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Structured modeling, analysis, and control of complex railway operations*

Ton J.J. van den Boom¹, Bart Kersbergen¹, and Bart De Schutter¹

Abstract—In this paper we discuss the rescheduling of trains on a large-scale railway network in the case of perturbations using a max-plus-linear system description. We study the structure of the system matrices and derive how this structure can be manipulated by the control variables. In addition, we show that this leads to a system matrix that is affine in the control variables and the timing parameters. We also consider additional constraints for scheduling trains on multiple tracks and constraints for joining and splitting trains.

I. INTRODUCTION

Propagation of delays in the railway networks is a popular topic of recent research in railway traffic management [6], [8], [10], [11], [15]. A railway network with rigid connection constraints and a fixed routing schedule can be modeled using max-plus-linear models [2], [3], [4], [5], [9]. A max-plus-linear model is ‘linear’ in the max-plus algebra [1], which has maximization and addition as its basic operations. In [15] we have modeled a controlled railway system using the *switching* max-plus-linear system description of [13], in which we use a number of max-plus linear models, each model corresponding to a specific mode and describing the network by a different set of connection and order constraints. In [12], [15] we have discussed the control of the railway system by switching between different modes, allowing us to break train connections and to change the order of trains.

In this paper we study the structure of the system matrices and show that the matrices can be partitioned into submatrices. Each submatrix represents the constraints related to either the running times, the dwell times, the headway times or the connection times. We also consider additional constraints for scheduling trains on multiple tracks and constraints for coupling and splitting trains.

II. RAILWAY OPERATIONS MODEL

Consider a railway operations system, with a periodic railway timetable with a cycle time T . The network consists of a set of tracks where overtaking is not possible and a set of stations. Each track starts and ends at a station. In this paper we generalize the concept of ‘station’ to places where the train order may alter, either because of a shunt or a junction (possibly without a ‘real’ station). By taking into account constraints that forbid overtaking on a track, we guarantee a feasible schedule. The operation of the schedule can be divided into a set of train runs, where each run starts with a

departure and ends with an arrival. A number of train runs, performed by the same ‘physical’ train will be denoted as a line. In the remainder of this paper we will simply refer to a ‘train run’ as a ‘train’.

Let $d_i(k)$ denote the time that train i depart for the k th time and let $r_i^d(k)$ be the scheduled departure time. Let $a_i(k)$ denote the time that train i arrives for the k th time, let $\tau_{\text{run},i}(k)$ be its running time and let $r_i^a(k)$ be the scheduled arrival time. We will call k the cycle counter. A departure may not occur before its scheduled departure time, so we have to satisfy the timetable constraint

$$d_i(k) \geq r_i^d(k) \quad (1)$$

The running time constraint now becomes

$$a_i(k) \geq d_i(k) + \tau_{\text{run},i}(k) \quad (2)$$

Note that for a cyclic timetable there holds $r_i^d(k) = r_i^d(0) + kT$ and $r_i^a(k) = r_i^a(0) + kT$ where we choose $0 \leq r_i^d(0) < T$ with T is the cycle time of the timetable. If the arrival of the train may not occur before the scheduled arrival time then we have to satisfy the timetable constraint

$$a_i(k) \geq r_i^a(k) \quad (3)$$

With the above definition it may happen that $r_i^a(0) \geq T$.

Let p_i be the preceding train of train i , which means that train i and p_i are physically the same train. Let $\tau_{\text{dwell},i}(k)$ be the minimum dwell time between arrival of train p_i and the departure of train i , then we have to satisfy the dwell time constraint

$$d_i(k) \geq a_{p_i}(k - \delta_{ip_i}^p) + \tau_{\text{dwell},i}(k) \quad (4)$$

where $\delta_{ij}^p = 0$ if each cycle k train p_i in proceeds as train i in that cycle, and $\delta_{ip_i}^p = \mu$ if train p_i in cycle $(k - \mu)$ proceeds as train i in cycle k .

Let $\mathcal{C}_i(k)$ be the set of trains to which train i gives a connection, and define a minimum connection time $\tau_{\text{connect},ic}$ for passengers to get from train c to train i . Then for each train $c \in \mathcal{C}_i(k)$ we have the connection constraint

$$d_i(k) \geq a_c(k - \delta_{ic}^c) + \tau_{\text{connect},ic} \quad (5)$$

(δ_{ic}^c is defined similarly as above).

