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Technical report 13-023

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If you want to cite this report, please use the following reference instead:
N. Groot, B. De Schutter, and H. Hellendoorn, "Optimal leader functions for the reverse Stackelberg game: Splines and basis functions," *Proceedings of the 2013 European Control Conference*, Zürich, Switzerland, pp. 696–701, July 2013.

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Optimal Leader Functions for the Reverse Stackelberg Game: Splines and Basis Functions

Noortje Groot, Bart De Schutter, and Hans Hellendoorn

Abstract— **In order to deal with the control of large-scale infrastructures, a multi-level approach may be required in which several groups of decision makers have different objectives. A game formulation can help to structure such a control task. The reverse Stackelberg game has a hierarchical structure in which the follower player acts subsequent to the leader's disclosure of her leader function, which maps the follower decision space into the leader decision space. The problem of finding a leader function such that the leader's objective function is optimized, given an optimal response w.r.t. the follower objective function, is in general a difficult problem. So far, the set of optimal affine leader functions has been delineated. However, for the more general class of nonlinear leader functions, no structured solution approach exists yet. In this paper, we consider several nonlinear structures for an optimal leader function based on basis functions as well as based on interpolating splines and we show how these approaches can be adopted to find an optimal leader function.**

I. Introduction

In multi-level control problems, a hierarchical decision structure can be adopted as an alternative to e.g., a decentralized and distributed communication framework [1]. In this paper, we focus on a specific type of hierarchical game that has been studied in problem settings like traffic tolling and route guidance problems [2], [3] and electricity network pricing [4]. In this reverse Stackelberg game [5], also known as inverse Stackelberg game [2], [6] or as a Stackelberg game with incentive strategies [7], a leader player first provides a mapping of her decision space into the follower's decision space, who subsequently decides upon his optimal variable along with the associated leader variable. The original Stackelberg game [8] is a special case in which the follower's decision space is mapped into a singleton, i.e., the leader acts directly by making her leader decision available.

In the current literature, mostly games are considered in which the follower objective function is quadratic and in which the state update equation in a dynamic game is linear [7], [9], [10]. In such a game, a leader function with an affine structure is automatically optimal [11]. To the best knowledge of the authors, only a few papers consider nonlinear leader functions; however, they appear in specific, numerical examples [4], [6], [12]. A general nonlinear leader function is considered though in [13], i.e., in the form of a hyperparaboloid or hypersphere tangent to the follower's objective function $\mathcal{J}_\mathrm{F}$ at the desired equilibrium $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ of leader and follower decisions, in a (linear) quadratic reverse Stackelberg game. There, the so-called quantity of threat is analyzed, which represents the likelihood of the follower player to chose his optimal variable for a given leader function. In particular, the hypersphere tangent is parameterized by its radius and it has a center on the gradient line. It should however be noted that in [13] the follower objective functional is assumed to be strictly convex and differentiable, and that therefore, the results cannot said to be sufficiently general for the purpose of deriving a systematic solution approach. It is stated that the nonlinear strategies proposed in [13] can solve problems for which no optimal affine leader function exists. However for the given problem instance of the reverse Stackelberg game, unless the follower objective functional is insensitive to the leader's decision, an optimal affine leader function always exists.

Further, evolutionary approaches have been proposed for finding equilibria that are optimal for the leader in original Stackelberg games or multi-level programming problems [14], [15]. These proposed iterative learning methods could also be applied for deriving nonlinear leader functions in reverse Stackelberg games. In [16], neural networks have been adopted to determine coefficients of a 2-degree polynomial leader function that leads to an equilibrium that is close-to-optimal for the leader, the value of which is evaluated and updated throughout the learning process. While this approach yields a feasible leader function and the associated leader objective function value, it may result in a local optimum; in addition, no guarantee is given on the rate of convergence and the gap from the leader's global optimum. Moreover, the decision spaces considered in [16] are unconstrained. Differently, in this paper, we propose methods to derive leader functions that are guaranteed to yield the desired equilibrium (which is assumed to be known a priori, e.g., as a result of the leader's optimization of her objective function).

In order to make the theory of reverse Stackelberg games more easily applicable in control settings, our aim is to build forward towards a more general and structured solution approach. Thus far, we have relaxed the conditions in which an affine leader function structure automatically leads to optimality for the leader [17], and delineated the full set of optimal affine leader functions [18]. In many cases however, no optimal affine solution exists; we therefore further extend that approach in this paper by also including nonlinear functions in a constrained decision space. The proposed methods are mainly based on deriving optimal parameters of a particular set of basis functions as well as on selecting suitable points of the decision space in order to compute an interpolating spline.

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This paper is structured as follows. After a brief definition of the reverse Stackelberg game in Section II, Section III includes the basis function, gridding, and spline approaches we propose to derive an optimal nonlinear leader function. The methods are illustrated in Section IV and subsequently the paper is concluded in Section V.

II. The Reverse Stackelberg Game

The basic single-leader single-follower, static reverse Stackelberg game can be defined through the leader and follower decision variables $u_p \in \Omega_p \subseteq \mathbb{R}^{n_p}$, $n_p \in \mathbb{N}$ and the objective (cost) functions $\mathcal{J}_p : \Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}} \to \mathbb{R}$, $p \in \{\mathrm{L}, \mathrm{F}\}$.

