated to a constant accuracy by the fuzzy model, i.e., the approximation error is bound by a constant, this result is obtained. Moreover, since in most cases only a constant upper bound on the approximation error can be determined, this is the most frequently found result.

The above presented cases are illustrated on the following example.

Example 1: Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -1.1 & x_1^2 - x_2 \\ 2x_2 & -4.1 + x_2^2 \end{pmatrix} \boldsymbol{x} \quad (16)$$

with the state variables $x_1, x_2 \in [-1, 1]$.

The system has one equilibrium point, $\boldsymbol{x} = 0$. This equilibrium point is asymptotically stable on the domain defined by $x_1, x_2 \in [-1, 1]$. The stability is provable with the Lyapunov function $V = \boldsymbol{x}^T\boldsymbol{x}$. Note that if the sector nonlinearity approach is used to obtain an exact TS representation of this system, one of the local matrices is $\begin{pmatrix} -1.1 & 2 \\ 2 & -3.1 \end{pmatrix}$, which has a positive eigenvalue 0.1361. Therefore, the stability of the so obtained TS model cannot be established.

A TS approximation of the system (16) can be obtained using the approach of [10]. Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1, x_2)|x_1, x_2 \in \{-1, 0, 1\}\}$. Therefore, 9 local models are obtained, and each one is asymptotically stable. Moreover, with this approximation we have the approximation error either $\|\overline{\boldsymbol{f}}\| \leq 0.58\|\boldsymbol{x}\|$, or $\|\overline{\boldsymbol{f}}\| \leq 0.53$.

If the bound $\|\overline{\boldsymbol{f}}\| \leq 0.58\|\boldsymbol{x}\|$ is used, with P and Q computed[1] as $P = \begin{pmatrix} 14.4874 & 0.0211 \\ 0.0211 & 7.2243 \end{pmatrix}$, and $Q = \begin{pmatrix} 9.7100 & -2.9048 \\ -2.9048 & 10.4225 \end{pmatrix}$ the exponential stability of the nonlinear system is proven (case 3).

If the approximation error bound $\|\overline{\boldsymbol{f}}\| \leq 0.53$ is used, with $P = \begin{pmatrix} 0.4945 & 0.0379 \\ 0.0379 & 0.2188 \end{pmatrix}$, $Q = 0.3920\,I$ the ultimate bound $\gamma = 1.0469$ is obtained (case 4). □

IV. STATE ESTIMATION

Design of estimators in the presence of model uncertainties is one of the most important issues in fault detection and identification. However, observer design as such for nonlinear systems using TS fuzzy models when the TS model is only an approximation and the guarantees that can be expected for the original nonlinear system are rarely discussed in the literature. It is important to note that in the context of robust output-feedback fuzzy control, observers are used. However, it is generally assumed that the controller compensates for or attenuates the estimation error resulting from the observer model–true system mismatch, without actually analyzing how this error affects the stability of the closed-loop system.

In this paper, we consider that an observer is designed based on a TS approximation of a given nonlinear system. This observer is afterwards applied to the nonlinear system. We investigate when and what guarantees can be expected on the convergence of the estimation error, in particular, when the scheduling vector of the TS model depends on unmeasured states.

Therefore, consider the nonlinear system (1), with the approximation:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\ \boldsymbol{y} &= \boldsymbol{h}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x}, \boldsymbol{u})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i) \end{aligned} \quad (17)$$

so that the approximation errors $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^\diamond$ and $\overline{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}^\diamond$ satisfy[2]

$$\begin{aligned} \|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_f \qquad \forall \boldsymbol{x}, \boldsymbol{u} \\ \|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_h \qquad \forall \boldsymbol{x}, \boldsymbol{u} \end{aligned} \quad (18)$$

where σ_f and σ_h are nonnegative finite constants. Recall that such a bound can always be obtained on a compact set, and therefore (18) is a valid assumption.

The observer considered in this section is of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}}, \boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}}, \boldsymbol{u})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i) \end{aligned} \quad (19)$$

[1]To solve LMI problems, in this paper Yalmip's [12] *sedumi* solver has been used.

[2]See after the derivation of the error system why a Lipschitz condition like (3) cannot be used.