Let $\mathcal{F}_i(k)$ be the set of trains that move over the same track and in the same direction as train i , and are scheduled before train i . Let $f \in \mathcal{F}_i(k)$ and let $\tau_{\text{headway},if}$ denote the minimum headway time between train f and train i . For

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each train $f \in \mathcal{F}_i(k)$ we have headway constraints for both departure and arrival

$$d_i(k) \geq d_f(k - \delta_{if}^h) + \tau_{\text{headway},if} \quad (6)$$

$$a_i(k) \geq a_f(k - \delta_{if}^h) + \tau_{\text{headway},if} \quad (7)$$

(δ_{if}^h is defined similarly as above). Note that due to the fact that train i cannot overtake train f headway constraints (for trains running in the same direction) always come in pairs, one constraint for headway at the departure and one constraint for headway at the arrival.

Let $\mathcal{W}_i(k)$ be the set of trains that move over the same track and in the opposite direction as train i , and are scheduled before train i . Let $w \in \mathcal{W}_i(k)$ and let $\tau_{\text{wait},iw}$ denote the minimum separation time between arrival of train w and departure of train i . For each train $w \in \mathcal{W}_i(k)$ we have separation constraint

$$d_i(k) \geq a_w(k - \delta_{iw}^w) + \tau_{\text{wait},iw} \quad (8)$$

(δ_{iw}^w is defined similarly as above).

Since we let a train depart as soon as all connection conditions are satisfied, we have [14], [15]:

$$\begin{aligned} d_i(k) = \max \left\{ r_i^d(k), a_{p_i}(k - \delta_{ip_i}^p) + \tau_{\text{dwell},p_i}(k), \right. \\ \max_{c \in \mathcal{C}_i(k)} (a_c(k - \delta_{ic}^c) + \tau_{\text{connect},ic}), \\ \max_{f \in \mathcal{F}_i(k)} (d_f(k - \delta_{if}^h) + \tau_{\text{headway},if}), \\ \left. \max_{w \in \mathcal{W}_i(k)} (a_w(k - \delta_{iw}^w) + \tau_{\text{wait},iw}) \right\}, \quad (9) \\ a_i(k) = \max \left\{ r_i^a(k), d_i(k) + \tau_{\text{run},ic}(k), \right. \\ \left. \max_{f \in \mathcal{F}_i(k)} (a_f(k - \delta_{if}^h) + \tau_{\text{headway},if}) \right\}. \quad (10) \end{aligned}$$

Note that in an undisturbed, well-defined time schedule the term $r_i(k)$ in (9) and (10) will be the largest. However, if due to unforeseen circumstances (an incident, a late departure, etc.) one of the trains (p_i , c , f , or w) has a delay, the corresponding term can become larger than the others, and then train i will depart later than the scheduled departure time $r_i(k)$ and will therefore also be delayed. Now let us consider a network with n trains (train runs) and define the vectors

$$x(k) = [d_1(k) \ \cdots \ d_n(k) \ a_1(k) \ \cdots \ a_n(k)]^T \in \mathbb{R}^{2n}$$

$$r(k) = [r_1^d(k) \ \cdots \ r_n^d(k) \ r_1^a(k) \ \cdots \ r_n^a(k)]^T \in \mathbb{R}^{2n}.$$

By defining appropriate matrices $A_\mu \in \mathbb{R}^{2n \times 2n}$ for $\mu = 0, 1, \dots, \mu_{\text{max}}$ we can rewrite equations (9) and (10) as:

$$\begin{aligned} x_i(k) = \max \left\{ r_i(k), \max_j (x_j(k) + [A_0]_{ij}), \right. \\ \max_j (x_j(k-1) + [A_1]_{ij}), \dots, \\ \left. \max_j (x_j(k - \mu_{\text{max}}) + [A_{\mu_{\text{max}}}]_{ij}) \right\} \quad (11) \end{aligned}$$

where $[A_\mu]_{ij}$ are the (i, j) th entries of the matrices A_μ , $\mu = 0, 1, \dots, \mu_{\text{max}}$, respectively. The matrices A_μ can be completed by adding $[A_\mu]_{ij} = -\infty$ for all combinations (μ, i, j) that do not appear in (11).

Now we introduce some notation from max-plus algebra. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition (\oplus) and multiplication (\otimes) are defined as follows [1]:

$$x \oplus y = \max(x, y) \quad x \otimes y = x + y$$

for $x, y \in \mathbb{R}_\varepsilon$ and

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^n a_{ik} \otimes c_{kj} = \max_{k=1, \dots, n} (a_{ik} + c_{kj})$$

$$[A \odot B]_{ij} = a_{ij} + b_{ij}$$

for $A, B \in \mathbb{R}_\varepsilon^{m \times n}$, $C \in \mathbb{R}_\varepsilon^{n \times p}$. The matrix \mathcal{E} is the max-plus-algebraic zero matrix: $[\mathcal{E}]_{ij} = \varepsilon$ for all i, j . A max-plus diagonal matrix $S = \text{diag}_{\oplus}(s_1, \dots, s_n)$ has elements $[S]_{i,j} = \varepsilon$ for $i \neq j$ and diagonal elements $[S]_{i,i} = s_i$ for $i = 1, \dots, n$. A max-plus permutation matrix $T \in \mathbb{R}_\varepsilon^{m \times m}$ has one zero in each row and one zero in each column and ε elsewhere.

