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Optimal trajectory planning for trains using mixed integer linear programming*

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Optimal Trajectory Planning for Trains Using Mixed Integer Linear Programming

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***Abstract*— The optimal trajectory planning for trains under constraints and fixed maximal arrival time is considered. The variable line resistance (including variable grade profile, tunnels, and curves) and arbitrary speed restrictions are included in this approach. The objective function is a trade-off between the energy consumption and the riding comfort. First, the nonlinear train model is approximated by a piece-wise affine model. Next, the optimal control problem is formulated as a mixed integer linear programming (MILP) problem, which can be solved efficiently by existing solvers. The good performance of this approach is demonstrated via a case study.**

I. INTRODUCTION

Because of the rising energy prices and environmental concerns, the energy efficiency of transportation systems becomes more and more important, and this also includes railway networks [1], [2]. Meanwhile, the interest of railway operators in energy efficiency has been rising more and more in recent years. Therefore, some systems have been developed to supervise drivers to drive the train optimally, such as FreightMiser [3] and driving style manager [4].

On the other hand, the automatic train operation system of advanced train control systems plays a key role in ensuring accurate stopping, operation punctuality, energy saving, and riding comfort [5]. It is responsible for calculating the optimal speed-position reference trajectory based on the information collected by train control systems, such as line conditions, traction, and braking performance, etc. Therefore, an efficient algorithm for calculating the reference trajectory is significant to the driving performance of the automatic train operation system.

The research of the optimal control of train operations began in the 1960s. A simplified train optimal control problem was studied by Ichikawa [6], [7], which was solved by the maximum principle. Later on, a lot of researchers explored this optimal control problem by various methods, which can be grouped into the following two main categories [4]:

- Analytical solution,
- Numerical optimization.

A. Analytical solution

The train is usually modeled as a mass point in the optimal control problem. According to whether the traction and braking force is continuous or discrete, there are two kinds of models, i.e. continuous-input models and discrete-input models. The research on discrete-input models is mainly done by the SCG group of the University of South Australia [3], [6]. A type of diesel-electric locomotive is considered, whose throttle can take only a finite number of positions. Each position determines a constant level of power supply to the wheels. Several results, which include consideration of varying grades and speed restrictions, were presented. But nowadays many locomotives or motor cars can provide a continuous traction and braking force. For a continuous-input model, Khmelnitsky [7] described the mathematical model of the train by using the kinetic energy as the state variable. In that study, the optimal control problem is solved under varying gradient and speed restrictions. An analytical solution, which contains the sequence of optimal controls and the change points, was obtained by Liu and Golovicher [2] for the continuous-input model.

The optimal driving style of analytical solution contains four optimal control regimes: maximum acceleration, cruising at constant speed, coasting, and maximum deceleration. It is worth to note that the analytical methods often meet difficulties if more realistic conditions are considered that introduce complex nonlinear terms into the model equations and the constraints [8].

B. Numerical optimization

A number of advanced techniques such as fuzzy and genetic algorithms have been proposed to calculate the optimal reference trajectory of train operation. Chang and Xu [9] proposed a modified differential evolution algorithm to optimally tune the fuzzy membership functions that provide a trade-off of punctuality, riding comfort, and energy consumption. The implementation of a genetic algorithm to optimize the coast control is demonstrated by Chang and Sim [10]. Han et al. [11] also use a genetic algorithm to construct the optimal reference trajectory. They conclude that the performance of their genetic algorithm is better than that of the analytic solution obtained by Howlett and Pudney [3].

The train optimal control problem is solved by nonlinear programming and dynamic programming in [4]. The performance of a sequential quadratic programming algorithm and discrete dynamic programming are evaluated. Ko et al. [8] apply Bellman's dynamic programming to optimize the optimal reference trajectory. In [12] multi-parametric quadratic programming is used to calculate the optimal control law of train operation.

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Furthermore, due to the comparable high computing power available nowadays, more and more researchers are applying numerical optimization approaches to the train optimal control problem. But in these approaches, the optimal solution is not always guaranteed. In addition, the computation is often too slow, e.g. the computation time in [12] is 12 hours. Therefore, we propose to solve this optimal control problem as an MILP problem, which can be solved efficiently using existing commercial and free solvers [13], [14]. However, we have to make some approximations in order to construct an MILP formulation of the nonlinear train operation model.

