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On a spatiotemporally discrete urban traffic model

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Abstract: In order to control urban traffic with model-based control methods, a proper traffic model is very important. This traffic control model needs to have enough descriptive power to reproduce relevant traffic phenomena, and it also has to be fast enough to be used in practice. Therefore, macroscopic urban traffic flow models are usually applied as control models. These models are normally sampled temporally and spatially into discrete models so as to be simulated using digital computers. In this paper, a spatiotemporally discrete urban traffic model with a variable sampling time interval is proposed for model-based predictive control, which allows to balance modeling accuracy and computational complexity. The model is analyzed and evaluated based on the model requirements for control purposes. In addition, conditions are given to selecting suitable sampling time intervals for the models that are used to control urban traffic networks.

Keywords: Macroscopic traffic modeling; Urban traffic control; Urban traffic network.

1. INTRODUCTION

Traffic models can be mainly classified into three categories based on the modeling details: microscopic models, macroscopic models, and mesoscopic models. Microscopic models are detailed traffic models that describe the dynamics of each individual vehicle, like car-following models. In contrast, macroscopic models are much rougher models focusing only on the dynamics of traffic flows, i.e. the average behavior of groups of vehicles instead of individual vehicles. Mesoscopic models combine both the properties of the microscopic models and the macroscopic models. A first-order macroscopic model was proposed by Lighthill and Whitham (1955) to describe the dynamic of traffic flows, and it was extended into second-order macroscopic models by Payne (1971). But, this model was criticized for not being able to reproduce enough descriptive accuracy for modeling the phenomena of real traffic by Daganzo (1995). In general, macroscopic models are approximations of traffic dynamics, and they ignore some details of individual vehicles and make a lot of simplifications, so macroscopic traffic models are in general not as accurate as the models with higher level-of-detail. However, this statement does not always hold in practice. On some occasions, macroscopic modeling approaches may provide better results than modeling approaches with a higher level-of-detail (Hoogendoorn and Bovy, 2001). In addition, macroscopic models open a way for efficiently running the models using digital computers, and thus they are applied in applications that are characterized by high computational requirements, such as traffic control.

Macroscopic models also have different level-of-details. Usually, the more detailed the traffic dynamics is modeled, the more complex the model will be, and the heavier computational burden the model will have. Therefore, when selecting a traffic model in practice, a criterion needs to be followed. The criterion (Papageorgiou, 1998) is that the model should have sufficient descriptive power to reproduce all important phenomena for the intended application, and at the same time the execution speed of a simulation should be fast enough for this particular application. Thus, we need to find a trade-off between the descriptive accuracy of the model and the computational complexity.

In urban areas, the traffic flows are influenced a lot by the traffic signals. Therefore, the store-and-forward model (Gazis and Potts, 1963) was proposed to describe the stop-and-go traffic flow dynamics controlled by the traffic lights for urban roads. The store-and-forward model, later used for control by Diakaki et al. (2002), is a simple model with a low computational complexity, but it only applies for saturated traffic, i.e. when the vehicle queues resulting from the red phase cannot be dissolved completely at the end of the following green phase. The model proposed by Barisone et al. (2002) and extended by Dotoli et al. (2006) can describe vehicle queues and the time delay for vehicles reaching the queues in a link, and is able to describe different scenarios, i.e. unsaturated, saturated, and oversaturated traffic scenarios. The model proposed by Kashani and Saridis (1983) has a lower modeling power, but cannot describe scenarios other than saturated traffic either. The model of van den Berg et al. (2003); Lin and Xi (2008) is capable of simulating the evolution of traffic dynamics (including vehicle queues) in all traffic scenarios by updating the discrete-time model in small simulation steps. To reduce the computational complexity of this model, Lin et al. (2009) proposed a model with a longer sampling time interval based on the previous model, but has intersection cycle times that can differ from intersection to intersection. The model is much faster than the previous model, with only a limited loss in modeling accuracy.

Actually, all the macroscopic urban traffic models mentioned above are spatiotemporally discrete models, which are spatially sampled into road segments and temporally sampled with a sampling time interval. For urban areas, the roads are comparatively short and divided by intersections with traffic lights, and thus an urban road is usually taken as a road segment. The sampling time interval can vary for different urban traffic models. A trade-off also needs to be made when selecting the sampling time interval for the spatiotemporally discrete urban traffic model. Normally, a higher sampling frequency results in a more accurate model, but also gives rise to more computations because of having to update the model more frequently. When the sampling time interval becomes too large, the spatiotemporally discrete model cannot represent the continuous traffic flow behavior anymore. Therefore, an additional criterion (Courant-Friedrichs-Lewy (CFL) condition (Courant et al., 1967) for urban traffic models) needs to be satisfied when sampling urban traffic models into spatiotemporally discrete models, so as to keep the descriptive ability of these models.