In max-plus notation, equation (11) becomes

$$x_i(k) = r_i(k) \oplus \bigoplus_{\mu=0}^{\mu_{\text{max}}} \bigoplus_{j=1}^n x_j(k - \mu) \otimes [A_\mu]_{ij}$$

and in matrix-notation we obtain

$$x(k) = \bigoplus_{\mu=0}^{\mu_{\text{max}}} A_\mu \otimes x(k - \mu) \oplus r(k) \quad (12)$$

as was already shown in [2], [7], [12]. Model (12) describes the uncontrolled operation of the railway network in which some trains should give pre-defined connections to other trains and the order of trains on the same track is fixed and no control is applied to the network.

In practice the running times, dwell times, headway times, and connection times are not always constant, but may deviate from their nominal values, which may cause delays in the network. Furthermore, if the delay of a train becomes too large, then it is sometimes better — from a global performance viewpoint — to change the departure order on a specific track, to let a connecting train depart anyway, or to reroute the train over another parallel track. This is done in order to prevent an accumulation of delays in the network. Now consider a railway network at time instant t . We define a parameter vector $\theta(k, t)$ consisting of all running times and dwell times in the network in cycle k with all information known at time t . We assume that all events that have taken place up to time instant t have been measured and that we can substitute these measured values of running times and dwell times into the vector $\theta(k, t)$. The times for events that have not taken place we can use estimates from observers or if no estimates are available we can use the nominal value.

Define at time instant t the control vector $u(k, t)$ that determines the operation mode of the network in cycle k , where each mode corresponds to a different set of pre-defined or broken connections and a specific order of train departures. We allow the system to switch between different modes, allowing us to break train connections and to change the order of trains. Note that any broken connection or change of train order leads to a new model, similar to the nominal equation (12), but now with adapted system matrices.

With the parameter vector $\theta(k, t)$ and the control vector $u(k, t)$ at time t in cycle k we obtain the system matrices $A_\mu(\theta(k, t), u(k, t))$. Often we use $A_\mu(k, t)$ as short notation. Note that in controlled operation it may occasionally happen that a delayed train of the k th cycle is rescheduled behind a train in the $(k + \alpha)$ th cycle with $\alpha > 0$. Such a value $\alpha > 0$ corresponds to a value $\mu = -\alpha < 0$. This means that the lower bound μ in (12) may become a negative number and we obtain the following system equation for the controlled operation:

$$x(k) = \bigoplus_{\mu=\mu_{\min}}^{\mu_{\max}} A_\mu(k, t) \otimes x(k - \mu) \oplus r(k) \quad (13)$$

with $\mu_{\min} \leq 0$.

III. STRUCTURE ANALYSIS OF THE SYSTEM MATRICES

From (13) we can observe that in each mode the railway system is described by the matrices A_μ , $\mu = 0, \dots, \mu_{\max}$. In this section we will study the structure of the matrices. Consider the max-plus linear system (13). Now define the matrices $A_{\mu,1}(k, t)$, $A_{\mu,2}(k, t)$, $A_{\mu,3}(k, t)$, $A_{\mu,4}(k, t)$, $A_{\mu,5}(k, t) \in \mathbb{R}_\varepsilon^{n \times n}$ at time t in cycle k :

Running matrix:

$$A_{\mu,1}(k, t) = \begin{cases} \begin{bmatrix} \tau_{\text{run},1}(k, t) & \varepsilon & \dots & \varepsilon \\ \varepsilon & \tau_{\text{run},2}(k, t) & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon & \dots & \dots & \tau_{\text{run},n}(k, t) \end{bmatrix} & \text{for } \mu = 0 \\ \varepsilon & \text{for } \mu \neq 0 \end{cases}$$

Dwell matrix:

$$[A_{\mu,2}(k, t)]_{ij} = \begin{cases} \tau_{\text{dwell},p_i}(k, t) & \text{if } j = p_i \text{ and } \delta_{ij}^p = \mu \\ \varepsilon & \text{elsewhere} \end{cases}$$