The remainder of this paper is structured as follows. In Section II the nonlinear model of train operation is presented. Section III formulates the optimal control problem of train operation where the position is chosen as independent variable instead of time. In Section IV a mixed logical dynamic (MLD) model is formulated and the optimal control problem is written into an MILP problem. Section V illustrates how to calculate the optimal reference trajectory by the MILP approach with a case study. We conclude with a short discussion of some topics for future work in Section VI.

II. Train Model

In the literature of train optimal control, the mass-point model of train is usually used [15]. The motion of a train can be described by the following simple continuous-time model [2]:

$$m\rho \frac{\mathrm{d}v}{\mathrm{d}t} = u(t) - R_{\mathrm{b}}(v) - R_{\mathrm{l}}(s), \tag{1}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v, \tag{2}$$

where m is the mass of the train, ρ is a factor to consider the rotating mass [1], v is the velocity of the train, s is the position of the train, u is the control variable, i.e. the traction or braking force, which is bounded by the maximum traction force $u_{\max}$ and the maximum braking force $u_{\min}$, $u_{\min} \le u \le u_{\max}$, $R_{\mathrm{b}}(v)$ is the basic resistance including roll resistance and air resistance, and $R_{\mathrm{l}}(s)$ is the line resistance caused by track grade, curves and tunnels.

In practice, according to the Strahl formula [16] the basic resistance $R_{\mathrm{b}}(v)$ can be described as

$$R_{\mathrm{b}}(v) = m(a_1 + a_2 v^2),$$

where m is the train's mass, the coefficients a_1 and a_2 depend on the train characteristics and the wind speed, which can be calculated from the data known about the train.

The line resistance $R_{\mathrm{l}}(s)$ caused by track slope, curves, and tunnels can be described by [17]

$$R_{\mathrm{l}}(s) = mg\sin\alpha(s) + f_{\mathrm{c}}(r(s)) + f_{\mathrm{t}}(l_{\mathrm{t}}(s), v),$$

where g is the gravitational acceleration, $\alpha(s)$, $r(s)$ and $l_{\mathrm{t}}(s)$ are the slope, the radius of the curve, and the length of the tunnel along the track, respectively. The curve resistance $f_{\mathrm{c}}(\cdot)$ and the tunnel resistance $f_{\mathrm{t}}(\cdot)$ are given by empirical formulas. An example of such an empirical formula of the curve resistance is Roeckl's formula [18]

$$f_{\mathrm{c}}(r(s)) = \frac{6.3}{r(s) - 55} m \quad \text{for} \quad r(s) \ge 300\text{m},$$

$$f_{\mathrm{c}}(r(s)) = \frac{4.91}{r(s) - 30} m \quad \text{for} \quad r(s) < 300\text{m}.$$

When running in tunnels, the train experiences a higher air resistance that depends on the tunnel form, the smoothness of tunnel walls, the exterior surface of the train, and so on. The expression for tunnel resistance is [18]

$$f_{\mathrm{t}}(l_{\mathrm{t}}(s), v) = a_{\mathrm{t}}(l_{\mathrm{t}}(s)) v^2,$$

where a_{t} is the tunnel factor, which depends on tunnel length and train type. For the tracks outside the tunnels, the coefficient a_{t} is equal to zero.