In this paper, a spatiotemporally discrete urban traffic model with a variable sampling time interval is proposed for model-based predictive control, which allows to balance modeling accuracy and computational complexity. The model is derived by sampling the first-order continuous traffic flow model spatially and temporally. A CFL condition is deduced for the spatiotemporally discrete model to make sure the descriptive ability of the model can be still guaranteed. Experiments are designed to verify whether the models have sufficient descriptive power to reproduce the important phenomena for traffic control, and whether the computation speeds of models are fast.

2. NOTATIONS

In order to describe the model, we define J as the set of nodes (intersections), and L as the set of links (roads) in the urban traffic network. Link (u,d) is marked by its upstream node u ($u \in J$) and downstream node d ($d \in J$). The sets of the upstream nodes of input links and downstream nodes of output links for link (u,d) are $I_{u,d} \subset J$ and $O_{u,d} \subset J$ (e.g., for the situation of Fig. 1 we have $I_{u,d} = \{i_1, i_2, i_3\}$ and $O_{u,d} = \{o_1, o_2, o_3\}$).

The variable notations (see also Fig. 1) used in the model are listed as follows:

- $I_{u,d}$: set of upstream nodes of input links of link (u,d),
- $O_{u,d}$: set of downstream nodes of output links of link (u,d),
- k : simulation step counter for the urban traffic model,
- $n_{u,d}(k)$: number of vehicles in link (u,d) at step k,
- $q_{u,d}(k)$: queue length (expressed as the number of vehicles) at step k in link (u,d), $q_{u,d,o}$ is the queue length of the sub-stream turning to link (d,o),
- $m^{\mathrm{l}}_{u,d,o}(k)$: number of vehicles leaving link (u,d) and turning to link (d,o) at step k,
- $m^{\mathrm{a}}_{u,d}(k)$: number of vehicles arriving *at the (tail of the) queue* in link (u,d) at step k, $m^{\mathrm{a}}_{u,d,o}(k)$ is the number of arriving cars in the sub-stream going towards link (d,o),
- $m^{\mathrm{e}}_{u,d}(k)$: number of cars entering link (u,d) at step k,
- $S_{u,d}(k)$: available storage space of link (u,d) at step k expressed in number of vehicles,
- $\alpha^{\mathrm{l}}_{u,d}(k)$: average flow rate leaving link (u,d) at step k, $\alpha^{\mathrm{l}}_{u,d,o}(k)$ is the leaving average flow rate of the sub-stream going towards link (d,o),
- $\alpha^{\mathrm{a}}_{u,d}(k)$: average flow rate arriving at the tail of the queue in link (u,d) at step k, $\alpha^{\mathrm{a}}_{u,d,o}(k)$ is the arriving average flow rate of the sub-stream going towards link (d,o),
- $\alpha^{\mathrm{e}}_{u,d}(k)$: average flow rate entering link (u,d) at step k,
- $\beta_{u,d,o}(k)$: fraction of the traffic in link (u,d) anticipating to turn to link (d,o) at step k,
- $\mu_{u,d,o}$: saturation flow rate leaving link (u,d) turning to link (d,o),
- $g_{u,d,o}(k)$: green time length during step k for the traffic stream towards link (d,o) in link (u,d),
- $b_{u,d,o}(k)$: boolean value indicating whether the traffic signal at intersection d for the traffic stream in link (u,d) turning to link (d,o) is green (1) or red (0) at step k,
- $v^{\mathrm{free}}_{u,d}$: free-flow vehicle speed in link (u,d),
- $C_{u,d}$: capacity of link (u,d) expressed in number of vehicles,
- $N^{\mathrm{lane}}_{u,d}$: number of lanes in link (u,d),
- $\Delta c_{u,d}$: offset between node u and node d, which represents the offset time between the cycles of the upstream and the downstream intersections at the beginning of every control time step,
- l_{veh} : average vehicle length.

