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On the convergence of ant colony optimization with stench pheromone[*]

Z. Cong, B. De Schutter, and R. Babuška

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Zhe Cong, Bart De Schutter, and Robert Babuška

Abstract— Ant Colony Optimization (ACO) has proved to be a powerful metaheuristic for combinatorial optimization problems. From a theoretical point of view, the convergence of the ACO algorithm is an important issue. In this paper, we analyze the convergence properties of a recently introduced ACO algorithm, called ACO with stench pheromone (SACO), which can be used to solve dynamic traffic routing problems through finding the minimum cost routes in a traffic network. This new algorithm has two different types of pheromone: the regular pheromone that is used to attract artificial ants to the arc in the network with the lowest cost, and the stench pheromone that is used to push ants away when too many ants converge to that arc. As a first step of a convergence proof for SACO, we consider a network with two arcs. We show that the process of pheromone update will transit among different modes, and finally stay in a stable mode, thus proving convergence for this given case.

I. INTRODUCTION

Ant Colony Optimization (ACO) is inspired by collective behavior of foraging ants searching paths between their nest and a food source. It was first proposed by Dorigo in his PhD thesis [1]. Since then, the field of ACO has grown tremendously. The Ant System (AS) [2] is an early example of the ACO algorithm, and it achieved promising results in solving traveling salesman problems. Several important ACO variants, e.g., Ant Colony System [3] and MAX-MIN Ant System [4], have been developed to extend and improve the original AS algorithm. These variants mainly differ from the AS algorithm in the pheromone update rules, where respectively the iteration-best rule and the best-so-far rule are used to replace the original AS update rule. Some other ACO variants focus on special issues. For instance, AntNet [5] is designed to solve the routing problem in telecommunication networks, and Ant Colony Learning [6] is used for finding optimal control policies for automatic control systems. A broad overview on ACO can be found in [7], [8].

From a theoretical point of view, convergence is an important topic. For ACO, convergence analysis has been introduced in [9], [10], [11]. Gutjahr [9], [10] presented a convergence proof for an ACO algorithm called graph-based ant system. The proof shows that the algorithm can generate an optimal solution at least once during the optimization. Stützle and Dorigo [11] prove that the $ACO_{gb,\tau_{min}}$ algorithm, which employs the global-best update rule, asymptotically converges to the optimal solution. However, because of the complexity and the diversity of ACO, there is no a general method to prove convergence for the entire class of ACO algorithms.

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In this paper, we prove the convergence of a novel ant-based algorithm — ACO with stench pheromone [12], abbreviated as ACO-SP. Motivated by the similarities between traffic systems and ant systems, we have developed this new algorithm in [12] to solve the dynamic traffic routing problem in freeway networks, for which ants are used to determine appropriate routes for vehicles according to a global objective. Generally speaking, this new algorithm has two types of pheromone with opposite functions: the same regular pheromone as in the standard ACO algorithm and the newly introduced stench pheromone. The former is used to attract ants to the best route in the network so as to guarantee the effectiveness of the algorithm, while the latter is used to decrease the total pheromone levels on the arcs to prevent the regular pheromone from being accumulated too much on the same arc. In such a way, some of the ants are pushed away when too many ants are crowded on the same route, and those ants then choose an alternative route in the network. Although ACO-SP is introduced for freeway networks, it can be applied to solve optimal flow distribution and routing problems in any other type of network subject to (soft) capacity constraints. Therefore, in this paper, we analyze the convergence properties in a general network. For the sake of simplicity and as this paper presents a first step towards a full convergence proof for ACO-SP, we consider a network with only two arcs.

The rest of this paper is structured as follows. Section II recapitulates the ACO-SP algorithm. Next, in Section III, we state the convergence problem, as well as the assumptions for the proof. Next, we prove the convergence of an auxiliary function in Section IV, and consequently prove the convergence of the pheromone levels in ACO-SP for a two-arc network in Section V. Finally, we illustrate the convergence of ACO-SP by simulation for a large-scale network in Section VI. Proofs of supporting lemmas are presented in the appendix.

II. ACO WITH STENCH PHEROMONE

The ACO-SP algorithm [12] originates from the AS algorithm [2]. Ants in ACO-SP can independently choose arcs with a pre-defined probability, and deposit pheromone according to the cost on each arc when traveling in the network. In ACO-SP, there is a critical number of ants on each arc in the network. Once the critical number is exceeded, congestion is considered to occur on that arc. Therefore, the goal of ACO-SP is to determine an optimal assignment of ants in the network so as to find minimum cost routes and to prevent congestion. The function of the stench pheromone is to decrease the pheromone levels on

Algorithm 1 ACO-SP

Input: N^{total}, $N_{i,j}^{\text{crit}}$, τ_0, ρ, T, $\tau_{\min}$

1: $\tau_{i,j} \leftarrow \tau_0$, $\forall (i,j)$
2: **for** $t = 0,\dots,T$ **do**
3:     $\mathscr{R}_{\text{upd}} \leftarrow \emptyset$
4:     $\mathscr{A} \leftarrow \{1,2,\dots,N^{\text{total}}\}$
5:     **for all** ants $a \in \mathscr{A}$ in parallel **do**
6:       $r_a \leftarrow \emptyset$
7:       put ant a in one origin vertex i
8:       **repeat**
9:        select the next vertex j according to (1)
10:        $r_a \leftarrow r_a \cup \{(i,j)\}$
11:       **until** ant a has reached a destination vertex
12:       $\mathscr{R}_{\text{upd}} \leftarrow \mathscr{R}_{\text{upd}} \cup \{r_a\}$
13:     **end for**
14:     calculate the stench function values $G_{i,j}(N_{i,j})$ based on the number of ants $N_{i,j}$ that traveled each on arc (i,j)
15:     update $\tau_{i,j}$: apply (3) for all (i,j)
16: **end for**

Output: $\tau_{i,j}$, $\forall (i,j)$

the congested arcs so that ants will be pushed away and start to search an alternative route. The description of ACO-SP is presented in Algorithm 1. For the inputs, N^{total} is the total number of ants traveling in the network, τ_0 is the initial pheromone level, ρ is the evaporation rate, T is the maximum number of iteration steps, and $\tau_{\min}$ is a given constant preventing the denominator of (1) from becoming zero. The algorithm is explained next.