III. Optimal Control Problem

As stated in [2], reference trajectory planning for trains can be formulated as an optimal control problem. The traction or braking force u then is the control variable. The state variables are the train position s and speed v. The objective function to be minimized could be the trip time, the energy consumption for a given trip time, or the total operation cost (a weighted sum of energy consumption and trip time). In this paper, we consider the energy consumption for a fixed amount of time T as objective criterion. In addition, the riding comfort is considered, which is expressed as a function of the change of the control variable u since reducing the number of transitions and the rate of change of u may improve passenger comfort [9]. The objective function can be written as:

$$J = \int_0^T \left(u(t) \cdot v(t) + \lambda \cdot \left| \frac{\mathrm{d}u(t)}{\mathrm{d}t} \right| \right) \mathrm{d}t \to \min \tag{3}$$

subject to the train dynamics (1) and (2), the following constraints

$$u_{\min} \le u(t) \le u_{\max} \tag{4}$$

$$0 \le v(t) \le V_{\max}(s) \tag{5}$$

and the boundary conditions

$$s(0) = s_{\text{start}}, \quad v(0) = v_{\text{start}}, \tag{6}$$

$$s(T) = s_{\text{end}}, \quad v(T) = v_{\text{end}}, \tag{7}$$

where J is the weighted sum of the energy consumption and riding comfort of the train operation; the maximum allowable velocity $V_{\max}(s)$ depends on the train characteristics and the line conditions, and as such it is usually a piecewise constant function of the coordinate s [2], [7]; s_{start} and v_{start} are the positon and the velocity at the beginning of the route; s_{end} and v_{end} are the position and the velocity at the end of the route. The duration of the trip T is usually given by the timetable.

As proposed in some previous works [2], [15], [7], [6], it is better to choose the position s as an independent variable rather than the time t. On the one hand, the choice of the position s as the independent variable will simplify the consideration of track-related data, such as line resistance and speed limits. On the other hand, the analytical and numerical study of the optimal control problem will be significantly simplified. Furthermore, Khmelnitsky [7] chose the total energy of the train and time t as states where the total energy includes kinetic and potential energy. Similarly, Franke et al. [15] used kinetic energy per mass unit and time as states. The choice of kinetic energy instead of speed v will facilitate the study of the optimal control problem, because this choice eliminates some of the model nonlinearities, but not all model nonlinearities. Therefore, we will also choose kinetic energy per mass unit $E = 0.5v^2$ and time t as states, and the position s as the independent variable. The continuous-time model (1) and (2) can then be rewritten as the following continuous-space model:

$$m\rho \frac{\mathrm{d}E}{\mathrm{d}s} = u(s) - R_{\mathrm{b}}(\sqrt{2E}) - R_{\mathrm{l}}(s), \tag{8}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\sqrt{2E}}. \tag{9}$$

The optimal control problem (3)- (7) can be stated as:

$$J = \int_{s_{\mathrm{start}}}^{s_{\mathrm{end}}} \left(u(s) + \lambda \cdot \left| \frac{\mathrm{d}u(s)}{\mathrm{d}s} \right| \right) \mathrm{d}s \to \min \tag{10}$$

subject to the model (8) and (9), the following constraints

$$u_{\min} \le u(s) \le u_{\max}, \tag{11}$$

$$0 \le E(s) \le E_{\max}(s), \tag{12}$$

and boundary conditions, which are rewritten as

$$E(s_{\mathrm{start}}) = E_{\mathrm{start}}, \;\; E(s_{\mathrm{end}}) = E_{\mathrm{end}}, \tag{13}$$

$$t(s_{\mathrm{start}}) = 0, \;\; t(s_{\mathrm{end}}) = T, \tag{14}$$

where $E_{\max}(s) = 0.5V_{\max}^2(s)$, $E_{\mathrm{start}} > 0$, and $E_{\mathrm{end}} = 0.5v_{\mathrm{end}}^2$. An assumption should be noted for the above equations. It is assumed that the unit kinetic energy $E(s) > 0$, which means the train's speed is always strictly larger than zero, i.e. the train travels nonstop [7]. Khmelnitsky states that this assumption is not restrictive in practice for two reasons. First, the speed of the initial start and the terminal stop can be approximated by small nonzero velocities. Second, stops at an intermediate point of the trip will not be planned deliberately in the optimal control design for a single train's operation.