3. DISCRETE TIME DELAY

In this paper, the urban traffic models are discrete-time models with a time delay, during which a vehicle travels from the beginning of the road until it reaches the queues waiting in the road. In Åström and Wittenmark (1996), a method is presented to sample a continuous-time system with a time delay into a discrete-time system. Given this method, the discrete time delay that the vehicles take to reach the end of the queues in a link, will be obtained. Let a linear continuous time-invariant system with time delay $\gamma \in \mathbb{R}^+$ be described by [1]

$$\dot{\tilde{\mathbf{X}}}(t) = \mathbf{A}\tilde{\mathbf{X}}(t) + \mathbf{B}\tilde{\mathbf{U}}(t-\gamma) \ . \tag{1}$$

Let us now sample this system using a sampling period T. Define

$$\delta = \mathrm{floor}\left\{\frac{\tau}{T}\right\}, \qquad \gamma = \mathrm{rem}\{\tau, T\} \ , \tag{2}$$

where floor$\{x\}$ refers to the largest integer smaller than or equal to x, and rem$\{x,y\}$ is the remainder of the division of x by y. So δ is an integer, and the time delay τ can be expressed as

$$\tau = \delta \cdot T + \gamma \quad 0 \le \gamma < T \ . \tag{3}$$

If the input of the system ($\tilde{\mathbf{U}}(t)$) is assumed to be piecewise constant during each sampling time interval, the sampled discrete-time system will be

$$\mathbf{X}(k+1) = \Phi\mathbf{X}(k) + \Gamma_0\mathbf{U}(k-\delta) + \Gamma_1\mathbf{U}(k-\delta-1) \ , \tag{4}$$

where

$$\Gamma_0 = \int_0^{T-\gamma} e^{\mathbf{A}s}\mathrm{d}s\,\mathbf{B} \tag{5}$$

[1] $\tilde{\cdot}$ represents a continuous variable.

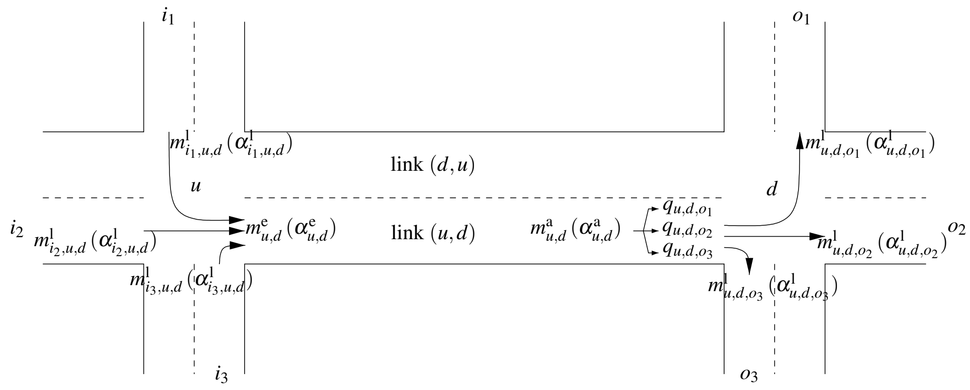


Fig. 1. A link connecting two traffic-signal-controlled intersections

$$\Gamma_1 = e^{\mathbf{A}(T-\gamma)}\int_0^{\gamma} e^{\mathbf{A}s}\mathrm{d}s\,\mathbf{B}\ . \tag{6}$$

Thus, the vehicles that enter into a link normally will run with free-flow speed for a certain time, and finally join the tail of the queues. This time period is a time delay that is needed before the vehicles join the queues waiting at the stop-line of the link. Then, the queue length in a link is updated by the number of vehicles leaving the link and the number of delayed vehicles entering the link. The differential equation describing the evolution of the queue length can be therefore written as

$$\dot{\tilde{q}}_{u,d,o}(t) = \tilde{\beta}_{u,d,o}(t)\tilde{\alpha}^{\mathrm{e}}_{u,d}(t-\tau) - \tilde{\alpha}^{\mathrm{l}}_{u,d,o}(t)\,, \tag{7}$$

i.e. the rate of changing for the queue length ($\dot{\tilde{q}}_{u,d,o}(t)$) is equal to the difference between the input flow rate (delayed by τ and then divided by multiplying the current turning rate) and the output flow rate. In (7), the traffic flow turning rate ($\tilde{\beta}_{u,d,o}(t)$), and the traffic flow rate entering or leaving the queue ($\tilde{\alpha}^{\mathrm{e}}_{u,d}(t)$ and $\tilde{\alpha}^{\mathrm{l}}_{u,d,o}(t)$), are all piece-wise constant during the sampling time intervals. Then, according to the addition principle of linear equations, (7) can be divided into two equations, as