An ant network is presented by a graph consisting of vertices and arcs that connect vertices. In this network, an ant a constructs a route r_a by moving from one vertex to another vertex and adding the corresponding arc into r_a, until it reaches a destination vertex. More specifically, ant a staying at vertex i chooses vertex j based on a probability $p_a(j|i)$ w.r.t the pheromone $\tau_{i,j}$ associated with arc (i,j). Since the stench pheromone may result in the total pheromone level $\tau_{i,j}$ becoming negative, a lower bound $\tau_{\min} > 0$ is used to prevent the pheromone levels from becoming negative. The probability $p_a(j|i)$ is calculated by:

$$p_a(j|i) = \begin{cases} \dfrac{\max(\tau_{\min},\tau_{i,j})^{\alpha}}{\sum\limits_{l \in \mathcal{N}_{i,a}} \left(\max\{\tau_{\min},\tau_{i,l}\}\right)^{\alpha}}, & \forall j \in \mathcal{N}_{i,a} \\ 0, & \forall j \notin \mathcal{N}_{i,a}, \end{cases} \quad (1)$$

with $\alpha \geq 1$ a parameter, and $\mathcal{N}_{i,a}$ the set of nodes that are connected to i and that have not yet been visited by ant a.

After the ants reached their destination vertices, pheromone is deposited on arc (i,j) according to the fitness function f:

$$\Delta\tau_{i,j}(r_a) = \begin{cases} f(r_a) & \text{if } (i,j) \in r_a \\ 0 & \text{otherwise} \end{cases}$$

where f assigns strictly positive values to each route r_a,

and a higher value of f corresponds to a better solution. At the same time, the stench pheromone is also deposited by a stench function $G(\cdot)$, which can e.g. be defined as the following piecewise affine function:

$$G_{i,j}(N_{i,j}(t)) = \max(0, P(N_{i,j}(t) - N_{i,j}^{\text{crit}})) \quad (2)$$

where $N_{i,j}(t)$ denotes the number of ants that traveled on arc (i,j) in iteration t, $N_{i,j}^{\text{crit}}$ denotes the critical number of ants on arc (i,j), and $P > 0$ denotes a slope. Therefore, the pheromone update equation can be formulated as:

$$\tau_{i,j}(t+1) = (1-\rho)\tau_{i,j}(t) + \sum_{r_a \in \mathscr{R}_{\text{upd}}} \Delta\tau_{i,j}(r_a) - G(N_{i,j}(t)) \quad (3)$$

where ρ denotes the evaporation rate and $\mathscr{R}_{\text{upd}}$ denotes the set of routes that ants constructed from an origin vertex to a destination vertex.

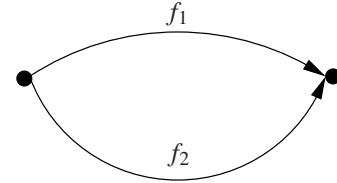## III. PROBLEM STATEMENT

A. Notations and formulations

Fig. 1. Simple network with two arcs

Definition 1: The ACO-SP algorithm is said to converge, if the pheromone levels on all arcs converge as the iteration step $t \to \infty$.

As the first step of the convergence proof, in this paper we investigate a simple network with two arcs as shown in Figure 1. Without loss of generality, we assume that arc 1 is a better solution than arc 2, so the value of the fitness function f_1 of arc 1 is larger than the value of the fitness function f_2 of arc 2:

Assumption 1: $f_1 > f_2$.

Before we proceed with the convergence proof, we first introduce some important notation. The pheromone levels in iteration t on arc 1 and 2 are denoted by $\tau_1(t)$ and $\tau_2(t)$, respectively. According to (2) and (3), they are given by:

$$\begin{aligned} \tau_1(t+1) &= (1-\rho)\tau_1(t) + N_1(t) \cdot f_1 \\ &\quad - \max\left(0, P(N_1(t) - N_1^{\text{crit}})\right) \\ \tau_2(t+1) &= (1-\rho)\tau_2(t) + N_2(t) \cdot f_2 \\ &\quad - \max\left(0, P(N_2(t) - N_2^{\text{crit}})\right) \end{aligned} \quad (4)$$

where $N_1(t)$ and $N_2(t)$ are the numbers of ants that chose arc 1 and arc 2 in iteration t, respectively. Since there are only two arcs in the network, it always holds that $N_1(t) + N_2(t) = N^{\text{total}}$, $t = 1,2,\dots$ According to the attraction mechanism of the ACO algorithm, the arc accumulated with more pheromone has a higher probability to be chosen by ants. The probabilities $p_1(t)$ and $p_2(t)$ for selecting respectively

arc 1 and arc 2 are computed based on[1] (1):

$$p_1(t) = \frac{\max(\tau_{\min}, \tau_1(t))}{\max(\tau_{\min}, \tau_1(t)) + \max(\tau_{\min}, \tau_2(t))}$$

$$p_2(t) = \frac{\max(\tau_{\min}, \tau_2(t))}{\max(\tau_{\min}, \tau_1(t)) + \max(\tau_{\min}, \tau_2(t))} \quad (5)$$

We make the following assumptions regarding the parameters of the ACO-SP algorithm:

Assumption 2:

1) $P \leqslant \min\left(\dfrac{N^{\text{total}} \cdot f_1 - \rho\tau_{\min}}{N^{\text{total}} - N_1^{\text{crit}}}, \dfrac{N^{\text{total}} \cdot f_2 - \rho\tau_{\min}}{N^{\text{total}} - N_2^{\text{crit}}}\right)$

2) $\tau_{\min} \leqslant \frac{f_2}{\rho}$ and $\tau_{\min} \leqslant \tau_0$

3) N^{total} is sufficiently large such that $N_1(t), N_2(t) \geqslant 1, \forall t$

4) $N^{\text{total}} \neq N_1^{\text{crit}}$ and $N^{\text{total}} \neq N_2^{\text{crit}}$

Lemma 1: If Assumptions 2.1 to 2.3 are satisfied, $\tau_1(t), \tau_2(t) \geqslant \tau_{\min}$ holds for all t.

Proof: For $t = 0$, we have $\tau_1(t) = \tau_2(t) = \tau_0 \geqslant \tau_{\min}$ according to Assumption 2.2. For $t = 1, 2, \ldots$ the proof will be done by introducing 3 cases.