IV. Solution Approach—MILP

Vašak et al. [12] proposed a discrete-time model of the train operation to calculate the optimal control law by dynamic programming. They split the time period into K intervals and assume the traction force or braking force to be constant on each interval $[kT, (k+1)T)$, where T is the sampling time. Franke et al. [15] similarly split the position horizon $[s_{\mathrm{start}}, s_{\mathrm{end}}]$ into N intervals to get a discrete-space model. They assumed that the track and train parameters as well as traction or breaking force can be considered as constant in each interval $[s_k, s_{k+1}]$ with length $\Delta s_k = s_{k+1} - s_k$, for $k = 1, 2, \ldots, N$. Note that $s_1 = s_{\mathrm{start}}$ and $s_{N+1} = s_{\mathrm{end}}$. In this paper, we will get the discrete-space model similarly as [15], since the optimal control problem is stated by the choice of s as the independent variable.

A. The mixed logical dynamic (MLD) model

In addition, we assume that $R_{\mathrm{l}}(s)$ is a piecewise constant function. By redefining the discretization of the interval $[s_{\mathrm{start}}, s_{\mathrm{end}}]$ if necessary, we can assume without loss of generality that $R_{\mathrm{l}}(s)$ is of the following form:

$$R_{\mathrm{l}}(s) = R_{\mathrm{l},k} \quad \text{for} \quad s \in [s_k, s_{k+1}].$$

In the interval $[s_k, s_{k+1}]$, the differential equation of the kinetic energy (8) can then be rewritten as

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \frac{1}{m\rho} u(k) - \frac{a_2}{\rho} E(s) - \frac{1}{\rho}(a_1 + R_{\mathrm{l},k}),$$

where $u(k)$ is a constant in interval $[s_k, s_{k+1}]$. By defining $\zeta = \frac{1}{m\rho}$, $\eta = -\frac{a_2}{\rho}$, $\gamma_k = -\frac{1}{\rho}(a_1 + R_{\mathrm{l},k})$, this equation can be rewritten as

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \zeta u(k) + \eta E(s) + \gamma_k. \tag{15}$$

Solving this differential equation with initial condition $E(s_k) = E(k)$, we obtain the following formula[1] for $E(s_{k+1})$

$$E(k+1) = e^{\eta \Delta s_k} E(k) + (e^{\eta \Delta s_k} - 1) \frac{\zeta}{\eta} u(k) + (e^{\eta \Delta s_k} - 1) \frac{\gamma_k}{\eta}$$

with $E(1) = E_{\mathrm{start}}$. Defining $a_k = e^{\eta \Delta s_k}$, $b_k = (e^{\eta \Delta s_k} - 1)\frac{\zeta}{\eta}$ and $c_k = (e^{\eta \Delta s_k} - 1)\frac{\gamma_k}{\eta}$, the above equation can be simplified as follows:

$$E(k+1) = a_k E(k) + b_k u(k) + c_k. \tag{16}$$

Note that this is an affine equation. For the differential equation of time (9), we approximate it by using a trapezoidal integration rule [19], written as the following equation

$$t(k+1) = t(k) + \frac{1}{2}\Big(\frac{1}{\sqrt{2E(k)}} + \frac{1}{\sqrt{2E(k+1)}}\Big)\Delta s_k \tag{17}$$

with $t(1) = 0$. In addition, the nonlinear part in this equation is approximated by a piece-wise affine (PWA) function. There are various methods for approximating functions in a PWA way, see e.g., the overview by Azuma et al. [20]. In this paper, we first select the number of regions of the PWA function and then optimize the interval lengths and parameters of the affine functions using least-squares optimization, minimizing the squared difference between the original function and the approximation. For example, if we consider an approximation using 3 affine subfunctions (cf. Figure 1), the PWA approximation[2] of the nonlinear function $f(E) = \frac{1}{2\sqrt{2E}}$ can be written as

$$f_{\mathrm{PWA}}(E) = \begin{cases} \alpha_1 E + \beta_1 & \text{for} \quad E_0 = E_{\min} \le E \le E_1, \\ \alpha_2 E + \beta_2 & \text{for} \quad E_1 \le E \le E_2, \\ \alpha_3 E + \beta_3 & \text{for} \quad E_2 \le E \le E_{\max} = E_3. \end{cases}$$

[1] For the sake of simplicity of notation we use $E(k)$ as a short-hand notation for $E(s_k)$ from now on.

[2] The approximation error can be reduced by taking more regions.

