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First steps towards finding a solution of a dynamic investor-bank game*

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Abstract— The subject of this paper is a one-leader-one-follower dynamic inverse Stackelberg game with a fixed duration between a bank acting as the leader and an investor acting as the follower. The investor makes her transaction decisions with the bank as intermediary and the bank charges her transaction costs that are dependent on the investor's transactions. The goal of both players is to maximize their profits. The problem is to find a closed-form ε-optimal strategy for the bank. This problem belongs to the realm of composed functions and therefore is very difficult to solve. In this paper we first propose general guidelines for finding such an ε-optimal strategy for the bank and then apply these guidelines on specific academic examples. First we present an example in which we are able to find a closed-form ε-optimal solution, but we also introduce an example in which it is impossible to find such a solution and one has to proceed in a numerical way.

Keywords: game theory, (inverse) Stackelberg games, pricing

I. INTRODUCTION & LITERATURE OVERVIEW

This paper deals with a continuous dynamic game between an investor making her transaction decisions and a bank setting transaction costs for the investor as a transaction-decision-dependent mapping. The bank maximizes the transaction costs over the considered time period, while the investor minimizes her losses, which are increasing with the transaction costs.

As the players' objectives are conflicting, the game is non-cooperative [1], [2], [3], [4], as opposed to the cooperative games [5], [6]. If the transactions costs were independent of the investor's decision, the problem would fit into the framework of the so-called Stackelberg games [1], [2], [7], [8], [9]. The investor-bank can be also simplified into a "take-it-or-leave-it" principal-agent type of problem belonging to the so-called theory of incentives [9], [10], [11].

With the leader's decision being a mapping from the followers decision space into her own decision space the investor-bank problem fits within the framework of the so-called inverse Stackelberg games (ISG) [9], [12].

Although the ISG structure is recognized in a wide range of applications [9], [13], [14], only a limited amount of theory about these games exists and this theory focuses on exploring specific phenomena by means of examples. In this paper we first propose general guidelines on how to proceed in a general investor-bank game and subsequently we focus on applying these guidelines in solving two specific academic examples. While in the first considered example a closed-form solution can be found, in the second example only an implicit formulation of the solution is found and numerical techniques have to be adopted in order to find an approximate solution.

This paper is composed as follows. In Section II the general game formulation is given. In Section III the static variant of the game is dealt with as a first step to solve the dynamic problem. In Section IV first specific dynamic problem is considered. In Section V a more complicated variant of the problem is dealt with. In Section VI the achieved results, conclusions, and possibilities for future research are discussed.

II. THE INVESTOR-BANK GAME

A. Preliminaries: Dynamic Inverse Stackelberg Games

Let us consider a continuous dynamic game with pre-specified duration T ($0 < T < +\infty$) between two players. We will refer to these players as the leader (L) and the follower (F). The leader and the follower have decision variables $u_L(\cdot) \in U_L$ and $u_F(\cdot) \in U_F$, respectively, where U_L and U_F are decision spaces for the leader and for the follower, respectively, defined on the interval $[0, T]$ with a prespecified structure and known to both players. Moreover, there exist functions $P_L : U_L \times U_F \to \mathbb{R}$ and $P_F : U_L \times U_F \to \mathbb{R}$ so that integrals

$$\int_0^T P_L(u_L(t), u_F(t)) \, dt, \quad \int_0^T P_F(u_L(t), u_F(t)) \, dt, \quad (1)$$

representing the leader's and the follower's profits, respectively, are feasible for all $u_L(\cdot) \in U_L$ and $u_F(\cdot) \in U_F$. Additionally, the system dynamics may evolve according to certain dynamics. If the so-called inverse Stackelberg game is played, the leader announces her decision u_L as a mapping from the follower's decision space into her own decision space, i.e., $u_L(\cdot) = \gamma_L(u_F(\cdot))$ with $\gamma_L : U_F \to U_L$. Subsequently, the γ_L-mapping is made known to the follower and the follower chooses her decision $u_F^*(\cdot)$ so that

$$u_F^*(\cdot) = \arg\max_{u_F(\cdot)} \left(\int_0^T P_F(\gamma_L(u_F(t)), u_F(t)) \, dt \right). \quad (2)$$

If the leader knows the optimal response of the follower (2) to any choice of the γ_L-function, the leader's aim can be symbolically written as[1]

$$\gamma_L^*(\cdot) = \arg\max_{\gamma_L(\cdot)} \left(\int_0^T P_F(\gamma_L(u_F^*(\gamma_L(\cdot))), u_F^*(\gamma_L(\cdot))) \, dt \right). \quad (3)$$

Even if u_F^* in (2) is unique for any choice of γ_L and even if it can be guaranteed that $\gamma_L^*(\cdot)$ in (3) is unique, the problem (3)

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[1] Here for the sake of simplicity $u(t)$ is referred to as u.

is a very difficult one belonging to the realm of composed functions [15].