Case A. If there is no stench pheromone, we have $G(N_{i,j}(t)) = 0$ according to (2). Therefore, based on (3), the pheromone update equations are given by:

$$\tau_1(t+1) = (1-\rho)\tau_1(t) + N_1(t) \cdot f_1$$
$$\tau_2(t+1) = (1-\rho)\tau_2(t) + N_2(t) \cdot f_2 \quad (6)$$

Since we want to prove that $\tau_1(t+1), \tau_2(t+1) \geqslant \tau_{\min}$, when $\tau_1(t), \tau_2(t) \geqslant \tau_{\min}$, we should show that

$$(1-\rho)\tau_1(t) + N_1(t) \cdot f_1 \geqslant \tau_{\min}$$
$$(1-\rho)\tau_2(t) + N_2(t) \cdot f_2 \geqslant \tau_{\min} \quad (7)$$

According to Assumption 2.3, $N_1(t) \geqslant 1$ and $N_2(t) \geqslant 1$. Therefore, a sufficient condition for (7) to hold is

$$(1-\rho)\tau_{\min} + f_1 \geqslant \tau_{\min}$$
$$(1-\rho)\tau_{\min} + f_2 \geqslant \tau_{\min}$$

To satisfy the inequalities above, we need $\tau_{\min} \leqslant \frac{f_1}{\rho}$ and $\tau_{\min} \leqslant \frac{f_2}{\rho}$. Because of Assumption 1, we only need $\tau_{\min} \leqslant \frac{f_2}{\rho}$ in Assumption 2.2 as the sufficient condition.

Case B. If the stench pheromone is deposited on arc 1, i.e., $N_1(t) > N_1^{\text{crit}}$. According to (2) and (3), the pheromone update equation is then given by:

$$\tau_1(t+1) = (1-\rho)\tau_1(t) + N_1(t) \cdot f_1 - P(N_1(t) - N_1^{\text{crit}})$$

Since we want to prove that $\tau_1(t+1) \geqslant \tau_{\min}$, when $\tau_1(t) \geqslant \tau_{\min}$, we should show that

$$(1-\rho)\tau_1(t) + N_1(t) \cdot f_1 - P(N_1(t) - N_1^{\text{crit}}) \geqslant \tau_{\min}$$

This inequality holds if

$$P \leqslant \frac{(1-\rho)\tau_1(t) - \tau_{\min} + N_1(t) \cdot f_1}{N_1(t) - N_1^{\text{crit}}}$$
$$\leqslant \frac{(1-\rho)\tau_1(t) - \tau_{\min}}{N_1(t) - N_1^{\text{crit}}} + \frac{N_1(t) \cdot f_1}{N_1(t) - N_1^{\text{crit}}} \quad (8)$$

Given a rational function defined by $y(x) = \dfrac{ax}{x-b}$, with $x > b$ and $a, b > 0$, it is easy to verify that $y(\cdot)$ is a monotonically decreasing function. Since we know that $\tau_1(t) \geqslant \tau_{\min}$, and $N_1(t) \leqslant N^{\text{total}}$, a sufficient condition for (8) to hold is

$$P \leqslant \frac{(1-\rho)\tau_{\min} - \tau_{\min}}{N^{\text{total}} - N_1^{\text{crit}}} + \frac{N^{\text{total}} \cdot f_1}{N^{\text{total}} - N_1^{\text{crit}}}$$
$$\leqslant \frac{N^{\text{total}} \cdot f_1 - \rho\tau_{\min}}{N^{\text{total}} - N_1^{\text{crit}}}$$

Case C. Similarly, if stench pheromone is deposited on arc 2, a sufficient condition for $\tau_2(t) \geqslant \tau_{\min}$ is that

$$P \leqslant \frac{N^{\text{total}} \cdot f_2 - \rho\tau_{\min}}{N^{\text{total}} - N_2^{\text{crit}}}$$

As a conclusion, we need Assumption 2.1:

$$P \leqslant \min\left(\frac{N^{\text{total}} \cdot f_1 - \rho\tau_{\min}}{N^{\text{total}} - N_1^{\text{crit}}}, \frac{N^{\text{total}} \cdot f_2 - \rho\tau_{\min}}{N^{\text{total}} - N_2^{\text{crit}}}\right)$$

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With Lemma 1, we can further simplify (5) as:

$$p_1(t) = \frac{\tau_1(t)}{\tau_1(t) + \tau_2(t)}$$
$$p_2(t) = \frac{\tau_2(t)}{\tau_1(t) + \tau_2(t)} \quad (9)$$

The expected values of the numbers of ants that will select arc 1 and arc 2 in iteration t can be computed based on (9):

$$N_1(t) = p_1(t) \cdot N^{\text{total}} = \frac{\tau_1(t)}{\tau_1(t) + \tau_2(t)} \cdot N^{\text{total}}$$
$$N_2(t) = p_2(t) \cdot N^{\text{total}} = \frac{\tau_2(t)}{\tau_1(t) + \tau_2(t)} \cdot N^{\text{total}} \quad (10)$$

B. Cases and modes

According to the relationship between the total number of ants N^{total} and the critical numbers of ants on arcs 1 and 2, N_1^{crit} and N_2^{crit}, we have four different cases:

1) **Case 1**: $N^{\text{total}} < \min(N_1^{\text{crit}}, N_2^{\text{crit}})$;
2) **Case 2**: $N_1^{\text{crit}} < N^{\text{total}} < N_2^{\text{crit}}$;
3) **Case 3**: $N_2^{\text{crit}} < N^{\text{total}} < N_1^{\text{crit}}$;
4) **Case 4**: $N^{\text{total}} > \max(N_1^{\text{crit}}, N_2^{\text{crit}})$;

where N^{total} neither equals to N_1^{crit} nor N_2^{crit} due to Assumption 2.4. In each case, we can divide the process of pheromone updating into four different modes based on whether stench pheromone is deposited or not on the arcs:

1) M1: No stench pheromone is deposited;
2) M2: Stench pheromone is only deposited on arc 1;
3) M3: Stench pheromone is only deposited on arc 2;
4) M4: Stench pheromone is deposited on both arcs.