Connection matrix:

$$[A_{\mu,3}(k, t)]_{ij} = \begin{cases} \tau_{\text{connect},ij} & \text{if } j \in \mathcal{C}_i(k, t) \text{ and } \delta_{ij}^c = \mu \\ \varepsilon & \text{elsewhere} \end{cases}$$

Headway matrix (same direction):

$$[A_{\mu,4}(k, t)]_{ij} = \begin{cases} \tau_{\text{headway},ij} & \text{if } j \in \mathcal{F}_i(k, t) \text{ and } \delta_{ij}^h = \mu \\ \varepsilon & \text{elsewhere} \end{cases}$$

Headway matrix (opposite direction):

$$[A_{\mu,5}(k, t)]_{ij} = \begin{cases} \tau_{\text{wait},ij} & \text{if } j \in \mathcal{W}_i(k, t) \text{ and } \delta_{ij}^w = \mu \\ \varepsilon & \text{elsewhere} \end{cases}$$

Now we find the following structure in the matrices:

$$A_\mu(k, t) = \begin{bmatrix} A_{\mu,4}(k, t) & A_{\mu,2}(k, t) \oplus A_{\mu,3}(k, t) \oplus A_{\mu,5}(k, t) \\ A_{\mu,1}(k, t) & A_{\mu,4}(k, t) \end{bmatrix} \quad (14)$$

where the matrix $A_{\mu,1}$ represents the running constraints, the matrices $A_{\mu,2}$ represent the dwell constraints, the matrices $A_{\mu,3}$ represent the connection constraints, the matrices $A_{\mu,4}$ represent the headway constraints for trains in the same direction, and the matrices $A_{\mu,5}$ represent the headway constraints for trains in the opposite directions. We will now discuss and analyze the structure of the five matrices $A_{\mu,1}, \dots, A_{\mu,5}$ in more detail.

The running matrix $A_{\mu,1}$:

This matrix represents the running constraints. We assume that for every train the arrival has the same cycle index as the departure, and furthermore we assume that each arrival $a_i(k)$ is related to the departure $d_i(k)$. Therefore the matrix $A_{\mu,1}$ will be a max-plus diagonal matrix $A_{\mu,1}(k, t) = \text{diag}_{\oplus} [\tau_{\text{run},1}(k, t) \quad \tau_{\text{run},2}(k, t) \quad \dots \quad \tau_{\text{run},n}(k, t)]$ for $\mu = 0$ and a zero matrix $A_{\mu,1}(k, t) = \varepsilon$ for $\mu \neq 0$. The values $\tau_{\text{run},1}(k, t), \dots, \tau_{\text{run},n}(k, t)$ are entries of the vector $\theta(k, t)$ and may change in time and per cycle. The matrix $A_{\mu,1}(k, t)$ only depends on $\theta(k, t)$ and not on the control vector $u(k, t)$.

The dwell matrix $A_{\mu,2}$:

This matrix represents the dwell constraints. Let n_L be the number of lines in the network and let $n_{l,i}$ be the number of trains on track i , $i = 1, \dots, n_L$, so $n_1 + n_2 + \dots + n_L = n$. Now we can define a max-plus permutation matrix E_D that reshuffles the states by line in chronological order:

$$A_{\mu,2}(k, t) = E_D \otimes \begin{bmatrix} \bar{A}_{\mu,2,1}(k, t) & \varepsilon & \dots & \varepsilon \\ \varepsilon & \bar{A}_{\mu,2,1}(k, t) & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon & \dots & \dots & \bar{A}_{\mu,2,n_L}(k, t) \end{bmatrix} \otimes E_D^T$$

For each line the initial station and terminal station are the same, leading to a cyclic behavior of the line, and we obtain for line i a sub-matrix of the form:

$$\bar{A}_{\mu,2,i}(k, t) = \begin{bmatrix} \varepsilon & \dots & \varepsilon & \bar{\tau}_{\text{dw},\mu,i,n_{l,i}}(k, t) \\ \bar{\tau}_{\text{dw},\mu,i,1}(k, t) & \dots & \varepsilon & \varepsilon \\ \vdots & \ddots & & \vdots \\ \varepsilon & \dots & \bar{\tau}_{\text{dw},\mu,i,n_{l,i}-1}(k, t) & \varepsilon \end{bmatrix}$$

The dwell matrix $A_{\mu,2}(k, t)$ only depends on $\theta(k, t)$ and not on the control vector $u(k, t)$. All finite values $\bar{\tau}_{\text{dw},\mu,i,\ell}(k, t)$, $i = 1, \dots, n_L$, $\ell = 1, \dots, n_{l,i}$ are entries of the vector $\theta(k, t)$ and may change in time and per cycle.