Note that if the leader's decision is a constant mapping (i.e., independent of u_F), the inverse Stackelberg game is simplified to the Stackelberg game [9], [12].

B. The Problem

The main topic of this paper is a dynamic inverse Stackelberg game between a bank, acting as the leader, and an investor, acting as the follower. The investor makes transactions with the bank as intermediary, while the bank tries to profit from the transaction costs that the investor pays to her.

In the following text we will simplify the notation as follows: The follower's decision will be denoted by u and the leader's decision will be denoted by γ. We will also refer to $u(t)$ and $x(t)$ by u and x, respectively, when no confusion can be caused by this simplification. The goal of the bank is to find a function $\gamma : \mathbb{R} \to [0, +\infty]$, $\gamma(0) = 0$, $\gamma(\cdot) \geq 0$, $\gamma(u) = \gamma(-u)$, which maximizes

$$J_L = \int_0^T \gamma(u(t)) \mathrm{d}t, \tag{4}$$

where $u(t) \in \mathbb{R}$ is the investor's transaction density, i.e., during the time interval $[t, t + \mathrm{d}t]$ the number of transactions equals $u(t)\mathrm{d}t$. The expression $\gamma(u)$ represents transaction costs that the investor has to pay when making transaction decision u. Possible additional restriction on γ is that $\gamma(u)$ is nondecreasing with respect to $|u|$. This would correspond to the situation in which the bank wants to impose higher transaction costs on the investor if the investor makes more transactions.

After the bank has announced the γ-function, the investor chooses $u \in U_F$, $U_F \overset{\text{def}}{=} \mathbb{R}$ in order to minimize her losses

$$J_F^c = q(x(T)) + \int_0^T g(x, u) \mathrm{d}t + \int_0^T \gamma(u(t)) \mathrm{d}t, \tag{5}$$

with $x(t) \in \mathbb{R}$, $q : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$. The term $q(x(T))$ in (5) represents the losses of the investor at the final time T and the term $\int_0^T g(x, u) \mathrm{d}t$ represents her consumption during the time interval $[0, T]$. Note that the transaction costs (the third term) are added to the costs of the investor. We assume that the investor does not participate in the game if her profit (5) is lower than her profit when doing nothing. Additionally, we assume that the system dynamics evolves according to the following state equation:

$$\dot{x} = f(x, u, \gamma(u)), \tag{6}$$

where $t \in [0, T]$, the initial state $x(0) = x_0$ is known a priori, $\dot{x}$ denotes derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$, and f is the state function, given a priori. In this general formulation we assume that u, x, q, g, γ, f satisfy all properties needed to allow (4)–(6) to be feasible.

As it is not intuitively clear how the leader should choose an optimal γ-function in a dynamic setting, we will start our analysis with a static variant of the problem and will try to extend the results of the static problem into the dynamic one.

III. THE STATIC VARIANT OF THE PROBLEM

Let us consider the following game, which is a static variant of the problem (4)–(6):

$$\min_u (f(u) + \gamma(u)), \quad \max_{\gamma(\cdot)} \gamma(u),$$

subject to $\gamma(\cdot) \geq 0$ and $\gamma(0) = 0$. With the same interpretation as before, the investor is secured of a maximum cost $f(0)$ by playing $u = 0$. Therefore she will consider only u-values for which $f(u) \leq f(0)$. This admissible set of u-values is denoted by U.