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[1]For illustration purposes, we set $\alpha = 1$ in this paper, but the proof is similar for $\alpha \geqslant 1$

In this section, we introduce an auxiliary function $F(t) = \frac{\tau_1(t)}{\tau_2(t)}$. Since both $\tau_1(t)$ and $\tau_2(t)$ are positive, we have $F(t) > 0$ for all t. This function can be used to mathematically define the four modes M1 to M4. Taking M1 as an example, if there is no stench pheromone deposited on either of the arcs, the numbers $N_1(t)$ and $N_2(t)$ are not greater than the corresponding critical number on each arc, that is $N_1(t) \leqslant N_1^{\text{crit}}$ and $N_2(t) \leqslant N_2^{\text{crit}}$. By using (10), we then have:

$$\frac{\tau_1(t)}{\tau_1(t) + \tau_2(t)} N^{\text{total}} = \frac{F(t)}{F(t)+1} N^{\text{total}} \leqslant N_1^{\text{crit}}, \qquad (11)$$

$$\frac{\tau_2(t)}{\tau_1(t) + \tau_2(t)} N^{\text{total}} = \frac{1}{F(t)+1} N^{\text{total}} \leqslant N_2^{\text{crit}}. \qquad (12)$$

For the sake of compactness, we let $F_1^{\text{b}} = \frac{N_1^{\text{crit}}}{N^{\text{total}} - N_1^{\text{crit}}}$ and $F_2^{\text{b}} = \frac{N^{\text{total}} - N_2^{\text{crit}}}{N_2^{\text{crit}}}$. In Case 1, we simplify (11) and (12) to obtain $F(t) \geqslant F_1^{\text{b}}$ and $F(t) \geqslant F_2^{\text{b}}$. Since in Case 1 both F_1^{b} and F_2^{b} are negative, and $F(t) > 0$ always holds by definition, in Case 1 only M1 is possible, whereas the other modes cannot occur. Using a similar reasoning, each of the 4 cases can be summarized as follows:

Case 1: $N^{\text{total}} < \min(N_1^{\text{crit}}, N_2^{\text{crit}})$

$$\text{M1}: F(t) > 0 ,$$
$$\text{M2}: \text{NR} ,$$
$$\text{M3}: \text{NR} ,$$
$$\text{M4}: \text{NR}$$

where NR indicates "not reachable". The process of pheromone updating will stay in M1 as shown in Figure 2.

Case 2: $N_1^{\text{crit}} < N^{\text{total}} < N_2^{\text{crit}}$

$$\text{M1}: 0 < F(t) \leqslant F_1^{\text{b}} ,$$
$$\text{M2}: F(t) > F_1^{\text{b}} ,$$
$$\text{M3}: \text{NR} ,$$
$$\text{M4}: \text{NR}$$

The process of pheromone updating will transit from M1 to M2, and stay in M2 as shown in Figure 3.

Case 3: $N_2^{\text{crit}} < N^{\text{total}} < N_1^{\text{crit}}$

$$\text{M1}: F(t) \geqslant F_2^{\text{b}} ,$$
$$\text{M2}: \text{NR} ,$$
$$\text{M3}: 0 < F(t) < F_2^{\text{b}} ,$$
$$\text{M4}: \text{NR}$$

The process of pheromone updating will transit from M3 to M1, and stay in M1 as shown in Figure 4.

Case 4: $N^{\text{total}} > \max(N_1^{\text{crit}}, N_2^{\text{crit}})$

$$\text{M1}: F_2^{\text{b}} \leqslant F(t) \leqslant F_1^{\text{b}} ,$$
$$\text{M2}: F(t) > \max(F_1^{\text{b}}, F_2^{\text{b}}) ,$$
$$\text{M3}: 0 < F(t) < \min(F_1^{\text{b}}, F_2^{\text{b}}) ,$$
$$\text{M4}: F_1^{\text{b}} < F(t) < F_2^{\text{b}} .$$

[2]The transition indicated by the dashed line can only occur once.
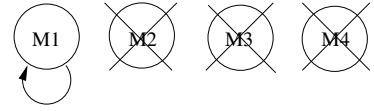

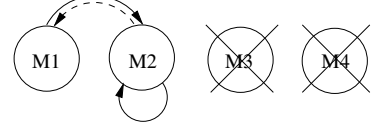Fig. 2.   Mode transitions in Case 1

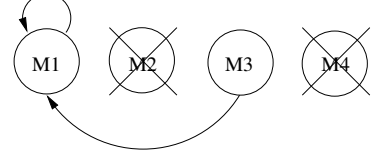

Fig. 3.   Mode transitions in Case[2]2

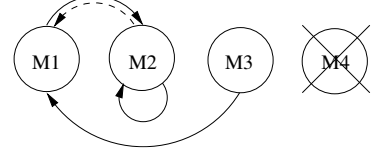Fig. 4.   Mode transitions in Case 3

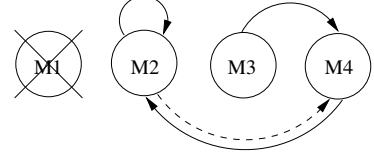Fig. 5a.   Mode transitions of Subcase (a) in Case[2] 4

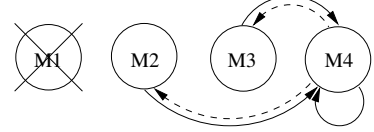Fig. 5b.   Mode transitions of Subcase (b) in Case[2] 4

Fig. 5c.   Mode transitions of Subcase (c) in Case[2] 4

The process of pheromone updating has three sub-cases as shown in Figure 5a–5c:

Subcase 4.a

$$N^{\text{total}} \leqslant N_1^{\text{crit}} + N_2^{\text{crit}}$$

Subcase 4.b

$$N^{\text{total}} > N_1^{\text{crit}} + N_2^{\text{crit}} \text{ and } P \leqslant \frac{(f_1 - f_2)(N^{\text{total}} - N_2^{\text{crit}})}{N^{\text{total}} - N_1^{\text{crit}} - N_2^{\text{crit}}}$$

Subcase 4.c

$$N^{\text{total}} > N_1^{\text{crit}} + N_2^{\text{crit}} \text{ and } P > \frac{(f_1 - f_2)(N^{\text{total}} - N_2^{\text{crit}})}{N^{\text{total}} - N_1^{\text{crit}} - N_2^{\text{crit}}}$$

This will be explained in more detail in Section V.

IV. CONVERGENCE PROPERTIES OF F

In this section, we investigate the properties of F. All lemmas in this section are proven in the appendix. Given in Section III-B, we see that F_1^{b} and F_2^{b} represent the mode

boundaries of F. If the value of F becomes larger or smaller than F_1^{b} or F_2^{b}, it means a transition occurs from one mode to another.

Lemma 2: In M1 and M3, F is a monotonically increasing function.