$$\dot{\tilde{q}}^{1}_{u,d,o}(t) = -\tilde{\alpha}^{\mathrm{l}}_{u,d,o}(t) \tag{8}$$

$$\dot{\tilde{q}}^{2}_{u,d,o}(t) = \tilde{\beta}_{u,d,o}(t)\tilde{\alpha}^{\mathrm{e}}_{u,d}(t-\tau), \tag{9}$$

such that

$$\tilde{q}_{u,d,o}(t) = \tilde{q}^{1}_{u,d,o}(t) + \tilde{q}^{2}_{u,d,o}(t)\,. \tag{10}$$

To sample differential equation (8) without a time delay into a discrete equation, we define $A=0$ and $B=-1$, then according to (4), (5), and (6), we have

$$q^{1}_{u,d,o}(k+1) = \Phi q^{1}_{u,d,o}(k) + \Gamma\alpha^{\mathrm{l}}_{u,d,o}(k) \tag{11}$$

where

$$\Phi = e^{AT} = 1$$

$$\Gamma = \int_0^{T} e^{\mathbf{A}s}\mathrm{d}s\,\mathbf{B} = -T \tag{12}$$

Similarly, we can sample differential equation (9) with a time delay τ into a discrete equation. Since, in Section 4 the time delay τ will vary slowly with time t, then according to (2) and (3) we can approximately have

$$\delta(k) = \mathrm{floor}\left\{\frac{\tau(k)}{T}\right\}, \qquad \gamma(k) = \mathrm{rem}\{\tau(k),T\}\ , \tag{13}$$

and

$$\tau(k) = \delta(k)\cdot T + \gamma(k) \quad 0 \le \gamma(k) < T\ . \tag{14}$$

Next, we define $A_\tau = 0$ and $B_\tau = 1$, and then according to (4), (5), and (6), (9) results in

$$\begin{aligned} q^{2}_{u,d,o}(k+1) =& \Phi_\tau q^{2}_{u,d,o}(k) + \beta_{u,d,o}(k)\big(\Gamma_0\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)) \\ &+ \Gamma_1\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)-1)\big), \end{aligned} \tag{15}$$

where

$$\Phi_\tau = e^{AT} = 1$$

$$\Gamma_0 = \int_0^{T-\gamma(k)} e^{\mathbf{A}s}\mathrm{d}s\,\mathbf{B}_\tau = T-\gamma(k)$$

$$\Gamma_1 = e^{\mathbf{A}(T-\gamma(k))}\int_0^{\gamma(k)} e^{\mathbf{A}s}\mathrm{d}s\,\mathbf{B}_\tau = \gamma(k) \tag{16}$$

Therefore, by adding (11) and (15) together, we derive

$$\begin{aligned} q_{u,d,o}(k+1) = q_{u,d,o}(k) - T\alpha^{\mathrm{l}}_{u,d,o}(k) \\ + \beta_{u,d,o}(k)\big((T-\gamma(k))\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)) \\ + \gamma(k)\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)-1)\big)\,, \end{aligned} \tag{17}$$

and the arriving average traffic flow at the tail of the queues

$$\alpha^{\mathrm{a}}_{u,d}(k) = \frac{T-\gamma(k)}{T}\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)) + \frac{\gamma(k)}{T}\alpha^{\mathrm{e}}_{u,d}(k-\delta(k)-1)\,. \tag{18}$$

4. SPATIOTEMPORALLY DISCRETE URBAN TRAFFIC MODEL

4.1 Traffic dynamics on a link

Suppose the sampling time interval for intersection $d \in J$ and all the links that connect to intersection d is T_d and k_d is the corresponding time step counter. The cycle time of intersection j ($\in J$) can be defined as

$$c_j = M_j T_j, \tag{19}$$

where M_j and T_j are integers, and $0 < T_j \le c_j$. Sampling time intervals and cycle times can be different for intersections.