Example 3.1: As a specific f take $f(u) = (u-1)^2 + 1$, then $U = [0, 2]$ and an upper bound for the investor's criterion is $f(0) = 2$. This corresponds to the situation in which the investor does not make any transactions. However, the investor will participate in the game if she has a nonnegative profit, even if this profit equals to a very small positive number. Suppose that the bank chooses

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1 - \varepsilon), & \text{if } 0 \leq u \leq 2; \\ \text{nonnegative} & \text{elsewhere,} \end{cases} \tag{7}$$

with $\varepsilon \downarrow 0$. With such choice of the transaction costs the optimal choice of the investor is $u^* = 1$, the investor's costs are $2 - \varepsilon$ and the bank's profit is $(1 - \varepsilon)$. The strategy of the bank leading to this outcome is referred to as an ε-optimal strategy for the bank, because the outcome of the game is the best possible outcome minus ε. The bank takes essentially all the investor's profits (The latter would have been $\min_u f(u) = 1$ if the transaction costs would have been identically zero and the investor's profit would be equal to 2 is she does not invest). Note that the 'optimal' γ-function of the bank is nonunique; another choice would be

$$\gamma(u) = \begin{cases} 1 - \varepsilon, & \text{if } u \neq 0; \\ 0, & \text{if } u = 0, \end{cases}$$

where $\varepsilon \downarrow 0$. If one wants to γ to be nondecreasing, (7) could be replaced by

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1 - \varepsilon), & \text{if } 0 \leq u \leq 1; \\ 1 - \varepsilon + (1 - u)^2, & \text{if } u \geq 1, \end{cases}$$

and for negative u: $\gamma(u) = \gamma(-u)$, without altering the results.

Proposition 3.1: An upper bound for the profit of the bank is $J_F(u = u^*) - J_F(u = 0)$, where u^* is the optimal control of the investor in absence of transaction costs.

Remark 3.1: Although Proposition 3.1 can be easily proven, it establishes only an upper bound of the profit of the bank. This upper bound cannot be often reached (see [9] for examples of this).

Let us now formulate the guidelines which we propose in order to find an ε-optimal solution for the bank in the original dynamic investor-bank game.

The general guidelines on finding an ε-optimal solution for the bank:

1) Find an upper bound of the profit of the bank according to Proposition 3.1. We talk about an ε-optimal strategy of the bank if this strategy implies the profit of the bank equal to this upper bound minus some ε-value.

2) Find a set of strategies Γ that are likely candidates for an ε-optimal strategy.
3) Check which of these candidates are indeed ε-optimal strategies for the bank.

Point 2. of this procedure is critical. There is no clear way of picking the candidates for an ε-optimal strategy. In the following examples these candidates are chosen depending on a structure of Hamiltonian of the investor's losses function without the transaction costs.

Remark 3.2: In the rest of the paper we may also use the term ε-optimal strategy if this strategy leads to the upper bound value minus multiple ε-terms. Because we assume that $\varepsilon \downarrow 0$, this simplification is acceptable.

IV. THE FIRST DYNAMIC EXAMPLE

This example is a dynamic extension of the static example introduced in Section III. The dynamics of the system is described as

$$\dot{x} = u, \; x(0) = 1.$$

In order to compute an upper bound of the profit of the bank according to Proposition 3.1 we need to know what are the minimum losses for the investor. They can be computed as the outcome of the minimization of the investor's objective function when the transaction costs are set to zero, i.e., as minimization of J_{F} given by

$$J_{\mathrm{F}} = \frac{1}{2} \int_0^1 u^2(t)\mathrm{d}t + \frac{1}{2}x^2(1).$$

We use the Pontryagin minimum principle [16], [17], [18] with the Hamiltonian

$$H = \lambda u + \frac{1}{2}u^2. \tag{8}$$

It can be seen that

$$u^* = -\lambda, \dot{\lambda} = 0, \lambda(1) = x(1), u(t) = -x(1), t \in [0,1],$$

and hence

$$u^* = -\frac{1}{2}, \; x^* = 1 - \frac{1}{2}t, \; J_{\mathrm{F}}(u = u^*) = \frac{1}{4}, \; J_{\mathrm{F}}(u = 0) = \frac{1}{2}.$$

The above optimal control problem is now extended to a game theoretic problem by adding transaction costs. The criterion is changed into $\min_u J_{\mathrm{F}}^{\mathrm{c}}$, where

$$J_{\mathrm{F}}^{\mathrm{c}} = \frac{1}{2} \int_0^1 u^2(t)\mathrm{d}t + \frac{1}{2}x^2(1) + \int_0^1 \gamma(u(t))\mathrm{d}t.$$

The function γ satisfies restrictions $\gamma(\cdot) \geq 0$, $\gamma(0) = 0$. There is another criterion for the second player: $\max_{\gamma(\cdot)} J_{\mathrm{L}}$, with $J_{\mathrm{L}} = \int_0^1 \gamma(u(t))\mathrm{d}t$.