Define

$$F_1^{\mathrm{equ}} = \frac{PN_1^{\mathrm{crit}}}{N^{\mathrm{total}}(f_2 - f_1) + P(N^{\mathrm{total}} - N_1^{\mathrm{crit}})},$$

$$F_2^{\mathrm{equ}} = \frac{1}{2PN_2^{\mathrm{crit}}}\left(N^{\mathrm{total}}(f_1 - f_2) + P(N_1^{\mathrm{crit}} - N_2^{\mathrm{crit}}) + \right.$$

$$\left. \sqrt{\left(N^{\mathrm{total}}(f_1 - f_2) + P(N_1^{\mathrm{crit}} - N_2^{\mathrm{crit}})\right)^2 + 4P^2 N_1^{\mathrm{crit}} N_2^{\mathrm{crit}}}\right)$$

Lemma 3: If $F_1^{\mathrm{equ}} \geqslant \max(F_1^{\mathrm{b}}, F_2^{\mathrm{b}})$, F_1^{equ} is the only equilibrium point of F in M2. In M2, when $F(t) > F_1^{\mathrm{equ}}$, then $F(t+1) < F(t)$, and when $F(t) < F_1^{\mathrm{equ}}$, then $F(t+1) > F(t)$. If $F_1^{\mathrm{b}} < F_2^{\mathrm{equ}} < F_2^{\mathrm{b}}$, F_2^{equ} is the only equilibrium point of F in M4. In M4, when $F(t) > F_2^{\mathrm{equ}}$, then $F(t+1) < F(t)$, and when $F(t) < F_2^{\mathrm{equ}}$, then $F(t+1) > F(t)$.

Assumption 3:

$$P > \frac{N^{\mathrm{total}}(f_1 - f_2)}{N^{\mathrm{total}} - N_1^{\mathrm{crit}}}$$

With Assumption 3, it is easy to verify $F_1^{\mathrm{equ}} > 0$, and $F_2^{\mathrm{equ}} > 0$ always holds based on its formulation. Therefore, if $F(t) < F_1^{\mathrm{equ}}$ when $F(t)$ is in M2, or if $F(t) < F_2^{\mathrm{equ}}$ when $F(t)$ is in M4, $F(t)$ is a monotonically increasing function, while if $F(t) > F_1^{\mathrm{equ}}$ when $F(t)$ is in M2, or if $F(t) > F_2^{\mathrm{equ}}$ when $F(t)$ is in M4, F is a monotonically decreasing function. In the other words, F always moves towards an equilibrium point when it stays in M2 or M4. However, Lemma 3 does not guarantee that F will converge to either F_1^{equ} or F_2^{equ} as it could still oscillate around F_1^{equ} or F_2^{equ}. Therefore, we introduce another lemma.

Lemma 4: In M2, $\lim_{t \to \infty}|F(t) - F_1^{\mathrm{equ}}| = 0$, and in M4, $\lim_{t \to \infty}|F(t) - F_2^{\mathrm{equ}}| = 0$.

From Lemma 4, we know that F will asymptotically converge to these equilibrium points. However, Lemma 4 can only be applied if F always stays in M2 or M4. It is possible that F jumps out of M2 or M4 due to the mode transition. In such case, the convergence is still not guaranteed.

Lemma 5: In all four modes, if F transits from $\mathrm{M}^{\dagger}$ to M^*, where $\mathrm{M}^{\dagger}$ denotes a mode without an equilibrium point, and M^* denotes a mode with an equilibrium point, F will stay in M^*.

Lemma 5 shows that F will not jump out of M2 or M4 when it enters these modes. More specifically, as shown in Figure 3 and 5a, when F transits from M1 to M2, F will not go back to M1. The case that F may transit from M2 to M1 (shown by the dash line) can only occur when F is initialized in M2, and such a transition can only occur once. Similarly, in Figure 5b, when F transits from M4 to M2, F will not go back to M4, and in Figure 5c, when F transits from M2 and M3 to M4, F will not go back to neither M2 nor M3.

V. CONVERGENCE OF THE PHEROMONE LEVELS

Proposition 1: In **Case 1**, the pheromone levels on both arcs asymptotically converge to a finite value.

Proof: In Case 1, F will only stay in M1. According to Lemma 2, F is a monotonically increasing function in M1, so the value of $F(t)$ will monotonically converge when $t \to \infty$. Let[3] $F' = \lim_{t \to \infty}F(t)$. According to (10),

$$\lim_{t \to \infty}N_1(t) = \lim_{t \to \infty}\frac{F(t)}{F(t)+1}\cdot N^{\mathrm{total}} = \frac{1/F'}{1/F'+1}\cdot N^{\mathrm{total}},$$

$$\lim_{t \to \infty}N_2(t) = \lim_{t \to \infty}\frac{1}{F(t)+1}\cdot N^{\mathrm{total}} = \frac{1}{F'+1}\cdot N^{\mathrm{total}}.$$

Therefore, the numbers of ants $N_1(t)$ and $N_2(t)$ also converge.

Given a difference equation $x(t+1) = ax(t) + b(t)$, with $0 < a < 1$, if we have $\lim_{t \to \infty}b(t) = B$, we know that:

$$\forall \varepsilon > 0, \; \exists T : B - \varepsilon < b(t) < B + \varepsilon, \forall t > T.$$

Therefore,

$$ax(t) + B - \varepsilon < ax(t) + b(t) < ax(t) + B + \varepsilon, \forall t > T.$$

This is equivalent to

$$ax(t) + B - \varepsilon < x(t+1) < ax(t) + B + \varepsilon, \forall t > T.$$

From $x(t+1) < ax(t) + B + \varepsilon$, we can conclude that

$$x(t+1) < a^t x(0) + (a^{t-1} + a^{t-2} + \cdots + 1)(B + \varepsilon)$$

Now select $T' > T$ such that

$$a^t x(0) < \frac{\varepsilon}{1-a}, \; \forall t > T'$$

Since $0 < a < 1$, we can always find such a T'. Then for all $t > T'$ we have

$$x(t+1) < \frac{\varepsilon}{1-a} + \frac{1}{1-a}(B + \varepsilon)$$

$$< \frac{B}{1-a} + \frac{2\varepsilon}{1-a}$$

Moreover, since $a > 0$, we have

$$x(t+1) > \frac{1}{1-a}(B - \varepsilon) > \frac{B}{1-a} - \frac{2\varepsilon}{1-a}$$

Defining $\varepsilon' = \frac{2\varepsilon}{1-a}$, we find

$$\forall \varepsilon' > 0, \; \exists T' : \frac{B}{1-a} - \varepsilon' < x(t+1) < \frac{B}{1-a} + \varepsilon', \forall t > T'.$$

Hence,

$$\lim_{t \to \infty}x(t+1) = \frac{B}{1-a} \qquad (13)$$

Using (13) for (6), it is proven that pheromone levels $\tau_1(t)$ and $\tau_2(t)$ converge. ∎

Proposition 2: In **Case 2**, the pheromone levels on both arcs asymptotically converge.