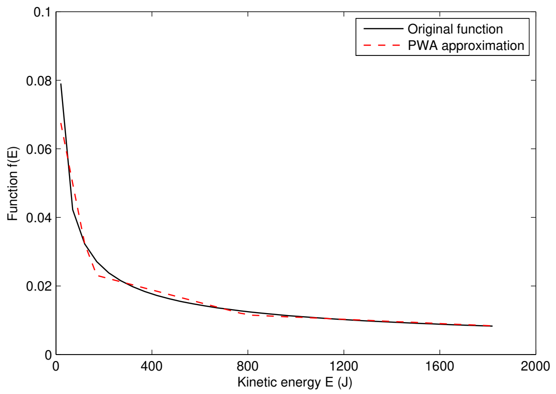

Fig. 1. The PWA approximation of the nonlinear function $f(E)$

Then the time dynamics (17) can be formulated as

$$t(k+1) = t(k) + (\alpha_l E(k) + \beta_l + \alpha_m E(k+1) + \beta_m)\Delta s_k, \quad (18)$$

when $E_{l-1} \le E(k) \le E_l$, $E_{m-1} \le E(k+1) \le E_m$, with $l, m \in \{1,2,3\}$. This PWA model can be transformed into an MLD model by introducing some auxiliary logical variables [21]. Here, we introduce auxiliary logical variables $\delta_1(k)$ and $\delta_2(k)$, defined as

$$[E(k) \le E_1] \Leftrightarrow [\delta_1(k) = 1],$$
$$[E(k) \le E_2] \Leftrightarrow [\delta_2(k) = 1].$$

Since the maximum and minimum values of $E(k)$ are $E_{\max}$ and $E_{\min}$, the logical conditions can be rewritten as [21]

$$\begin{aligned}(E_{\max} - E_1)\delta_1(k) &\le E_{\max} - E(k),\\ (E_{\min} - E_1 - \varepsilon)\delta_1(k) &\le E(k) - E_1 - \varepsilon,\\ (E_{\max} - E_2)\delta_2(k) &\le E_{\max} - E(k),\\ (E_{\min} - E_2 - \varepsilon)\delta_2(k) &\le E(k) - E_2 - \varepsilon,\end{aligned} \quad (19)$$

where ε is a small positive number (typically the machine precision) that is introduced to transform a strict equality into a non-strict inequality, which fits the MLD and MILP frameworks [21]. Furthermore, the auxiliary logical variable $\delta_3(k)$ is introduced to replace the product $\delta_1(k)\delta_2(k)$. This logical condition can be rewritten as the following system of linear inequalities [21]:

$$\begin{aligned}-\delta_1(k) + \delta_3(k) &\le 0,\\ -\delta_2(k) + \delta_3(k) &\le 0,\\ \delta_1(k) + \delta_2(k) - \delta_3(k) &\le 1.\end{aligned} \quad (20)$$

By defining new auxiliary variables $z_1(k) = \delta_1(k)E(k)$, $z_2(k) = \delta_2(k)E(k)$ and $z_3(k) = \delta_3(k)E(k)$, which can be expressed as [21]

$$\begin{aligned}z_i(k) &\le E_{\max}\delta_i(k),\\ z_i(k) &\ge E_{\min}\delta_i(k),\\ z_i(k) &\le E(k) - E_{\min}(1 - \delta_i(k)),\\ z_i(k) &\ge E(k) - E_{\max}(1 - \delta_i(k)),\end{aligned} \quad (21)$$

for $i = 1,2,3$, the expression $\alpha_l E(k) + \beta_l$ can be formulated as

$$\begin{aligned}\alpha_l E(k) + \beta_l = &\begin{bmatrix} -\alpha_3 & \alpha_2 - \alpha_3 & \alpha_1 - \alpha_2 + \alpha_3 \end{bmatrix} z(k)\\ &+ \begin{bmatrix} -\beta_3 & \beta_2 - \beta_3 & \beta_1 - \beta_2 + \beta_3 \end{bmatrix} \delta(k)\\ &+ \alpha_3 E(k) + \beta_3,\end{aligned}$$