Therefore, a spatiotemporally discrete urban traffic model can be derived as follows (for more details see (Lin et al., 2009)):

The leaving average flow rate over T_d is determined by:

$$\begin{aligned} \alpha^{\mathrm{l}}_{u,d,o}(k_d) = \min\Big(&\mu_{u,d,o}\cdot g_{u,d,o}(k_d)/T_d, \\ &q_{u,d,o}(k_d)/T_d + \alpha^{\mathrm{a}}_{u,d,o}(k_d), \\ &\frac{\mu_{u,d,o}}{\sum\limits_{u'\in I_{d,o}}\mu_{u',d,o}}\cdot\frac{C_{d,o}-n_{d,o}(k_d)}{T_d}\Big)\ , \end{aligned} \tag{20}$$

where $\mu_{u,d,o}$ is the saturation flow rate that can leave link (u,d) turning to link (d,o) depending on the physical structure of link (u,d). The leaving flow rate is the minimum value of three flow rate values, average saturated flow rate, average unsaturated flow rate, and average over-saturated flow rate, which are given respectively by the three terms in (20). The first term calculates the average saturated flow rate, which depends on the saturation flow rate $\mu_{u,d,o}$ and green time duration; the second term calculates the average unsaturated flow rate based on the vehicles waiting in and arriving the queues; the third term calculates the average over-saturated flow rate depending on the proportional storage capacity of the downstream link

The flow rate entering link (u,d) will arrive at the end of the queues after a continuous time delay $\tau(k_d) = \frac{(C_{u,d} - q_{u,d}(k_d)) \cdot l_{\text{veh}}}{N^{\text{lane}}_{u,d} \cdot v^{\text{free}}_{u,d} \cdot T_d}$.

Then with $\delta(k_d)$ and $\gamma(k_d)$ derived from formulas (2) and (3), according to (17) the delayed flow rate arriving at the end of queues is

$$\alpha^{\text{a}}_{u,d}(k_d) = \frac{c_d - \gamma(k_d)}{c_d} \cdot \alpha^{\text{e}}_{u,d}(k_d - \delta(k_d)) + \frac{\gamma(k_d)}{c_d} \cdot \alpha^{\text{e}}_{u,d}(k_d - \delta(k_d) - 1). \tag{21}$$

The flow rate entering link (u,d) is made up from the flow rates from all the input links:

$$\alpha^{\text{e}}_{u,d}(k_d) = \sum_{i \in I_{u,d}} \alpha^{\text{l}}_{i,u,d}(k_d). \tag{22}$$

4.2 Synchronization between two intersections

In (22), the flow rate entering link (u,d) is provided by the combination of the flow rates leaving the upstream links. Recall that we may have different sampling time intervals between upstream and downstream intersections ($T_u \neq T_d$). Thus, the simulation time steps may be not equal to each other. Therefore, we have to synchronize the leaving and entering flow rates. First of all, a least common multiple time interval has to be defined for integer N_j as

$$T_{\text{lcm}} = N_j \cdot c_j \qquad \text{for all } j \in J, \tag{23}$$

Then, in each time interval T_{lcm}, we will recast the flow rates expressed in the timing of intersection u into the timing of intersection d. First, we transform the discrete-time leaving flow rates from the upstream links into continuous time, as

$$\tilde{\alpha}^{\text{l}}_{i,u,d}(t) = \alpha^{\text{l}}_{i,u,d}(k_u), \quad k_u \cdot T_u \leq t < (k_u + 1) \cdot T_u, \tag{24}$$

and then sample them again to obtain the average flow rates in time step k_d so as to be able used by the downstream link

$$\alpha^{\text{e}}_{i,u,d}(k_d) = \frac{1}{T_d} \int_{k_d \cdot T_d + \Delta c_{u,d}}^{(k_d+1) \cdot T_d + \Delta c_{u,d}} \tilde{\alpha}^{\text{l}}_{i,u,d}(t)\, \mathrm{d}t \ , \tag{25}$$

where $\Delta c_{u,d}$ represents the offset time between the cycle times of the upstream and the downstream intersections at the beginning of a control time step. Then, the flow rate entering link (u,d) can be computed by

$$\alpha^{\text{e}}_{u,d}(k_d) = \sum_{i \in I_{u,d}} \alpha^{\text{e}}_{i,u,d}(k_d). \tag{26}$$