Proof: We first prove the mode transition in Figure 3. In Case 2, M3 and M4 cannot be reached. Due to Lemma 2,

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[3]F' can be ∞. In that case, $\lim_{t \to \infty}N_1(t) = N^{\mathrm{total}}$, and $\lim_{t \to \infty}N_2(t) = 0$.

if F is initialized in M1, it will keep increasing until reaching F_1^{b}, and then it will transit to M2. If F is initialized in M2, it may transit from M2 to M1. However, Lemma 5 proves that after F transits from M1 to M2, it will stay in M2, because M2 has an equilibrium point F_1^{equ}, while M1 has no equilibrium point. In this way, the process described by Figure 3 is proved.

Since F finally stays in M2 , it will converge to F_1^{equ}, as stated in Lemma 4. According to (10),

$$\lim_{t\to\infty} N_1(t) = \lim_{t\to\infty} \frac{F(t)}{F(t)+1}\cdot N^{\text{total}} = \frac{F_1^{\text{equ}}}{F_1^{\text{equ}}+1}\cdot N^{\text{total}} ,$$

$$\lim_{t\to\infty} N_2(t) = \lim_{t\to\infty} \frac{1}{F(t)+1}\cdot N^{\text{total}} = \frac{1}{F_1^{\text{equ}}+1}\cdot N^{\text{total}} .$$

Therefore, the numbers of ants $N_1(t)$ and $N_2(t)$ also converge, which results in convergence of the pheromone levels $\tau_1(t)$ and $\tau_2(t)$. ∎

Proposition 3: In **Case 3**, the pheromone levels on both arcs asymptotically converge.

Proof: Similar to Proposition 2. ∎

Lemma 6: In **Case 4**, if $N_1^{\text{crit}} < N^{\text{total}} \leqslant N_1^{\text{crit}}+N_2^{\text{crit}}$, then $F_1^{\text{b}} \geqslant F_2^{\text{b}}$, and if $N^{\text{total}} > N_1^{\text{crit}}+N_2^{\text{crit}}$, then $F_1^{\text{b}} < F_2^{\text{b}}$.

Proof: We have

$$F_1^{\text{b}} - F_2^{\text{b}} = \frac{N_1^{\text{crit}}}{N^{\text{total}}-N_1^{\text{crit}}} - \frac{N^{\text{total}}-N_2^{\text{crit}}}{N_2^{\text{crit}}}$$

$$= \frac{N_1^{\text{crit}}N_2^{\text{crit}} - (N^{\text{total}}-N_1^{\text{crit}})(N^{\text{total}}-N_2^{\text{crit}})}{(N^{\text{total}}-N_1^{\text{crit}})N_2^{\text{crit}}}$$

$$= \frac{N^{\text{total}}(N_1^{\text{crit}}+N_2^{\text{crit}}-N^{\text{total}})}{(N^{\text{total}}-N_1^{\text{crit}})N_2^{\text{crit}}}$$

If $N_1^{\text{crit}} < N^{\text{total}} \leqslant N_1^{\text{crit}}+N_2^{\text{crit}}$, then $F_1^{\text{b}} \geqslant F_2^{\text{b}}$, and if $N^{\text{total}} < N_1^{\text{crit}}+N_2^{\text{crit}}$, then $F_1^{\text{b}} > F_2^{\text{b}}$. ∎

Proposition 4: In **Case 4**, the pheromone levels on both arcs asymptotically converge.

Proof: In Subcase 4.a, we know that $F_1^{\text{b}} \geqslant F_2^{\text{b}}$ from Lemma 6. The four modes M1 –M4 can be further described as:

$$M1 : F_2^{\text{b}} \leqslant F(t) \leqslant F_1^{\text{b}} ,$$
$$M2 : F(t) > F_1^{\text{b}} ,$$
$$M3 : 0 < F(t) < F_2^{\text{b}} ,$$
$$M4 : NR.$$

As proved in Lemma 2, F monotonically increases in both M1 and M3. Therefore, if F is initialized in M3, it will eventually transit to M1, and if F is initialized in M1, it will eventually transit to M2. Because $f_1 > f_2$ according to Assumption 1, $F_1^{\text{equ}} > F_1^{\text{b}}$ always holds, which means that F_1^{equ} is always located in the range of M2 due to Lemma 3. Therefore, similarly to Case 2, if F is initialized in M2, it may transit from M1 to M2, but after F transits from M1 to M2, it will stay in M2 according to Lemma 5. Moreover, F will finally converge to F_1^{equ}, which proves the mode transitions of Figure 5a.

In Subcase 4.b and Subcase 4.c, we know that $F_1^{\text{b}} < F_2^{\text{b}}$

from Lemma 6. The four modes M1–M4 can be further described as:

$$M1 : NR ,$$
$$M2 : F(t) \geqslant F_2^{\text{b}} ,$$
$$M3 : 0 < F(t) \leqslant F_1^{\text{b}} ,$$
$$M4 : F_1^{\text{b}} < F(t) < F_2^{\text{b}} .$$

In Subcase 4.b, we have $P \leqslant \dfrac{(f_1-f_2)(N^{\text{total}}-N_2^{\text{crit}})}{N^{\text{total}}-N_1^{\text{crit}}-N_2^{\text{crit}}}$, and one can prove that $F_1^{\text{equ}} > F_2^{\text{b}}$ and $F_2^{\text{equ}} > F_2^{\text{b}}$. As a result, F_1^{equ} is in M2, while F_2^{equ} is outside the range of M4. Since F is a monotonically increasing function in M4, we can use a method similar to that of Subcase 4.a to prove the mode transitions of Figure 5b, where F will also converge to F_1^{equ} in M2.

In Subcase 4.c, we have $P > \dfrac{(f_1-f_2)(N^{\text{total}}-N_2^{\text{crit}})}{N^{\text{total}}-N_1^{\text{crit}}-N_2^{\text{crit}}}$, and one can prove that $F_1^{\text{equ}} < F_2^{\text{b}}$, and $F_1^{\text{b}} < F_2^{\text{equ}} < F_2^{\text{b}}$. As a result, F_1^{equ} is outside the range of M2, while F_2^{equ} is in M4. Since F is a monotonically decreasing function in M2, we can also prove the mode transitions of Figure 5c similar to Subcase 4.a, where F will converge to F_2^{equ} in M4.