A. An ad hoc Approach

A likely candidate for the optimal γ is $\gamma(u) = -(\frac{1}{2} - \varepsilon)u(1+u)$ on the interval $[0,1]$ and $\gamma(u) \geq 0$ elsewhere, with $\varepsilon \downarrow 0$. This choice of γ mimics the idea for the first choice of γ in Example 3.1. Here γ is, if $\varepsilon = 0$, equal to $-H$ from

expression (8) on the essential interval, with $\lambda = \lambda^* = \frac{1}{2}$. It can be derived (with the new Hamiltonian $H^{(2)}$) that

$$H^{(2)} = \lambda u + \frac{1}{2}u^2 - \left(\frac{1}{2} - \varepsilon\right)u(1+u)$$

and therefore

$$u^* = -\frac{\lambda - \frac{1}{2} + \varepsilon}{2\varepsilon}, \quad \dot{\lambda} = 0, \quad \lambda(1) = x(1),$$

$$x^*(t) = 1 - \frac{1}{2}t, \quad J_{\mathrm{F}}^{\mathrm{c}} = \frac{3}{8} - \frac{1}{4}\varepsilon, \quad J_{\mathrm{L}} = \frac{1}{8} - \frac{1}{4}\varepsilon.$$

However, this choice of the γ is not optimal, because the bank can do better, even with a quadratic γ, as it will shown now. Let

$$\gamma(u) = \frac{1}{2}\beta u^2 + \alpha u,$$

on a certain interval to be determined. It follows that $\gamma(0) = 0$. Under the condition $\beta > -1$ it follows that

$$x^*(1) = \frac{1+\beta-\alpha}{2+\beta}; \; u^*(t) \equiv -\frac{1+\alpha}{2+\beta}.$$

Since $J_{\mathrm{F}}(u=0) = \frac{1}{2}$, the parameters α and β must necessarily satisfy $J_{\mathrm{F}}^{\mathrm{c}}(u = u^*) \leq \frac{1}{2}$. This leads to

$$\frac{1}{2}((u^*)^2 + (x^*(1))^2) + \gamma(u^*) = \frac{(2+\beta) - (1+\alpha)^2}{2(2+\beta)} \leq \frac{1}{2},$$

which is always fulfilled for $\beta > -1$. Consider

$$\max_{\alpha,\beta} \gamma(u^*) = \max_{\alpha,\beta} \frac{1}{2}\left(\beta\left(\frac{1+\alpha}{2+\beta}\right)^2 - 2\alpha\frac{1+\alpha}{2+\beta}\right)$$

$$= \max_{\alpha,\beta} \frac{\beta - 4\alpha - (4+\beta)\alpha^2}{2(2+\beta)^2}.$$

The maximization with respect to α leads to $\alpha = \frac{-2}{4+\beta}$, which leads to

$$\max_{\alpha,\beta} \gamma(u^*) = \max_{\beta} \frac{1}{2(4+\beta)}.$$

The best value for β is $\beta^* = -1 + \varepsilon$, with $\varepsilon \downarrow 0$. Subsequently, $\alpha = -\frac{2}{3} + \frac{2}{9}\varepsilon$ up to first order in ε, and with the same accuracy, $u^* = -\frac{1}{3} + \frac{1}{9}\varepsilon$. This leads to

$$J_{\mathrm{F}}^{\mathrm{c}} = \frac{4}{9} - \frac{1}{27}\varepsilon, \; J_{\mathrm{L}} = \frac{1}{6} - \frac{1}{18}\varepsilon,$$

which is the best result for the bank within the class of quadratic γ-functions. Without the transaction costs for the investor, its costs would be

$$J_{\mathrm{F}} = J_{\mathrm{F}}^{\mathrm{c}} - J_{\mathrm{L}} = \frac{5}{18} + \frac{1}{54}\varepsilon,$$

which is less than what he would have obtained by playing $u = 0$ and therefore, this is the outcome of the game for the investor. Now that α and β have given values, it can be checked that $g(u^*) > 0$ in a neighborhood of u^*. Further away from u^* the function γ can be adjusted such that $\gamma(\cdot) \geq 0$ everywhere.

We found the best possible transaction cost definition in the realm of quadratic functions. In order to validate whether the quadratic choice of γ is the optimal one, we will

discretize the problem with number of discretization steps equal to N, starting from two time steps and proceeding to $N \to \infty$.