5. CFL CONDITION FOR URBAN TRAFFIC MODELS

The maximum number of vehicles that can leave link $(u,d) \in L$ with a saturation flow rate (also called as link-intersection capacity) should not exceed the number of vehicles on this link

$$\mu_{u,d} T_d \leq n_{u,d}(k_d) \leq C_{u,d}, \tag{27}$$

where the number of vehicles on link (u,d) is bounded by its storage capacity $C_{u,d}$, and the link-intersection capacity $\mu_{u,d}$ is the sum of the saturation flow rates that leave link (u,d) turning into different directions:

$$\mu_{u,d} = \sum_{o \in O_{u,d}} \mu_{u,d,o}. \tag{28}$$

Then, by dividing the number of vehicles on link (u,d) into two parts, the number of vehicles in the queue ($q_{u,d}(k_d)$) and the number of vehicles running freely on the link ($f_{u,d}(k_d)$), we get

$$T_d \leq \frac{n_{u,d}(k_d)}{\mu_{u,d}} = \frac{q_{u,d}(k_d) + f_{u,d}(k_d)}{\mu_{u,d}}. \tag{29}$$

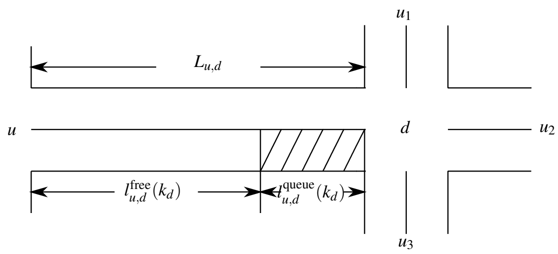


Fig. 2. Illustration of the free-speed flow and the queues

According to traffic theory, the traffic flow running with free-flow speed on the link always has a lower flow rate than the link-intersection capacity, i.e. $\alpha^{\text{free}}_{u,d}(k_d) \leq \mu_{u,d}$, and the traffic flow rate for the vehicles moving in queues is also lower than the link-intersection capacity, i.e. $\alpha^{\text{queue}}_{u,d}(k_d) \leq \mu_{u,d}$. Hence, (29) can be further written into

$$\begin{aligned} T_d &\leq \frac{q_{u,d}(k_d)}{\alpha^{\text{queue}}_{u,d}(k_d)} + \frac{f_{u,d}(k_d)}{\alpha^{\text{free}}_{u,d}(k_d)} \\ &= \frac{\rho^{\text{queue}}_{u,d}(k_d) l^{\text{queue}}_{u,d}(k_d)}{\alpha^{\text{queue}}_{u,d}(k_d)} + \frac{\rho^{\text{free}}_{u,d}(k_d) l^{\text{free}}_{u,d}(k_d)}{\alpha^{\text{free}}_{u,d}(k_d)} \\ &= \frac{l^{\text{queue}}_{u,d}(k_d)}{v^{\text{queue}}_{u,d}(k_d)} + \frac{l^{\text{free}}_{u,d}(k_d)}{v^{\text{free}}_{u,d}}, \end{aligned} \tag{30}$$

where $\rho^{\text{queue}}_{u,d}(k_d)$ and $\rho^{\text{free}}_{u,d}(k_d)$ are the density of the queue and the density of the free-running traffic flow on link (u,d) at time step k_d respectively. Furthermore, because the length of link (u,d) equals to the sum of the queue length and the free-running link length, i.e. $l^{\text{queue}}_{u,d}(k_d) + l^{\text{free}}_{u,d}(k_d) = L_{u,d}$, and since the average speed of the vehicles waiting in queues is bounded as $0 \leq v^{\text{queue}}_{u,d}(k_d) \leq v^{\text{free}}_{u,d}$, we have

$$T_d \leq \frac{L_{u,d}}{v^{\text{queue}}_{u,d}(k_d)}. \tag{31}$$

Since (31) should hold for all values of $v^{\text{queue}}_{u,d}(k_d)$, and since $v^{\text{queue}}_{u,d}(k_d) \leq v^{\text{free}}_{u,d}$, (31) can be further written as

$$T_d \leq \min\left(\frac{L_{u,d}}{v^{\text{queue}}_{u,d}(k_d)}\right) = \frac{L_{u,d}}{v^{\text{free}}_{u,d}}. \tag{32}$$

Hence, we derive a *sufficient condition* for the sampling time interval T_d of the model, as

$$T_d \leq \frac{L_{u,d}}{v^{\text{free}}_{u,d}}, \tag{33}$$

which is exactly a CFL condition. The condition can be interpreted intuitively as requiring that the distance $v^{\text{free}}_{u,d} T_d$ traveled