If the observer (19) is now used for the nonlinear system (1), the error dynamics can be expressed as:

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}^{\diamond}(\widehat{\boldsymbol{x}},\boldsymbol{u}) \\
&= \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\
&\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}}))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\boldsymbol{e} - L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\
&\quad + \sum_{i=1}^{m}(w_i(\boldsymbol{x},\boldsymbol{u}) - w_i(\widehat{\boldsymbol{x}},\boldsymbol{u}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\
&= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\boldsymbol{e} - L_i \\
&\quad \cdot (\sum_{j=1}^{m} w_j(\boldsymbol{x},\boldsymbol{u})(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\
&\quad - \sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(C_j\widehat{\boldsymbol{x}} + D_j\boldsymbol{u} + d_j))) \\
&\quad + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\
&= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\boldsymbol{e} - L_i(\sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})C_j\boldsymbol{e} \\
&\quad + \sum_{j=1}^{m}(w_j(\boldsymbol{x},\boldsymbol{u}) - w_j(\widehat{\boldsymbol{x}},\boldsymbol{u}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) \\
&\quad + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}))) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})
\end{aligned}
$$

or, simply as

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_iC_j)\boldsymbol{e} \\
&\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})
\end{aligned}
\tag{20}
$$

with

$$
\begin{aligned}
\Delta_{wf} &= \sum_{i=1}^{m}(w_i(\boldsymbol{x},\boldsymbol{u}) - w_i(\widehat{\boldsymbol{x}},\boldsymbol{u}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\
\Delta_{wh} &= \sum_{j=1}^{m}(w_j(\boldsymbol{x},\boldsymbol{u}) - w_j(\widehat{\boldsymbol{x}},\boldsymbol{u}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j)
\end{aligned}
$$

For the observer–TS fuzzy model mismatch, bounds similar to (3) are assumed:

$$
\begin{aligned}
\|\Delta_{wf}\| &\le \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\
\|\Delta_{wh}\| &\le \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\|
\end{aligned}
\tag{21}
$$

Using these bounds, in the worst case,

$$
\begin{aligned}
&\| - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\| \\
&\quad \le \max_i \|L_i\|(\sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \sigma_h) \\
&\qquad + \sigma_f + \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\
&\quad = \sigma + \delta\|\boldsymbol{e}\|
\end{aligned}
\tag{22}
$$

with

$$
\begin{aligned}
\sigma &= \max_i \|L_i\|(\sigma_{wh} + \sigma_h) + \sigma_f + \sigma_{wf} \\
\delta &= \max_i \|L_i\|\delta_{wh} + \delta_{wf}
\end{aligned}
\tag{23}
$$

To summarize, we have:

$$
\begin{aligned}
&\dot{\boldsymbol{e}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_iC_j)\boldsymbol{e} + \Delta \\
&\|\Delta\| \le \sigma + \delta\|\boldsymbol{e}\|
\end{aligned}
\tag{24}
$$

with δ and σ given by (23). Note however, that σ depends on the L_i, $i = 1, 2, \ldots, m$ to be designed, and in order to obtain the smallest possible bound on the estimation error, σ, and therefore $\|L_i\|$, $i = 1, 2, \ldots, m$ should be minimized.

Using the Lyapunov function $V = \boldsymbol{e}^T P \boldsymbol{e}$, similarly to Section III, and assuming that there exist $P = P^T > 0$, $Q = Q^T > 0$ so that

$$
\begin{aligned}
&\mathcal{H}(P(A_i - L_iC_i)) < -2Q \\
&\mathcal{H}(P(A_i - L_iC_j) + P(A_j - L_jC_i)) < -4Q \\
&j, i = 1, 2, \ldots, m
\end{aligned}
\tag{25}
$$

we get

$$
\begin{aligned}
\dot{V} &= \boldsymbol{e}^T\mathcal{H}(P\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_iC_j)\boldsymbol{e}) \\
&\quad + 2\boldsymbol{e}^T P\Delta \\
&\quad \le -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta)(1-\theta)\|\boldsymbol{x}\|^2 \\
&\quad - 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma)
\end{aligned}
$$

with $\theta \in (0, 1)$. Based on the results presented in Section III, by analyzing the expression of $\dot{V}$, when the observer (19) is applied to the nonlinear system (1), one of the following conclusions can be drawn regarding the estimation error:

1) ($\lambda_{\min}(Q) - \lambda_{\max}(P)\delta < 0$) or ($\lambda_{\min}(Q) - \lambda_{\max}(P)\delta = 0$ and $\sigma > 0$): no conclusion can be drawn;
2) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta = 0$ and $\sigma = 0$: under conditions similar to those in Section III, the estimation error dynamics are asymptotically stable.
3) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma = 0$: the nonlinear system (20) has a globally exponentially stable equilibrium point in $\boldsymbol{x} = 0$; However, this case can only be obtained if the fuzzy system is an exact representation of the nonlinear system, i.e., in (18) σ_f, $\sigma_h = 0$,
4) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma > 0$: the estimation error is uniformly ultimately bounded by

$$
\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta} \frac{\sigma}{\theta}. \tag{26}
$$

This is the result obtained in general.

The following example illustrates the computation of the bounds during observer design:

Example 2: Consider the nonlinear system

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_2^2 \end{pmatrix} \boldsymbol{x} \\
\boldsymbol{y} &= [1\ 0]\boldsymbol{x}
\end{aligned}
\tag{27}
$$

with $x_1,\, x_2 \in [-1,\, 1]$. Note that this system is unstable.

A TS approximation of the system (27) is obtained using the approach of [10]. Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1,\, x_2)|x_1,\, x_2 \in \{-1, 0, 1\}\}$, and 9 local models are obtained. The TS system can be written as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x}) A_i \boldsymbol{x} \\ \boldsymbol{y} &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{28}$$

The approximation errors are $\|\overline{\boldsymbol{f}}\| \leq \sigma_f = 0.407$ and $\|\overline{\boldsymbol{h}}\| = \sigma_h = 0$. With these membership functions, we have $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - \boldsymbol{x}(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq 6.3$ and $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq 6.3\|\boldsymbol{e}\|$. Combining the two bounds, we can actually use $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq \alpha \cdot 6.3 + (1 - \alpha) \cdot 6.3\|\boldsymbol{e}\|$, with α arbitrarily chosen in $[0,\ 1]$. Consequently, $\delta = (1 - \alpha) \cdot 6.3$, and $\sigma = \alpha \cdot 6.3 + 0.407$.

Solving (25) such that simultaneously $\lambda_{\max}(P)$ is minimized and $\lambda_{\min}(P)$ and $\lambda_{\min}(Q)$ are maximized, one obtains: $\lambda_{\min}(P) = 0.33$, $\lambda_{\max}(P) = 0.33$ and $Q = I$. Consequently, $\delta = (1 - \alpha) \cdot 6.3$, and $\sigma = \alpha \cdot 6.3 + 0.407$. With these values, the cases presented above become:

- for $\alpha < \frac{20}{21}$, i.e., $(1-\alpha) \cdot 6.3 > 0.3$ we have $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta < 0$ and no conclusion can be drawn
- since $\sigma_f > 0$, the conclusion of "asymptotic stability" (Cases 2) and 3) above) using the observer (19) is excluded.
- for $\alpha \geq \frac{20}{21}$, we obtain that the estimation error is uniformly ultimately bounded by

$$\begin{aligned}\gamma &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta} \frac{\sigma}{\theta} \\ &= \frac{0.33 \cdot (6.3\alpha + 0.407)}{(1 - 2.08(1 - \alpha))\theta} \\ &< 2.2\end{aligned}$$

with $\theta \in (0,\ 1)$ and $\alpha \in [\frac{20}{21},\ 1]$.

A large part of the value of the bound is due to observer model mismatch, i.e., the dependency of the scheduling vector on the non-measured states. For instance, if the considered system is

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_1^2 \end{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{29}$$

instead of (27), the scheduling variable is x_1 only, which is measured. Therefore, $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| = 0$, and the bound on the estimation error is simply