B. Two Time Steps

Here we consider a discretized version of the continuous-time problem with two time steps. The model is

$$x_1 = x_0 + \frac{1}{2}u_1 = 1 + \frac{1}{2}u_1, \quad x_2 = x_1 + \frac{1}{2}u_2 = 1 + \frac{1}{2}(u_1 + u_2),$$

and the criteria are

$$J_F^c = \frac{1}{4}\left(u_1^2 + u_2^2\right) + \frac{1}{2}\left(1 + \frac{1}{2}(u_1 + u_2)\right)^2 + \frac{1}{2}(\gamma(u_1) + \gamma(u_2)),$$

$$J_L = \frac{1}{2}(\gamma(u_1) + \gamma(u_2)).$$

1) First Attempt: A natural assumption is that J_F^c is minimized for the u_i-values, which minimize

$$J_F \stackrel{\text{def}}{=} \frac{1}{4}(u_1^2 + u_2^2) + \frac{1}{2}(1 + \frac{1}{2}(u_1 + u_2))^2.$$

These values are $u_1 = u_2 = -\frac{1}{2}$. Since

$$J_F(u_1 = -\frac{1}{2}, u) = J_F(u, u_2 = -\frac{1}{2}) = \frac{3}{8}u^2 + \frac{3}{8}u + \frac{11}{32},$$

we consider γ-function

$$\frac{1}{2}\gamma(u) = -(\frac{3}{8}u^2 + \frac{3}{8}u)(1 - \varepsilon), \tag{9}$$

with a small positive ε. With this γ-function and with $u_1 = u_2 = -\frac{1}{2}$ it is easily shown that

$$J_F^c = \frac{1}{4} + 2(1 - \varepsilon)\frac{3}{32} < J_F(u_1 = 0, u_2 = 0) = \frac{1}{2},$$

$$J_L = 2(1 - \varepsilon)\frac{3}{32}.$$

Note that $\frac{1}{2}(J_F(u_1 = 0, u_2 = -\frac{1}{2}) - J_F(u_1 = -\frac{1}{2}, u_2 = -\frac{1}{2}))$ exactly equals $\frac{3}{32}$ which is the same fraction as which appeared in the previous formula. However, with this γ the Hessian of J_F with respect to u_1 and u_2 is not positive definite at the point $u_i = -\frac{1}{2}$, $i = 1, 2$, and, therefore, the follower can do better than choosing $u_i = -\frac{1}{2}$, $i = 1, 2$. To avoid this deviating behavior on part of the follower, the leader will adjust the γ-function in such a way that $u_1 = u_2 = -\frac{1}{2}$ is best for the follower as follows. On the interval $-\frac{1}{2} \leq u \leq 0$ γ remains as given by (9). For $u < -\frac{1}{2}$ we choose a decreasing function of u (i.e. increasing with $|u|$), which is continuous at $u = -\frac{1}{2}$, e.g.

$$\gamma(u) = -u - \frac{1}{2} + (1 - \varepsilon)\frac{3}{32}.$$

If we require the function γ to be even, then it is defined for $u > 0$ also. With this choice of γ the best the follower can do is to choose $u_i = -\frac{1}{2}$, $i = 1, 2$.

Remark 4.1: A different, discontinuous and non-monotonous choice for the leader is:

$$\gamma(u) = \begin{cases} 0 & \text{for } u = 0, \\ \frac{1}{4} - \varepsilon, & \text{for } |u| = \frac{1}{2}, \\ \gg 1, & \text{elsewhere.} \end{cases}$$

This choice of γ is ε-optimal, because it leads to $u_1 = 0$, $u_2 = -\frac{1}{2}$, or the other way around, and $J_F = \frac{15}{32} - \varepsilon$ and $J_L = \frac{1}{8} - \frac{1}{2}\varepsilon$.

2) Second Attempt: Inspired by the remark in the previous subsection, we may try to find α and β values that maximize

$$J_F(0, \beta) - J_F(\alpha, \beta). \tag{10}$$

Because of the symmetry with respect to u_1 and u_2, an equivalent problem is to maximize

$$J_F(\alpha, 0) - J_F(\alpha, \beta). \tag{11}$$

To simplify the expressions, it is possible to maximize the sum of (10), (11) instead of maximizing only one of them:

$$J_F(0, \beta) - J_F(\alpha, \beta) + J_F(\alpha, 0) - J_F(\alpha, \beta). \tag{12}$$