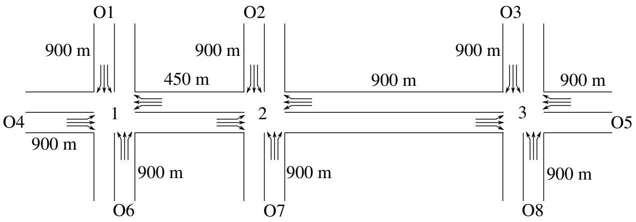


Fig. 3. Layout of an urban road network

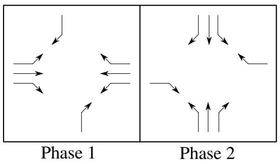


Fig. 4. Intersection traffic signal phases

by the traffic flow in one time step should not exceed one spatial step Δx. In practice, this CFL condition can be used as a criterion for selecting proper sampling time intervals for the spatiotemporally discrete traffic models.

However, for urban intersections, the sampling time intervals of intersection d not only depend on the link (u,d), but also on the rest of the links connecting to this intersection. We define $U_d \subset J$ as the set of the upstream intersections of intersection $d \in J$. Therefore, to guarantee that the spatiotemporally discrete urban traffic model can correctly represent the urban traffic dynamics, the simulation time interval T_d (i.e. sampling time interval) needs to satisfy condition:

$$T_d \leq \min_{u' \in U_d} \left(\frac{L_{u',d}}{v^{\text{free}}_{u',d}} \right). \tag{34}$$

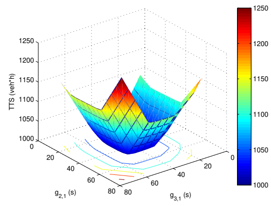
6. MODEL ASSESSMENT

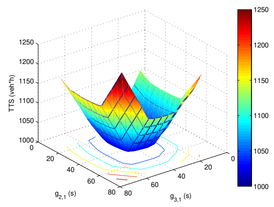
In this section, we will evaluate the effectiveness of the proposed spatiotemporally discrete urban traffic model, and analyze its sensitivity from a control point of view. Experiments are designed to demonstrate how the Total Time Spent (TTS, frequently selected as traffic control performance criterion) will change when varying the green time lengths of traffic signals. The evaluated urban road network is shown in Fig. 3.

As Fig. 3 shows, the length of the links in the network are 450 m and 900 m. The vehicle anticipating turning rates β are constant, i.e. 0.33 for left turn, through turn, and right turn respectively. The saturation flow rates μ are 1800 veh/h, 1600 veh/h, and 1500 veh/h respectively for turning through, left, and right in each link. The average vehicle length l_{veh} is set to 7 m, and the free-flow speed $v^{\text{free}}_{u,d}$ is 50 veh/h. Then the storage capacities C are 193 veh for link (1,2) and link (2,1), and 386 veh for the rest of the links in the network. Fixed-time control is executed for each intersection, where the phases, the cycle times, and the green time lengths are all constant during each simulation. The phases and their order for all the intersections are given in Fig. 4. The green time lengths and cycle times are listed in Table 1. The symbol $g_{j,i}$ stands for the green time length of the ith phase for intersection j. The offsets between intersections are specified as 0 s. The network input flow rates of the network are set to be equal to each other and constant in time (2000 veh/h). The simulation time duration is 30 min.

Table 1. Traffic signal fixed-time control setup

Intersection	Phase 1 (s)	Phase 2 (s)	Cycle time (s)
1	45	45	90
2	$g_{2,1}$	$90 - g_{2,1}$	90
3	$g_{3,1}$	$90 - g_{3,1}$	90


(a) TTS ($T = 1$ s)


(b) TTS ($T = 30$ s)

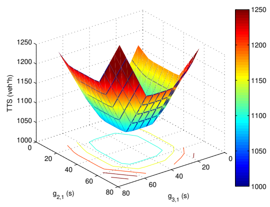
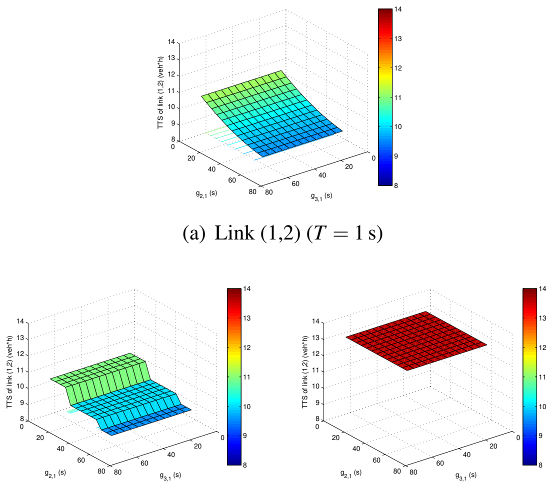