The connection matrix $A_{\mu,3}$:

This matrix represents the connection constraints and only depends on the control vector $u(k, t)$ and not on the parameter vector $\theta(k, t)$. Now collect all minimum connection times

in one matrix C with $[C]_{ij} = \tau_{\text{connect},ij}$ for all $i = 1, \dots, n$, $j \in \mathcal{C}_i(k, t)$ and ε elsewhere. Define the matrices $Q_\mu(k, t)$ such that

$$[Q_\mu(k, t)]_{ij} = \begin{cases} 0 & \text{if } j \in \mathcal{C}_i(k), v_{\mu,ij}(k, t) = 0, \text{ and } \delta_{ij}^c = \mu \\ \varepsilon & \text{if } j \in \mathcal{C}_i(k), v_{\mu,ij}(k, t) = 1, \text{ and } \delta_{ij}^c = \mu \\ \varepsilon & \text{elsewhere} \end{cases}$$

where the binary control variable $v_{\mu,ij}(k, t) \in \{0, 1\}$ determines whether the connection between train i in cycle k and train j in cycle $(k - \mu)$ will be broken or not ($v = 0$ means ‘make the connection’ and $v = 1$ means ‘break the connection’).

The elements $v_{\mu,ij}(k, t)$, $j \in \mathcal{C}_i(k)$, $i = 1, \dots, n$ can be stacked in one vector $v(k, t)$ which now controls the connections in the network and we obtain

$$A_{\mu,3}(k, t) = Q_\mu(k, t) \odot C$$

For computational reasons during the implementation we will introduce a variable $\beta \ll 0$ that replaces all ε values. If β is sufficiently negative, it will not have any influence on the final outcome of the control algorithm.

This will result in the matrix

$$[Q_\mu(k, t)]_{ij} = \begin{cases} v_{\mu,ij}(k, t)\beta & \text{if } j \in \mathcal{C}_i(k) \text{ and } \delta_{ij}^c = \mu \\ \beta & \text{elsewhere} \end{cases}$$

The headway matrices $A_{\mu,4}$ and $A_{\mu,5}$:

The matrix $A_{\mu,4}$ represents the headway constraints for trains in the same direction and $A_{\mu,5}$ represents the headway constraints for trains in the opposite directions.

Let n_T be the number of tracks in the network and let n_m be the number of trains on track m , $m = 1, \dots, n_T$, so $n_1 + n_2 + \dots + n_{n_T} = n$.

Now we can define a permutation matrix $E_H = [E_{H,1} \ E_{H,2} \ \dots \ E_{H,n_T}]$ with $E_{H,m} \in \mathbb{R}_\varepsilon^{n_m \times n_m}$ for $m = 1, \dots, n_T$, that reshuffles the states by track in nominal chronological order:

$$A_{\mu,4}(k, t) = E_{H \otimes} \begin{bmatrix} \bar{A}_{\mu,4,1}(k, t) & \varepsilon & \dots & \varepsilon \\ \varepsilon & \bar{A}_{\mu,4,2}(k, t) & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \bar{A}_{\mu,4,n_T}(k, t) \end{bmatrix} \otimes E_H^T$$

$$A_{\mu,5}(k, t) = E_{H \otimes} \begin{bmatrix} \bar{A}_{\mu,5,1}(k, t) & \varepsilon & \dots & \varepsilon \\ \varepsilon & \bar{A}_{\mu,5,2}(k, t) & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \bar{A}_{\mu,5,n_T}(k, t) \end{bmatrix} \otimes E_H^T$$

We collect all headway times $\tau_{\text{headway},ij}$ (for same direction) in the matrix H such that $[H]_{ij} = \tau_{\text{headway},ij}$ and we collect all headway times $\tau_{\text{wait},ij}$ (for opposite direction) in the matrix W such that $[W]_{ij} = \tau_{\text{wait},ij}$. Now let n_m be the number of trains on track m and define the matrices

$$\bar{H}_m, \bar{W}_m, M_m, N_m, D_m, \bar{D}_m \in \mathbb{R}_\varepsilon^{n_m \times n_m}:$$

$$\bar{H}_m = E_{H,m}^T \odot H \odot E_{H,m}$$

$$\bar{W}_m = E_{H,m}^T \odot W \odot E_{H,m}$$

$$[M_{\mu,m}(k, t)]_{ij} = \begin{cases} \varepsilon & \text{if train } i \text{ in cycle } k \text{ is scheduled} \\ & \text{before train } j \text{ in cycle } k - \mu \\ 0 & \text{if train } i \text{ in cycle } k \text{ is scheduled} \\ & \text{behind train } j \text{ in cycle } k - \mu \end{cases}$$