This leads to $\alpha = \beta = -\frac{2}{5}$. Subsequently the leader chooses $\gamma(u)$ in such a way that the follower will indeed choose $\alpha = \beta = -\frac{2}{5}$. A different way of how to obtain the same result is that the leader wants to maximize $\delta = \frac{1}{2}(\gamma(\alpha) + \gamma(\beta))$, with $\alpha = \beta$, subject to

$$J_F^c(\alpha, \beta) \leq J_F^c(0, \beta) \Rightarrow J_F(\alpha, \beta) + \frac{1}{2}\delta \leq J_F(0, \beta)$$

$$\Rightarrow \frac{1}{2}\delta \leq J_F(0, \beta) - J_F(\alpha, \beta),$$

$$J_F^c(\alpha, \beta) \leq J_F^c(\alpha, 0) \Rightarrow J_F(\alpha, \beta) + \frac{1}{2}\delta \leq J_F(\alpha, 0)$$

$$\Rightarrow \frac{1}{2}\delta \leq J_F(\alpha, 0) - J_F(\alpha, \beta),$$

$$J_F^c(\alpha, \beta) \leq J_F^c(0, 0) \Rightarrow J_F(\alpha, \beta) + \delta \leq J_F(0, 0)$$

$$\Rightarrow \delta \leq J_F(0, 0) - J_F(\alpha, \beta),$$

for suitably chosen $\alpha = \beta \neq 0$. The maximal δ is obtained for $\alpha = \beta = -\frac{2}{5}$ and thus $\delta = \frac{1}{5}$. Note that this is a better result for the leader than the one obtained with the first attempt with $\delta = \frac{3}{16}$.

C. Many Time Steps and the Limit to Infinity

In this subsection we consider the model

$$x_i = x_{i-1} + \frac{1}{N}u_i, \quad i = 1, 2, \ldots, N, \quad x_0 = 1,$$

and the criteria

$$J_F = \frac{1}{2N}\sum_{i=1}^{N} u_i^2 + \frac{1}{2}\left(1 + \frac{1}{N}\sum_{i=1}^{N} u_i\right)^2, \quad J_L = \frac{1}{N}\sum_{i=1}^{N} \gamma(u_i).$$

The expression equivalent to (12) becomes

$$\frac{1}{2}\left[\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{i=1, i\neq k}^{N} u_i^2\right) + \sum_{k=1}^{N}\left(1 + \frac{1}{N}\sum_{i=1, i\neq k}^{N} u_i\right)^2\right]$$

$$- \frac{1}{2}\sum_{i=1}^{N} u_i^2 - \frac{1}{2}N\left(1 + \frac{1}{N}\sum_{i=1}^{N} u_i\right)^2.$$

Minimization of this expression with respect to u_1 and subsequent substitution (again using the symmetry property) of $u_2 = \cdots = u_N = u_1$ lead to

$$u_i^* = -\frac{N}{3N - 1}, \quad i = 1, 2, \ldots, N.$$

For $N = 2$ this coincides with the results of the previous subsection. For $N \rightarrow +\infty$ we get $u_i^* = -\frac{1}{3}$, $i = 1, 2, \ldots, N$. It is easily shown that the profit for the bank is $\frac{N}{2(3N-1)}$. Note that for $N \rightarrow +\infty$ this profit converges to $\frac{1}{6}$, which equals the result obtained with the best quadratic γ-function. Let us consider J_F as a function of u_1 only and with $u_2 = \ldots = u_N = -\frac{N}{3N-1}$; $J_F(u_1, \omega) = \frac{1}{2}[\frac{1}{N}u_1^2 + \frac{N-1}{N}(\frac{N}{3N-1})^2 + (1 + \frac{1}{N}(u_1 - \frac{N(N-1)}{3N-1}))^2]$, where ω represents substitution $u_{2 \leq i \leq N} = -\frac{N}{3N-1}$. For this function it can be written

$$J_F(u_1 = 0, \omega) - J_F(u_1 = -\frac{N}{3N-1}, \omega)$$
$$= \frac{1}{2}\left[\frac{N(N-1) + (2N)^2}{(3N-1)^2} - \frac{N^2 + (2N-1)^2}{(3N-1)^2}\right],$$

which is the N-equivalent of (11). If we calculate $N(J_F(u_1 = 0, \omega) - J_F(u_1 = -\frac{N}{3N-1}, \omega))$, the result is $\frac{N}{2(3N-1)}$, which equals the profit of the bank (as already obtained earlier). From the point of view of the bank it is necessary that the investor makes some transactions, i.e.,

$$J_F(u_1 = -\frac{N}{3N-1}, \omega) + \frac{1}{N}\gamma \leq J_F(u_1 = 0, \omega),$$

or with a quadratic ε-term,

$$J_F + \frac{1}{N}\gamma = J_F(u_1 = 0, \omega) + \frac{1}{N}\varepsilon\left[\left(u_1 + \frac{N}{3N-1}\right)^2 - (\frac{N}{3N-1})^2\right].$$