(c) TTS ($T = 90$ s)

Fig. 5. TTS of the network in Fig. 3 for discrete model with different sampling time intervals

In order to evaluate how the evaluation performance (TTS) changes with the traffic signals, we allow the green time lengths of intersection 2 and 3 to change within a given interval, $g_{2,1}, g_{3,1} \in \{15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75\}$. The lower bound and the upper bound for a green time duration is 15 s and 75 s, and $g_{2,2}$ and $g_{3,2}$ change accordingly with $g_{2,1}$ and $g_{3,1}$, due to the cycle time constraint of each intersection, i.e. $g_{i,2} = T_i - g_{i,1}$. The proposed spatiotemporally discrete traffic model is sampled by different sampling time intervals (simulation time intervals), i.e. T =1 s, 30 s, and 90 s respectively. Then, for each set-up of the traffic signals, all the sampled discrete traffic models are run for the same period of time (30 min). According to the CFL condition for urban traffic models in Section 5, the upper bounds of the sampling time intervals for every intersection are $T_{1,\max} = T_{2,\max} = 32$ s and $T_{3,\max} = 64$ s. Therefore, when $T_1 = T_2 = T_3 = 30$, the urban CFL conditions are satisfied in all the three intersections, as $30 < T_{1,\max} = T_{2,\max} < T_{3,\max}$; when $T_1 = T_2 = T_3 = 90$, the urban CFL condition is violated. The results are shown in Fig. 5 and Fig. 6 for the TTS of the entire network and for the TTS of link (1,2), in which the urban CFL condition is easier noticed to be violated.

From Fig. 5 and Fig. 6, we can see that the spatiotemporally discrete traffic model can describe a more detailed variation of the TTS changing with the green time lengths, when the sampling time interval is small. For $T = 1$ s and $T = 30$ s, the shapes of the TTS curves are very similar to each other for both the entire network and the single link (1,2). Generally speaking, the larger the sampling time interval is, the faster the model will run. For the discrete traffic models with sampling time intervals as 1 s and 30 s, the time needed to run the simulation is 5.6 s and 0.4 s respectively. Consequently, the spatiotemporally discrete model with $T = 30$ s is a better choice for urban traffic network



(a) Link (1,2) ($T = 1$ s)

(b) Link (1,2) ($T = 30$ s)

(c) Link (1,2) ($T = 90$ s)

Fig. 6. TTS of link (1,2) in Fig. 3 for discrete model with different sampling time intervals

control, because it can guarantee a similar performance as the model with $T = 1$ s, but requires less computing time.

For the spatiotemporally discrete model with sampling time 90 s, the time needed to run the simulation is even less, 0.2 s. But, the sampling time becomes too large so as to violate the CFL condition. Thus, the model fails to describe the traffic phenomena correctly. In Fig. 5(c), the values of the TTS become much higher than that of the models with $T = 1$ s and $T = 30$ s. In Fig. 6(c), the TTS curve becomes very flat, which does not capture the variation of TTS values anymore. Therefore, even though the model with $T = 90$ s is very fast, but it does not have sufficient accuracy to be used as a control model. Consequently, in this case study, the spatiotemporally discrete urban traffic model with sampling time $T = 30$ s is comparatively more suitable to be used as a prediction model for the urban traffic controllers, as it gives a good trade-off between the modeling accuracy and the computational complexity.

7. CONCLUSIONS

In this paper, a macroscopic spatiotemporally discrete urban traffic model with a variable sampling time interval is proposed for traffic control. Applying varying sampling time intervals allows to balance the modeling accuracy and the computational complexity of the traffic models, and allows to search for the best trade-off for specific control requirements. A CFL condition is deduced for the spatiotemporally discrete urban traffic model to make sure the descriptive ability of the model can be still guaranteed, when the sampling time interval grows.

Experiments are designed to verify whether the model has sufficient descriptive power to reproduce the necessary phenomena for traffic control, and whether the computation speed of the model is high enough. The experiment results illustrate that the higher the sampling frequency is, the more detailed the model will be, but also the more computation time is needed. Hence, a trade-off can be made between the computation time and the accuracy by selecting a proper sampling time interval.

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