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \frac{\sigma}{\theta} = \frac{0.13}{\theta} < 0.13$$

with $\theta \in (0,\ 1)$. □

Note that in Example 2 a common measurement matrix has been considered. If the measurement matrix is not common for all the rules, σ depends on the L_i, $i = 1,\, 2,\, \ldots,\, m$ to be designed. In such a case, to facilitate the design, one can solve the multi-objective optimization problem:

$$\begin{aligned}&\text{maximize } \lambda_{\min}(Q),\ \lambda_{\min}(P), \\ &\text{minimize } \lambda_{\max}(P),\ \|L_i\|, i = 1,\, 2,\, \ldots,\, m \\ &\text{subject to} \\ &P = P^T > 0 \\ &Q = Q^T > 0 \\ &\mathcal{H}(P(A_i - L_iC_i)) \leq -2Q, \quad i = 1,\, 2,\, \ldots,\, m \\ &\mathcal{H}(P(A_i - L_iC_j) + P(A_j - L_jC_i)) \leq -4Q \\ &\quad j, i = 1,\, 2,\, \ldots,\, m\end{aligned}$$

Recall that instead of using the bound (3), for the observer (19), the constant bound on the approximation error has been used. This is because, if the bounds on $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ for observer design are not constants, but linear in $\boldsymbol{x}$, then, with the observer (19), the state itself has to be treated as a disturbance that affects the error dynamics. This would lead to a much larger bound on the estimation error.

A possible approach to still attain asymptotic stability is when $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ are known, and Lipschitz, i.e., there exist $\gamma_f,\, \gamma_h \geq 0$ such that $\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) - \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}}, \boldsymbol{u})\| \leq \gamma_f\|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|$, and $\|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u}) - \overline{\boldsymbol{h}}(\widehat{\boldsymbol{x}}, \boldsymbol{u})\| \leq \gamma_h\|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|$. In this case, instead of the observer (19), the observer

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}}, \boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) + \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}}, \boldsymbol{u}) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}}, \boldsymbol{u})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i) + \overline{\boldsymbol{h}}(\widehat{\boldsymbol{x}}, \boldsymbol{u})\end{aligned} \tag{30}$$

can be used.

Then, similarly to linear observer design for nonlinear systems with Lipschitz nonlinearities [14], [15], with γ_f and γ_h incorporated into δ, asymptotic stability of the estimation error can be obtained.

Example 3: Consider the nonlinear system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{pmatrix} 0.33x_1^2 + x_1 & 1 \\ 1 & -1 \end{pmatrix} \boldsymbol{x} \\ y &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{31}$$

with $x_1,\, x_2 \in [-1,\, 1]$.

A TS approximation of the system (31) can be obtained as the two-rule fuzzy system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{2} w_i(y) A_i \boldsymbol{x} \\ y &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{32}$$

with $A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $w_1(y) = \frac{1-y}{2}$, $w_2(y) = \frac{1+y}{2}$. The approximation error function is $\overline{\boldsymbol{f}} = \begin{pmatrix} 0.33x_1^2 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{x}$. For this function, we have $\|\overline{\boldsymbol{f}}(\boldsymbol{x}) - \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}})\| \leq \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|$, i.e., $\delta = 1$, and $\sigma = 0$. Note that since the membership functions depend on a measured variable, there is no observer-model mismatch.

Solving (25), one obtains $L_1 = L_2 = \begin{pmatrix} 15.1 \\ 7.3 \end{pmatrix}$, $P = \begin{pmatrix} 0.2109 & -0.3183 \\ -0.3183 & 0.8475 \end{pmatrix}$, $Q = I$, $\lambda_{\min}(P) = 0.08$, $\lambda_{\max}(P) = 0.97$. Then, using the observer

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}} &= \sum_{i=1}^{2} w_i(y)(A_i\hat{\boldsymbol{x}} + L_i(x - \hat{y})) + \overline{\boldsymbol{f}}(\hat{\boldsymbol{x}}) \\ e &= [1\ 0]\hat{\boldsymbol{x}}\end{aligned}$$

for the original nonlinear system (31), with a common quadratic Lyapunov function we obtain that there exist $P = P^T > 0$, $Q = Q^T > 0$ such that $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma = 0$ (i.e., case 3), and the estimation error is asymptotically stable. □

V. Conclusions

In this paper we have investigated whether and when stability guarantees can be obtained when an observer is designed for a fuzzy approximation of a nonlinear system and applied to the original nonlinear system. We have shown that unless the nonlinear system can be exactly represented or approximated up to a term that is Lipschitz continuous in the error, the dynamics of the estimation error are not globally asymptotically stable. However, the error is in general uniformly ultimately bounded. This bound can be computed using the estimation error and the Lyapunov matrix. In our future research we will investigate whether the estimation error can be reduced by employing other types of observers.

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