Hence,

$$\frac{1}{N}\gamma(u_1) = \frac{1}{2}\left[\frac{N(N-1) + (2N)^2}{(3N-1)^2} - \frac{1}{N}u_1^2 - \frac{N-1}{N}(\frac{N}{3N-1})^2\right.$$
$$\left. - \left(1 + \frac{1}{N}\left(u_1 - \frac{N(N-1)}{3N-1}\right)\right)^2\right] + \frac{\varepsilon}{N}\left(u_1^2 + \frac{2N}{3N-1}u_1\right)$$
$$= \frac{1}{2}\left[-\frac{1}{N}u_1^2 - \frac{1}{N^2}u_1^2 - \frac{4}{3N-1}u_1\right] + \frac{\varepsilon}{N}\left(u_1^2 + \frac{2N}{3N-1}u_1\right).$$

If we disregard ε-terms for $N \rightarrow \infty$ this leads to exactly the quadratic function obtained before. We write

$$J_F^c(u_1, \ldots, u_N) = J_F(u_1, \ldots, u_N) + \frac{1}{2}\left[\sum_{i=1}^{N}\left(-\frac{1}{N}u_i^2\right.\right.$$
$$\left.\left. - \frac{1}{N^2}u_i^2 - \frac{2}{3N-1}u_i\right) + \frac{2\varepsilon}{N}\left(u_i^2 + \frac{4N}{3N-1}u_i\right)\right].$$

For $N > \frac{1}{2\varepsilon}$ all eigenvalues of the Hessian lie in the right half plane. For $N \leq \frac{1}{2\varepsilon}$, however, the Hessian is not positive definite. In the latter case, one uses the trick of subsection IV-B.1, i.e. for $-\frac{N}{3N-1} \leq u \leq 0$, $\gamma(u)$ is as above, and for $u < -\frac{N}{3N-1}$ we choose it as a decreasing function.

The derivations carried out with the discretized variant of the problem show that the quadratic γ-function proposed in Section IV-A is a globally optimal choice. Note that its structure is the same as the structure of the Hamiltonian for the problem without transaction costs.

In the following section we will consider a more complicated dynamic investor-bank problem.

V. THE SECOND DYNAMIC EXAMPLE

Let us consider the dynamic model

$$\dot{x} = u, \, x(0) = 1,$$

with criterion

$$\min_u \frac{1}{2}\int_0^1 (x^2 + u^2)dt + \frac{1}{2}x^2(1).$$

An essential difference with the problem of the previous section is that the optimal decision is not constant anymore: $u^*(t) = -e^{-t}$ which leads to the minimal value $J_F^{c*} = \frac{1}{2}$. In the discretized problems (see the coming subsections) we cannot expect all u_i^* to be equal anymore. Consequently $\gamma(u)$ will have to be specified in the neighborhood of these different u_i^*-values.

When following the same guidelines in finding the solution as in the previous section it was found out that the slightly more complex nature of the problem makes it impossible to solve the problem in the closed-form.

For finding an optimal quadratic γ one can use the principle of optimality [1], leading to a value function minimization. Let $\gamma(u) \stackrel{\text{def}}{=} \frac{1}{2}\beta u^2 + \alpha u$. The value function, to be minimized with respect to α and β, is (assuming that $x(0) = 1$)

$$\frac{1}{2}S(0) + k(0) + m(0),$$

where $S(t)$, $k(t)$ and $m(t)$ satisfy [1]

$$\dot{S} = \frac{S^2}{1+\beta} - 1, \quad S(1) = 1,$$
$$\dot{k} = \frac{S}{1+\beta}(k + \alpha), \quad k(1) = 0,$$
$$\dot{m} = \frac{1}{1+\beta}(k\alpha + \frac{1}{2}k^2), \quad m(1) = 0.$$

This problem has to be solved numerically.