$$[D_m]_{ij} = \begin{cases} 0 & \text{if } i \text{ and } j \text{ run in the same direction on} \\ & \text{track } m \\ \varepsilon & \text{if } i \text{ and } j \text{ run in the opposite direction} \\ & \text{on track } m \end{cases}$$

$$[\bar{D}_m]_{ij} = \begin{cases} \varepsilon & \text{if } i \text{ and } j \text{ run in the same direction on} \\ & \text{track } m \\ 0 & \text{if } i \text{ and } j \text{ run in the opposite direction} \\ & \text{on track } m \end{cases}$$

With the above definitions we find for $\bar{A}_{\mu,4,m}$ and $\bar{A}_{\mu,5,m}$:

$$\bar{A}_{\mu,4,m}(k, t) = M_{\mu,m}(k, t) \odot D_m \odot \bar{H}_m \quad (15)$$

$$\bar{A}_{\mu,5,m}(k, t) = M_{\mu,m}(k, t) \odot \bar{D}_m \odot \bar{W}_m \quad (16)$$

The matrix \bar{H}_m and \bar{W}_m consists of all headway times between the trains on track m . The matrices D_m and \bar{D}_m determine the direction of the trains on track m and are complementary.

By ordering the trains on track m in nominal chronological order, the matrix $M_{0,m}$ for the uncontrolled case will become a lower-triangle matrix with 0 on the lower sub-diagonals and ε on the diagonal and upper sub-diagonals:

$$M_{0,m} = \begin{bmatrix} \varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon \\ 0 & \varepsilon & \ddots & \varepsilon & \varepsilon \\ 0 & 0 & \ddots & \varepsilon & \varepsilon \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \varepsilon \end{bmatrix} \in \mathbb{R}_\varepsilon^{n_m \times n_m}$$

For the uncontrolled case we also find $M_{\mu,m} = \varepsilon$ for $\mu < 0$ and $M_{\mu,m} = 0$ for $\mu > 0$ shows that in the nominal case all trains on track m in cycle k are scheduled behind all trains on track m in cycle $k - \mu$ and before all trains on track m in cycle $k + \mu$.

In controlled operation it can happen that a train from a previous cycle is scheduled behind a train in the present cycle. In that case the shift δ_{ij}^h in (10) becomes a negative number.

In controlled operation the matrices $M_{\mu,m}(k)$ may change. Now define the control variable $w_{\mu,m}(k, t) \in \{0, 1\}^{n_m \times n_m}$,

then the matrices $M_{\mu,m}$ can be parameterized by

$$M_{\mu,m}(k,t) = \beta \otimes \begin{bmatrix} [w_{\mu,m}(k,t)]_{1,1} & \cdots & [w_{\mu,m}(k,t)]_{1,n_m} \\ [w_{\mu,m}(k,t)]_{2,1} & \cdots & [w_{\mu,m}(k,t)]_{2,n_m} \\ \vdots & & \vdots \\ [w_{\mu,m}(k,t)]_{n_m,1} & \cdots & [w_{\mu,m}(k,t)]_{n_m,n_m} \end{bmatrix} \quad (17)$$

where, similar to the implementation of the connection matrix, we use the variable $\beta \ll 0$ to replace all ε values. The variables $[w_{\mu,m}(k,t)]_{i,j}$ have the following important properties

$$[w_0(k,t)]_{i,j} + [w_0(k,t)]_{j,i} = 1, \quad \text{for } i \neq j \quad (18)$$

$$[w_{-\mu,m}(k,t)]_{j,i} + [w_{\mu,m}(k,t)]_{i,j} = 1 \quad \text{for } \mu \neq 0 \quad (19)$$

$$[w_0(k,t)]_{i,i} = 1, \quad \forall i \quad (20)$$

This property can be explained as follows: Consider train i in cycle k and train j in cycle $k - \mu$. We find that either train i in cycle k is scheduled behind train j in cycle $k - \mu$, so $[w_{\mu,m}(k,t)]_{i,j} = 0$ and $[w_{-\mu,m}(k - \mu, t)]_{j,i} = 1$, or train j in cycle $k - \mu$ is scheduled behind train i in cycle k , so $[w_{\mu,m}(k,t)]_{i,j} = 1$ and $[w_{-\mu,m}(k - \mu, t)]_{j,i} = 0$.