where $z(k) = \begin{bmatrix} z_1(k) & z_2(k) & z_3(k) \end{bmatrix}^T$ and $\delta(k) = \begin{bmatrix} \delta_1(k) & \delta_2(k) & \delta_3(k) \end{bmatrix}^T$. Similarly, the expression $\alpha_m E(k+1) + \beta_m$ can be formulated as the form above. The dynamics of the system can then be rewritten as the following MLD model

$$\begin{aligned}x(k+1) = &A_k x(k) + B_k u(k) + C_k \delta(k) + C_k \delta(k+1)\\ &+ D_k z(k) + D_k z(k+1) + e_k,\end{aligned} \quad (22)$$

where $x(k) = \begin{bmatrix} E(k) & t(k) \end{bmatrix}^T$, $A_k = \begin{bmatrix} a_k & 0 \\ \Delta s_k \alpha_3 (a_k + 1) & 1 \end{bmatrix}$, $B_k = \begin{bmatrix} b_k \\ \Delta s_k \alpha_3 b_k \end{bmatrix}$, $C_k = \Delta s_k \begin{bmatrix} 0 & 0 & 0 \\ -\beta_3 & \beta_2 - \beta_3 & \beta_1 - \beta_2 + \beta_3 \end{bmatrix}$, $D_k = \Delta s_k \begin{bmatrix} 0 & 0 & 0 \\ -\alpha_3 & \alpha_2 - \alpha_3 & \alpha_1 - \alpha_2 + \alpha_3 \end{bmatrix}$, and $e_k = \begin{bmatrix} c_k \\ \Delta s_k(\alpha_3 c_k + 2\beta_3) \end{bmatrix}$. The MLD model is subject to the linear constraints (19), (20), and (21), which can be written more compactly as

$$\begin{aligned}R_{1,k}\delta(k) + R_{2,k}\delta(k+1) + R_{3,k}z(k) + R_{4,k}z(k+1)&\\ \le R_{5,k}u(k) + R_{6,k}x(k) + R_{7,k}&,\end{aligned} \quad (23)$$

where the coefficient matrixes $R_{i,k}$, for $i = 1,2,\ldots,7$, are defined appropriately. In addition, the upper bound and lower bound constraints for $E(k)$, $t(k)$, and $u(k)$ are also included in the coefficient matrices.

The objective function (10) can be discretized as

$$J = \sum_{k=1}^{N} u(k)\Delta s_k + \sum_{k=1}^{N-1} \lambda |\Delta u(k)|, \quad (24)$$

where $\Delta u(k) = u(k+1) - u(k)$. We introduce a new variable $\omega(k)$ to deal with the absolute value of $\Delta u(k)$, and we add the linear inequalities:

$$\omega(k) \ge u(k+1) - u(k),$$
$$\omega(k) \ge u(k) - u(k+1).$$

Then (24) can be rewritten as

$$J = \sum_{k=1}^{N} u(k)\Delta s_k + \sum_{k=1}^{N-1} \lambda \omega(k). \quad (25)$$

When we minimize the objective function (25), the optimal value of $\omega(k)$ will be equal to $|\Delta u(k)|$, so (24) will also be minimized.

B. The mixed linear programming problem (MILP)

Now the optimal control problem can be recast as a mixed integer linear programming (MILP) problem, where some of decision variables are binary and some are real variables. We define

$$\tilde{u} = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(N) \end{bmatrix}, \tilde{\delta} = \begin{bmatrix} \delta(1) \\ \delta(2) \\ \vdots \\ \delta(N+1) \end{bmatrix}, \tilde{\omega} = \begin{bmatrix} \omega(1) \\ \omega(2) \\ \vdots \\ \omega(N-1) \end{bmatrix},$$

and in a similar way as $\tilde{\delta}$ we also define $\tilde{z}$. Furthermore, if we define $\tilde{V} = \begin{bmatrix} \tilde{u}^T & \tilde{\delta}^T & \tilde{z}^T & \tilde{\omega}^T \end{bmatrix}^T$, the equivalent formulation of the optimal control problem is obtained as follows:

$$\min_{\tilde{V}} \quad C_J^T \tilde{V}, \tag{26}$$

subject to

$$F_1 \tilde{V} \leq F_2 x(1) + f_3 \tag{27}$$

$$F_4 \tilde{V} = F_5 x(1) + f_6 \tag{28}$$

where $C_J = \begin{bmatrix} \Delta s_1 & \cdots & \Delta s_N & 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}^T$. This can be shown as follows. The constraints for the MILP problem (23) are considered for $k = 1, 2, \ldots, N$. We can substitute $x(k)$ in the constraints by using the state equation (22) recursively. The substituted form is obtained as the following expression:

$$\begin{aligned} x(k) &= \Big[\prod_{j=1}^{k-1} A_j\Big] x(1) + \sum_{i=1}^{k-1} \Big[\prod_{j=i+1}^{k-1} A_j\Big] B_i u(i) + \Big[\prod_{j=2}^{k-1} A_j\Big] C_1 \delta(1) \\ &+ \sum_{i=2}^{k-1} \Big[\prod_{j=i+1}^{k-1} A_j\Big] (A_i C_{i-1} + C_i)\delta(i) + C_{k-1}\delta(k) \\ &+ \Big[\prod_{j=2}^{k-1} A_j\Big] D_1 z(1) + \sum_{i=2}^{k-1} \Big[\prod_{j=i+1}^{k-1} A_j\Big] (A_i D_{i-1} + D_i) z(i) \\ &+ D_{k-1} z(k) + \sum_{i=1}^{k-1} \Big[\prod_{j=i+1}^{k-1} A_j\Big] e_i. \end{aligned}$$

In addition, the end point condition $x(N+1) = [E_{\text{end}} \ \ T]^T$ needs to be considered in (28). Because we know the end value of $x(N+1)$, the values of α_m and β_m in (18) are also known. So the state equation at the end point can be written as

$$x(N+1) = A_N x(N) + B_N u(N) + C_N \delta(N) + D_N z(N) + e_N$$

where $A_N = \begin{bmatrix} a_N & 0 \\ \Delta s_N(\alpha_3 + \alpha_m a_N) & 1 \end{bmatrix}$, $B_N = \begin{bmatrix} b_N \\ \Delta s_N \alpha_m b_N \end{bmatrix}$, and $e_N = \begin{bmatrix} c_N \\ \Delta s_N(\alpha_m c_N + \beta_m + \beta_3) \end{bmatrix}$. By defining F_1, F_2, f_3, F_4, F_5, and f_6 properly, we can write all these constraints as (27) and (28).

The MILP problem (26)-(28) can be solved by several existing commercial and free solvers, such as CPLEX, Xpress-MP, GLPK (see e.g. [13], [14]).

V. Case Study

As a benchmark, we use the case study of [12]. The parameters for the train and rail path are given in Table I. The PWA approximation of $f(E) = \frac{1}{2\sqrt{2E}}$ is given in Figure 1. The parameters for the PWA approximation are given in Table II. The length Δs_k for interval $[s_k, s_{k+1}]$ depends on the speed limits, gradient profile, tunnels, and so on. In this paper, we assume that there is only one speed limit, i.e. $V_{\max}$ is equal to 50 m/s for the whole journey as is the case in [12]. In addition, the length of each interval is assumed to be the same, i.e. Δs_k is equal to 500 m for $k = 1, 2, \ldots, 20$. The objective function in this paper is a trade-off between energy consumption and riding comfort. The value of λ in (25) could be chosen properly according to the requirements, and is taken equal to 500 in this case study.

In this case study, we consider two cases: for the first one we choose the total running time equal to 315 s, indicated by T_1, while we select 600 s for the second case, indicated by T_2. The optimal reference trajectories and the traction and braking forces applied are shown in Figure 2. The running times and energy consumption are given in Figure 3. The black solid line shows the results for T_1 and the red dashed line the results for T_2. As we can see from the figures, the energy consumption with running time T_2 is lower than that with running time T_1, but at the cost of a longer travel time.