Since all of the three subcases of Case 4 lead to convergence of F, we can prove that in Case 4 the numbers of ants $N_1(t)$ and $N_2(t)$ converge, and accordingly the pheromone levels also converge. ∎

From Proposition 1–4, we know that in each case, the pheromone levels always converge, which satisfies Definition 1. In conclusion, the convergence of ACO-SP in a network with two arcs is proven.

VI. DISCUSSION

We have proven the convergence of ACO-SP in a simple network with only two arcs by using an auxiliary function F, as the first step of the convergence proof for the general case. In fact, in the current paper, $F(t)$ reflects the ratio of $\tau_1(t)$ and $\tau_2(t)$ on two arcs, and furthermore, it implicitly reflects the ratio of $N_1(t)$ and $N_2(t)$ on the arcs. Since the total number N^{total} is determined at the beginning, $N_1(t)$ and $N_2(t)$ can be obtained at iteration step t, and we hence know whether the stench pheromone will be deposited or not by comparing $N_1(t)$ with N_1^{crit} and $N_2(t)$ with N_2^{crit}. In that way, $F(t)$ reflects the deposition of stench pheromone on each arc, and we can use $F(t)$ to express the mode transitions.

However, the $F(t)$ method already results in four different cases, one of which has three sub-cases, in a simple network just with two arcs. It is very difficult to extend this method to a general network, because if the numbers of arcs and nodes in the network increase, the numbers of different cases and sub-cases will increase exponentially. Therefore, for the future work, we will try to find a different method to theoretically prove the convergence of ACO-SP in a general network.

Proof of Lemma 2: In M1, the pheromone update equations are formulated as:

$$\tau_1(t+1) = (1-\rho)\tau_1(t) + N_1(t)f_1$$
$$= (1-\rho)\tau_1(t) + \frac{\tau_1(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_1$$
$$\tau_2(t+1) = (1-\rho)\tau_2(t) + N_2(t)f_2$$
$$= (1-\rho)\tau_2(t) + \frac{\tau_2(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_2$$

The value $F(t+1)$ can thus be written as:

$$F(t+1) = \frac{(1-\rho)\tau_1(t) + \dfrac{\tau_1(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_1}{(1-\rho)\tau_2(t) + \dfrac{\tau_2(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_2}$$
$$= F(t)\frac{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1}N^{\text{total}}f_1}{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1}N^{\text{total}}f_2}$$

Because of Assumption 1, i.e., $f_1 > f_2$, we can conclude that $F(t+1) > F(t)$, which means $F(t)$ is a strictly monotonically increasing function in M1.

In M3, the pheromone update equations are formulated as:

$$\tau_1(t+1) = (1-\rho)\tau_1(t) + \frac{\tau_1(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_1$$
$$\tau_2(t+1) = (1-\rho)\tau_2(t) + N_2(t)(f_2-P) + PN_2^{\text{crit}}$$
$$= (1-\rho)\tau_2(t) + \frac{\tau_2(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}(f_2-P) + PN_2^{\text{crit}}$$

Hence, $F(t+1)$ can be written as:

$$F(t+1) = F(t)\frac{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_1}{F(t)+1}}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}(f_2-P)}{F(t)+1} + PN_2^{\text{crit}}}$$

Since we want to prove that $F(t+1) > F(t)$, it should hold

$$\frac{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_1}{F(t)+1}}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}(f_2-P)}{F(t)+1} + PN_2^{\text{crit}}} > 1$$

To satisfy the inequality above, we need

$$F(t) < \frac{N^{\text{total}}(f_1-f_2) + P(N^{\text{total}} - N_2^{\text{crit}})}{PN_2^{\text{crit}}}, \qquad (14)$$

Because of Assumption 1, the right-hand side of (14) is greater than F_2^{b}. Therefore, in M3, (14) always holds, and F is a strictly monotonically increasing function in M3. ∎

Proof of Lemma 3: In M2, the pheromone update equations are given by:

$$\tau_1(t+1) = (1-\rho)\tau_1(t) + N_1(t)f_1 - P(N_1(t) - N_1^{\text{crit}})$$
$$= (1-\rho)\tau_1(t) + N_1(t)(f_1-P) + PN_1^{\text{crit}}$$

$$= (1-\rho)\tau_1(t) + \frac{\tau_1(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}(f_1-P) + PN_1^{\text{crit}}$$
$$\tau_2(t+1) = (1-\rho)\tau_2(t) + N_2(t)f_2$$
$$= (1-\rho)\tau_2(t) + \frac{\tau_2(t)}{\tau_1(t)+\tau_2(t)}N^{\text{total}}f_2$$

The value $F(t+1)$ can be written as:

$$F(t+1) = F(t)\frac{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}(f_1-P)}{F(t)+1} + \dfrac{PN_1^{\text{crit}}}{F(t)}}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_2}{F(t)+1}} \quad (15)$$

Since we want to find the equilibrium point, we let $F(t+1) = F(t) = F_{\text{e}}$. Therefore, we have:

$$\frac{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}(f_1-P)}{F_{\text{e}}+1} + \dfrac{PN_1^{\text{crit}}}{F_{\text{e}}}}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_2}{F_{\text{e}}+1}} = 1 \quad (16)$$

This yields

$$F_{\text{e}} = \frac{PN_1^{\text{crit}}}{N^{\text{total}}(f_2-f_1) + P(N^{\text{total}} - N_1^{\text{crit}})}$$

This is the equilibrium point F_1^{equ}. Furthermore, if $F(t) < F_1^{\text{equ}}$, the factor that multiples $F(t)$ in (15) is larger than 1, which means $F(t+1) > F(t)$, and if $F(t) > F_1^{\text{equ}}$, that factor is smaller than 1, which means $F(t+1) < F(t)$.