The elements $[w_{\mu,m}(k,t)]_{i,j}$, $\mu = -\mu_{max}, \dots, \mu_{max}$, $m = 1, \dots, n_T$, $i, j = 1, \dots, n_m$ can be stacked in one vector $w(k,t)$, which now controls the headway constraints in the network. Now the order of the trains in cycle k is defined by the binary vector $w(k,t)$ and we obtain

$$\begin{aligned} A_{\mu,4}(k,t) &= \bigoplus_{m=1}^{n_T} E_{H,m} \otimes \bar{A}_{\mu,4,m} \otimes E_{H,m}^T \\ &= \bigoplus_{m=1}^{n_T} E_{H,m} \otimes (M_{\mu,m}(w_{\mu,m}(k,t)) \odot D_m \odot \bar{H}_m) \otimes E_{H,m}^T \end{aligned} \quad (21)$$

$$\begin{aligned} A_{\mu,5}(k,t) &= \bigoplus_{m=1}^{n_T} E_{H,m} \otimes \bar{A}_{\mu,5,m} \otimes E_{H,m}^T \\ &= \bigoplus_{m=1}^{n_T} E_{H,m} \otimes (M_{\mu,m}(w_{\mu,m}(k,t)) \odot \bar{D}_m \odot \bar{W}_m) \otimes E_{H,m}^T \end{aligned} \quad (22)$$

IV. DOUBLE TRACKS

In the previous section we have assumed that the index m refers to a single track between two stations, on which no overtaking is possible. However, very often there are multiple tracks between two stations. In busy railway networks, such as in the Netherlands, the major part has two tracks, one track for each direction. Furthermore, some parts of the network have four tracks, two for each direction, so that the slow and fast railway traffic can be split. In this paper we will handle a double track as a bundled track,

with two subtracks.

Consider a bundled track m with a track m_A and track m_B : Define n_m integers $s_{m,i}(k,t) \in \{0,1\}$, $i = 1, \dots, n_m$, such that $s_{m,i}(k,t) = 0$ if train i in cycle k runs over track m_A and $s_{m,i}(k,t) = 1$ if train i in cycle k runs over track m_B . Consider the matrix $M_{\mu,m}(w_{\mu,m}(k,t))$ from equation (17). If train i in cycle k and j in cycle $k + \mu$ (with $i \neq j$) are on the same subtrack (so $s_{m,i}(k,t) = s_{m,j}(k + \mu)$), then we have to determine the order between these two trains. We have either $[w_{\mu,m}(k,t)]_{i,j} = 0$ and $[w_{-\mu,m}(k,t)]_{j,i} = 1$ or we have $[w_{\mu,m}(k,t)]_{i,j} = 1$ and $[w_{-\mu,m}(k,t)]_{j,i} = 0$, so (compare (19))

$$[w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} = 1$$

If train i in cycle k and j in cycle $k + \mu$ (with $i \neq j$) are not on the same subtrack (so $s_{m,i}(k,t) \neq s_{m,j}(k + \mu)$), then the order between the two trains is irrelevant. We have $[w_{\mu,m}(k,t)]_{i,j} = 1$ and $[w_{-\mu,m}(k,t)]_{j,i} = 1$ and so

$$[w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} = 2$$

This results in the following inequality constraints:

$$\begin{aligned} [w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} &\geq s_{m,i}(k,t) + (1 - s_{m,j}(k + \mu, t)) \\ [w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} &\geq (1 - s_{m,i}(k,t)) + s_{m,j}(k + \mu, t) \\ [w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} &\leq 3 - s_{m,i}(k,t) - s_{m,j}(k + \mu, t) \\ [w_{\mu,m}(k,t)]_{i,j} + [w_{-\mu,m}(k,t)]_{j,i} &\leq 3 - (1 - s_{m,i}(k,t)) - (1 - s_{m,j}(k + \mu, t)) \end{aligned}$$

Let \mathcal{D} denote the set of all bundled tracks, then the elements $s_{m,i,j}(k,t)$, $m \in \mathcal{D}$, $i, j = 1, \dots, n$ can be stacked in one vector $s(k,t)$. This vector $s(k,t)$ is now a decision variable that decides which train is running over which track in the case of double tracks.