TABLE I

PARAMETERS OF TRAIN AND LINE PATH

Property	Symbol	Value
Train mass [kg]	m	$7 \cdot 10^5$
Basic resistance [N/kg]	R_b	$0.02 + 4.417 \cdot 10^{-5} v^2$
Mass factor	ρ	1.06
Maximum velocity [m/s]	$V_{\max}$	50
Line length [m]	s_T	10^4
Minimum kinetic energy [J]	$E_{\min}$	0.1
Maximum traction force [N]	$u_{\max}$	$4 \cdot 10^5$
Maximum braking force (regular)[N]	$u_{\min}$	$-3 \cdot 10^5$

TABLE II

THE COEFFICIENTS OF THE PWA APPROXIMATION WITH 3 REGIONS

Segment m	α_m	β_m	E_m
1	$-0.3525 \cdot 10^{-3}$	0.0746	$0 - 145$
2	$-0.0184 \cdot 10^{-3}$	0.0262	$145 - 800$
3	$-0.0031 \cdot 10^{-3}$	0.0140	$800 - 1250$

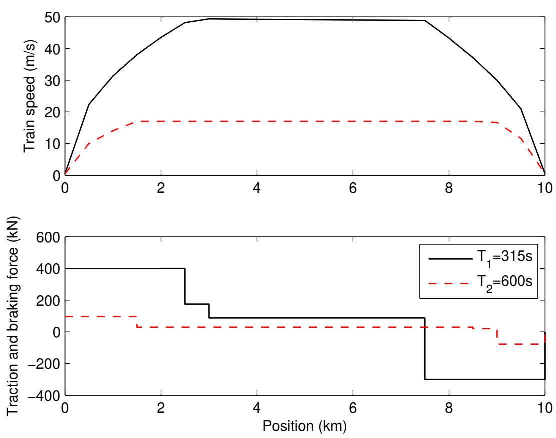

Fig. 2. The optimal position velocity curve and the input

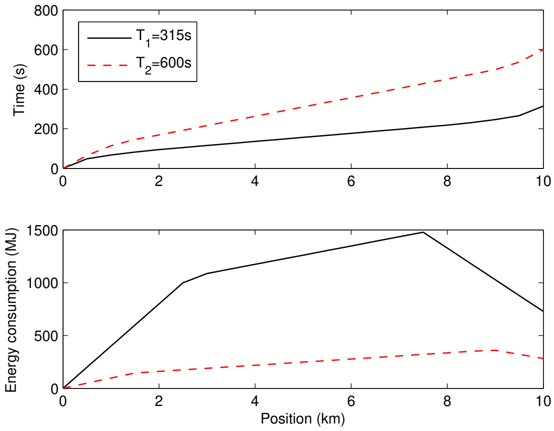

Fig. 3. The time and energy consumption

It is worth to note that the computation cost is very low for this MILP approach. The calculation time for the optimal control strategy is less than 10 minutes, which is much quicker than the 12 hours presented in [12].

VI. Conclusions and Future Work

In the current paper, we have considered the optimal trajectory planning problem for trains. To this aim, the nonlinear train operation model is formulated as a mixed logical dynamical (MLD) model by using piece-wise affine (PWA) approximation. The variable line resistance (including variable grade profile, tunnels, curves) and speed restrictions are considered, which are contained in the constraints of the MLD model. Furthermore, the optimal control problem is recast as a mixed integer linear programming (MILP) problem, which can be solved efficiently by existing solvers.

The maximum traction force is considered as constant in this paper. However, due to the maximum adhesion and the characteristics of the power equipment, in reality the maximum traction force is a nonlinear function of the velocity. For the maximum braking force, here, we just consider that the service braking force is 0.75 times the maximum braking force. However, the maximum braking force is also a nonlinear function of the velocity. In the future, those nonlinear functions of the traction and braking force will be approximated by PWA functions too., which can be included in the MILP problem by introducing more binary variables. Furthermore, the energy loss of the propulsion system [15] is not considered in this paper, which will also be approximated by PWA functions in future work. Finally, it is noted that the tunnel resistance is considered as constant in this paper. However, it is a nonlinear function of the velocity, which also needs to be included in our approach in the future.

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