Similarly, we can prove that F_2^{equ} is the only equilibrium point in M4 . When $F(t) < F_1^{\text{equ}}$, we have $F(t+1) > F(t)$, and when $F(t) > F_1^{\text{equ}}$, we have $F(t+1) < F(t)$. ∎

Proof of Lemma 4: We have

$$|F(t+1) - F_1^{\text{equ}}|$$
$$= \left| F(t)\frac{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}(f_1-P)}{F(t)+1} + \dfrac{PN_1^{\text{crit}}}{F(t)}}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_2}{F(t)+1}} - F_1^{\text{equ}} \right|$$
$$= \left| \frac{1}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_2}{F(t)+1}} \right| \cdot \left| (1-\rho)F(t)\tau_2(t) + PN_1^{\text{crit}} + \right.$$
$$\frac{F(t)}{F(t)+1}N^{\text{total}}(f_1-P) - F_1^{\text{equ}}\left((1-\rho)\tau_2(t) + \frac{N^{\text{total}}f_2}{F(t)+1} \right) \bigg|$$
$$= \left| \frac{1}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}}f_2}{F(t)+1}} \right| \cdot \left| (1-\rho)(F(t)-F_1^{\text{equ}})\tau_2(t) + \right.$$
$$\frac{1}{F(t)+1}\Big(F(t)(f_1N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}}) - F_1^{\text{equ}}N^{\text{total}}f_2 +$$
$$PN_1^{\text{crit}} \Big) \bigg| \qquad (17)$$

From the definition of F_1^{equ}, we know that

$$PN_1^{\text{crit}} = F_1^{\text{equ}}\left(N^{\text{total}}(f_2-f_1) + P(N^{\text{total}} - N_1^{\text{crit}}) \right) \quad (18)$$

Putting (18) into (17) yields:

$$|F(t+1) - F_1^{\text{equ}}|$$

$$= \left| \frac{1}{(1-\rho)\tau_2(t) + \dfrac{N^{\text{total}} f_2}{F(t)+1}} \right| \cdot \left| (1-\rho)(F(t) - F_1^{\text{equ}})\tau_2(t) + \right.$$

$$\left. \frac{1}{F(t)+1}(F(t) - F_1^{\text{equ}}) \left(f_1 N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}} \right) \right|$$

$$= |F(t) - F_1^{\text{equ}}| \cdot$$

$$\left| \frac{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1} \left(f_1 N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}} \right)}{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1} N^{\text{total}} f_2} \right| \tag{19}$$

From Assumption 3, we can derive from that $N^{\text{total}}(f_2 - f_1) + P(N^{\text{total}} - N_1^{\text{crit}}) > 0$, which means that $f_1 N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}} < N^{\text{total}} f_2$. Recall from Assumption 2 $f_1 N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}} > 0$. Therefore, we can conclude that

$$0 < \left| \frac{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1} \left(f_1 N^{\text{total}} - PN^{\text{total}} + PN_1^{\text{crit}} \right)}{(1-\rho)\tau_2(t) + \dfrac{1}{F(t)+1} N^{\text{total}} f_2} \right| < 1$$

Hence, (19) is a contraction to 0, $\lim\limits_{t\to\infty} |F(t+1) - F_1^{\text{equ}}| = 0$.

We can similarly prove that $\lim\limits_{t\to\infty} |F(t+1) - F_2^{\text{equ}}| = 0$. ∎

Proof of Lemma 5: When $N^{\text{total}} \leqslant N_1^{\text{crit}} + N_2^{\text{crit}}$ as in Subcase 4.a, the equilibrium point F_1^{equ} is in M2, and F can only transit from M1 to M2. We suppose that $F(t_0)$ is in M1, in which $F(t_0) < F_1^{\text{b}}$, and $F(t_0 + 1)$ is in M2, in which $F(t_0 + 1) > F_1^{\text{b}}$. From the contraction in (19), we know that F always moves towards F_1^{equ} in M2. If we can prove that $|F(t_0+1) - F_1^{\text{equ}}| < F_1^{\text{equ}} - F_1^{\text{b}}$, then we have proved that $F(t)$ will stay in M2 from iteration step t_0. Because $F(t_0 + 1) > F_1^{\text{b}}$, it is clear that $F_1^{\text{b}} - F_1^{\text{equ}} < F(t_0 + 1) - F_1^{\text{equ}}$. We only need to prove $F(t_0 + 1) - F_1^{\text{equ}} < F_1^{\text{equ}} - F_1^{\text{b}}$. Since $F(t_0)$ is in M1, $F(t_0 + 1)$ is calculated by the equations of M1:

$$(F(t_0+1) - F_1^{\text{equ}}) - (F_1^{\text{equ}} - F_1^{\text{b}})$$

$$= F(t_0) \frac{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_1}{F(t_0)+1}}{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_2}{F(t_0)+1}} + F_1^{\text{b}} - 2F_1^{\text{equ}}$$

$$< F_1^{\text{b}} \frac{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_1}{F(t_0)+1}}{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_2}{F(t_0)+1}} + F_1^{\text{b}} - 2F_1^{\text{equ}}$$

$$= F_1^{\text{b}} \frac{2(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}}(f_1 + f_2)}{F(t_0)+1}}{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_2}{F(t_0)+1}} - 2F_1^{\text{equ}}$$

$$= \frac{N_1^{\text{crit}}}{N^{\text{total}} - N_1^{\text{crit}}} \cdot \frac{2(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}}(f_1 + f_2)}{F(t_0)+1}}{(1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_2}{F(t_0)+1}} -$$

$$\frac{2PN_1^{\text{crit}}}{N^{\text{total}}(f_2 - f_1) + P(N^{\text{total}} - N_1^{\text{crit}})}$$

$$= \frac{N_1^{\text{crit}}}{(N^{\text{total}} - N_1^{\text{crit}}) \left((1-\rho)\tau_2(t_0) + \dfrac{N^{\text{total}} f_2}{F(t_0)+1} \right)} \cdot$$

$$\frac{1}{(N^{\text{total}}(f_2 - f_1) + P(N^{\text{total}} - N_1^{\text{crit}}))} \cdot$$

$$\left(2(1-\rho)\tau_2(t_0)N^{\text{total}}(f_2 - f_1) + \dfrac{N^{\text{total}}}{F(t_0)+1}(f_2 - f_1) \right.$$

$$\left. (N^{\text{total}}(f_1 + f_2) - P(N^{\text{total}} - N_1^{\text{crit}})) \right)$$

By using $f_1 > f_2$, as well as Assumption 2.1, we can prove that $(F(t_0 + 1) - F_1^{\text{equ}}) - (F_1^{\text{equ}} - F_1^{\text{b}}) < 0$. As a conclusion, we have $|F(t_0 + 1) - F_1^{\text{equ}}| < F_1^{\text{equ}} - F_1^{\text{b}}$.

We can also use the similar method to prove that when $N^{\text{total}} > N_1^{\text{crit}} + N_2^{\text{crit}}$, if $F(t)$ transit into M2 or M4, it will stay in those modes. ∎

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