V. JOINED TRAINS

In the Dutch railway network there are a few lines for which two trains are coupled on parts of the line and run separately on other parts of the line. If one of the trains has a delay at the station where the trains have to be coupled, the coupling may be canceled and both trains will run separately to the end station. Define a coupling variable $\sigma_{ij}(k,t)$ and let train i and j be coupled for $\sigma_{ij}(k,t) = 0$ and decoupled for $\sigma_{ij}(k,t) = 1$ with a minimum headway $\tau_{\text{headway},ij}$ if they are not coupled. If the trains are not coupled, we have the variables $[w_{0,m}(k,t)]_{i,j}$ and $[w_{0,m}(k,t)]_{j,i}$ to decide which train goes first (Section 2). Typically train i and train j are scheduled in the same cycle k so we only consider $[w_{0,m}(k,t)]_{i,j}$ and $[w_{0,m}(k,t)]_{j,i}$ in (17). If the trains are coupled ($\sigma_{ij}(k,t) = 0$) we must have $[w_{0,m}(k,t)]_{i,j} = [w_{0,m}(k,t)]_{j,i} = 0$. This makes train i to follow train j with headway 0 and train j will follow train i with headway

0, which means that train i and j have the same departure and arrival times. If train 1 precedes train 2 ($\sigma_{ij}(k, t) = 1$ and $u_{ij}(k, t) = 0$) we must have $[w_{0,m}(k, t)]_{i,j} = \tau$ and $[w_{0,m}(k, t)]_{j,i} = \beta + \tau$.

If train 2 precedes train 1 ($\sigma_{ij}(k, t) = 1$ and $u_{ij}(k, t) = 1$) we must have $[w_{0,m}(k, t)]_{i,j} = \beta + \tau$ and $[w_{0,m}(k, t)]_{j,i} = \tau$. This results in the following inequality constraints:

$$\begin{aligned} u_{ij}(k, t) &\leq \sigma_{ij}(k, t) \\ [w_{0,m}(k, t)]_{i,j} &= \sigma_{ij}(k, t)\tau + u_{ij}(k, t)\beta \\ [w_{0,m}(k, t)]_{j,i} &= \sigma_{ij}(k, t)(\tau + \beta) - u_{ij}(k, t)\beta \end{aligned}$$

Remark:

Note that in principle we can split and couple trains over the whole line now. In practice the decision to couple the trains or not is taken only once at a specific station, which means that the variables $\sigma_{ij}(k, t)$ will be equal for the rest of the line from that specific station on.

VI. AN AFFINE FORM FOR THE SYSTEM MATRIX

In the above the control measures are restricted to breaking connections between trains ($v(k, t)$), changing the order of trains running on the same track ($w(k, t)$), allocating trains to specific tracks ($s(k, t)$), and joining trains ($\sigma(k, t)$). For cycle k we collect all control variables in one vector:

$$u^T(k, t) = [w^T(k, t) \quad v^T(k, t) \quad s^T(k, t) \quad \sigma^T(k, t)]$$

Note that the matrices $A_{\mu,1}$, $A_{\mu,2}$ only depend on the parameter vector $\theta(k, t)$ and the matrices $A_{\mu,3}$, $A_{\mu,4}$, and $A_{\mu,5}$ depend on the control vector $u(k, t)$.

The matrices $M_{\mu,m}$ and Q_{μ} are affine in the control vector $u(k, t)$ which means that also the matrices $A_{\mu,3}(u(k, t))$, $A_{\mu,4}(u(k, t))$, and $A_{\mu,5}(u(k, t))$ are affine in the control vector $u(k, t)$.

Finally $A_{\mu,1}(\theta(k, t))$ is affine in the parameter vector $\theta(k, t)$ and the submatrices $\bar{A}_{\mu,2,i}$ are affine in $\theta(k, t)$, so we obtain that $A_{\mu,2}(\theta(k, t))$ is also affine in the parameter vector $\theta(k, t)$.

Now we obtain

$$A_{\mu}(k, t) = A_{\mu}^0 \odot \sum_{i=1}^{n_{\theta}} A_{\mu,i}^{\theta} \theta_i(k, t) \odot \sum_{\ell=1}^{n_u} A_{\mu,\ell}^u \cdot u_{\ell}(k, t) \quad (23)$$

where the constant matrix A_{μ}^0 contains the values ε , the matrices $A_{\mu,i}^{\theta}$ and $A_{\mu,\ell}^u$ have finite entries, n_{θ} is number of parameters in $\theta(k, t)$, and n_u is number of control variables in $u(k, t)$. Note that the multiplication in the affine part of (23) is the conventional multiplication.

VII. DISCUSSION

In this paper we use a max-plus-linear system description to describe a perturbed railway network. The system matrices are highly structured and the submatrices for running constraints, connection constraints, and headway constraints have been studied. All control actions, such as changing the train order, canceling connections, dispatching over double tracks, and joining and splitting trains are described by the

entries of a control vector $u(k, t)$. All minimum running times, minimum connection times, and minimum headway times are collected in a parameter vector $\theta(k, t)$. The system matrices are affine in the control vector u and the parameter vector θ . The derived model can be used to control the system with a model predictive control strategy ([12], [15]).

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