Another option is to solve the problem by the discretization as it was done in the previous section. Following the reasoning in Section IV, we can discretize the problem and can proceed from $N = 2$ to $N \rightarrow \infty$. Because of the space restrictions in the following we will show only the case $N \rightarrow \infty$. Then the system dynamics can be described as

$$x_i = x_{i-1} + \frac{1}{N}u_i, \, i = 1, 2, \ldots, N, \, x_0 = 1,$$

and the criteria are

$$J_F = \frac{1}{2N}\sum_{i=1}^{N}(u_i^2 + x_{i-1}^2) + \frac{1}{2}x_N^2 =$$
$$= \frac{1}{2N}\sum_{i=1}^{N}(u_i^2 + (1 + \frac{1}{N}\sum_{k=1}^{i-1}u_k)^2) + \frac{1}{2}(1 + \frac{1}{N}\sum_{i=1}^{N}u_i)^2;$$
$$J_L = \frac{1}{N}\sum_{i=1}^{N}\gamma(u_i).$$

First we want to solve $\min_u J_F$ subject to the model equations. This leads to a linear equation in u:

$$\begin{pmatrix} d+\zeta_1 & \zeta_2 & \cdots & \zeta_N \\ \zeta_2 & d+\zeta_2 & & \vdots \\ \zeta_3 & \zeta_3 & & \vdots \\ \vdots & & \ddots & \zeta_N \\ \zeta_N & \cdots & \zeta_N & d+\zeta_N \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix} = -N \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \vdots \\ \zeta_N \end{pmatrix}, \tag{13}$$

where

$$d = \frac{1}{N}, \ \zeta_i = \frac{1}{N^3}(N-i) + \frac{1}{N^2}. \tag{14}$$

Numerical computations indicate that the solution u indeed converges towards $-e^{-t}$ with $N \to \infty$. An upper bound of the leader's profit is

$$\begin{aligned} J_F(0, u_2, u_3, u_4, \ldots, u_N) &- J_F(u_1, u_2, u_3, \ldots, u_N) + \\ J_F(u_1, 0, u_3, u_4, \ldots, u_N) &- J_F(u_1, u_2, u_3, \ldots, u_N) + \\ &\vdots \\ J_F(u_1, u_2, u_3, \ldots, u_{N-1}, 0) &- J_F(u_1, u_2, u_3, \ldots, u_N), \end{aligned} \tag{15}$$

which can be rewritten as

$$-\left(\frac{1}{2N} + \frac{1}{2N^2}\right) \sum_{l=1}^{N} u_l^2 - \frac{1}{2N} \sum_{l=1}^{N} \sum_{i=l+1}^{N} \left(\frac{1}{N^2} u_l^2 + \frac{2}{N^2} u_l \sum_{k=1,\neq l}^{i-1} u_k\right) +$$

$$-\frac{1}{N^2} \sum_{l=1}^{N} \left(u_l \sum_{i=1,\neq l}^{N} u_i\right) - \frac{1}{2N} \sum_{l=1}^{N} \sum_{i=l+1}^{N} \frac{2}{N} u_l - \frac{1}{N} \sum_{l=1}^{N} u_l.$$

Then, for $j = 1, 2, \ldots, N$,

$$\begin{pmatrix} d+\zeta_1 & 2\zeta_2 & \cdots & 2\zeta_N \\ 2\zeta_2 & d+\zeta_2 & & \vdots \\ 2\zeta_3 & 2\zeta_3 & & \vdots \\ \vdots & & \ddots & 2\zeta_N \\ 2\zeta_N & \cdots & 2\zeta_N & d+\zeta_N \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix} = -N \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \vdots \\ \zeta_N \end{pmatrix},$$

with d and ζ_i defined as in (14). If this linear system of equations is symbolically written as $(\frac{1}{N}I + A)u = -N\zeta$, I being the identity matrix, then $u = -(I - NA + (NA)^2 - \cdots)N^2\zeta$.

It is impossible to derive the closed-form solution as we could do in the previous dynamic example and numerical computations have to be carried out in order to find an approximate solution.

VI. Discussion & Future Research

In this paper we introduced a dynamic investor-bank game. Using results obtained when solving the static variant of the problem, we formulated guidelines for finding an ε-optimal strategy for the bank. We showed one example in which we were able to find a closed-form ε-optimal strategy for the bank, but also an example of the situation in which only numerical solution can be found.

While it is often difficult to find a closed-form ε-optimal solution for the bank, finding a suboptimal solution within the prespecified class of γ-functions (e.g., quadratic functions) may be much less challenging.

Discretization of the problems helped to get more insights about the game structure as well as to validate an ε-optimality of certain strategies.

Additional research is needed to explore more general inverse Stackelberg problems. However, this research should be carried out only after we have solved the problems with a simple structure.

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