A common approach to the reverse Stackelberg problem is for the leader player to first determine a particular desired equilibrium that she seeks to achieve, e.g., her global optimum $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \in \arg\min_{(u_{\mathrm{L}}, u_{\mathrm{F}}) \in \Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}})$ [5], [19], [20]. Given such a desired point, the remaining problem can be written as:

$$\text{To find:} \quad \gamma_{\mathrm{L}} \in \Gamma_{\mathrm{L}}, \tag{1}$$

$$\text{s.t.} \quad \arg\min_{u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}}) = u_{\mathrm{F}}^{\mathrm{d}}, \tag{2}$$

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}^{\mathrm{d}}) = u_{\mathrm{L}}^{\mathrm{d}}, \tag{3}$$

with Γ_{L} the particular class of leader functions $\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$ that is allowed in the game context.

Given this formulation of the reverse Stackelberg game, the problem is reduced to finding a leader function that solves the game to optimality, i.e., that leads to the desired leader optimum. When assuming a particular leader function structure, the problem further reduces to finding parameters for which the given leader function is optimal. In this paper, nonlinear leader functions are analyzed that satisfy (2)–(3), where the constraint (2) implies that the function $\gamma_{\mathrm{L}}(u_{\mathrm{F}})$ should remain outside of the sublevel set

$$\Lambda_{\mathrm{d}} := \{(u_{\mathrm{L}}, u_{\mathrm{F}}) \in \Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}} | \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) \leq \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})\},$$

except in the point $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$. In the following, we assume $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ to be a boundary point of Λ_{d}; if this does not hold, no optimal leader function exists.

Necessary and sufficient existence conditions for an optimal affine leader function

$$u_{\mathrm{L}} := \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = u_{\mathrm{L}}^{\mathrm{d}} + B(u_{\mathrm{F}} - u_{\mathrm{F}}^{\mathrm{d}}), B \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{F}}} \tag{4}$$

for the unconstrained case, with $\Omega_p = \mathbb{R}^{n_p}, p \in \{\mathrm{L}, \mathrm{F}\}$ were proposed in [17], while a characterization of the full set of such optimal functions was given in [18]. This set may be further reduced in the constrained case that we consider, with $\Omega_p \subsetneq \mathbb{R}^{n_p}, p \in \{\mathrm{L}, \mathrm{F}\}$. Clearly, in case an optimal affine leader function exists, nonlinear functions can also be considered. This may be preferred in case the nonlinear alternatives are more robust to deviations of the follower to his optimal decision, which is even more strongly of interest when asymmetric, partial information applies.

III. Computation of a Nonlinear Leader function

Next, we consider γ_{L} to take the form of either a linear combination of basis functions, a linear combination of basis functions based on a gridding of the decision spaces, or of a piecewise polynomial. These three methods will be used to compute an optimal leader function in both a 2- and 3-dimensional decision space in Section IV.

A. Basis Functions

Basis functions are universal approximators, meaning that in principle each function can be described by means of a set of functions that form a basis for a particular class of functions. We denote the set of selected multidimensional basis functions [21] by:

$$\mathcal{B} = \{b_i(\cdot)\}_{i=1}^{n}, \tag{5}$$

for a finite set of basis functions $b_i : \mathbb{R}^{n_{\mathrm{F}}} \to \mathbb{R}^{n_{\mathrm{L}}}$, $i = 1, \ldots, n$, each of which is composed of n_{L} scalar basis functions. The leader function can then be represented by a linear combination of basis functions, with parameters $a_i \in \mathbb{R}^{n_{\mathrm{L}}}, i = 1, \ldots, n$, where the leader function can be written:

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}) = \sum_{i=1}^{n} a_i \odot b_i(u_{\mathrm{F}}), \tag{6}$$

where $\odot$ denotes the element-by-element matrix (Schur) product.

1) Choice of basis functions: The choice of a set of basis functions depends on the requirements of the setting to which a reverse Stackelberg approach is applied.

For instance, in multi-level control settings with precise instruments and an accurate knowledge of the desired equilibrium $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, nonsmooth optimal leader functions can be applied, whereas in settings where the true equilibrium may be in a neighborhood of $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, the control actions may need to be steered with a smooth leader function in order to prevent from a large deterioration of the objective function value; see Fig. 1 for an illustration. There, $\widehat{(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})}$ is the assumed desired leader equilibrium, while the true value is within an ellipsoidal set. Such analyses however strongly depend on the geometry of the sublevel set Λ_{d}.

Similarly, robustness of the leader function w.r.t. deviations from the optimal follower response could play a role in the determination of a particular leader function; such an analysis will be part of further research.

For different types of orthonormal basis functions and their properties, please refer to [21]. In principle, also combinations of basis functions can be considered. Similarly, the basis functions can be further parameterized, yet this extension is left for more elaborate analysis of possible leader function structures.

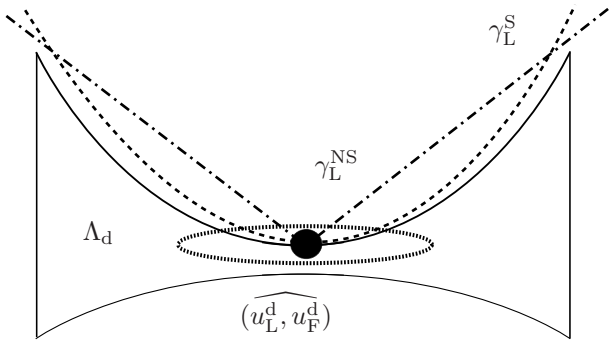


Fig. 1. Illustration of the effect of smoothness in case the 'true' desired solution $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ lies in an ellipse around the estimated solution $(\widehat{u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}})$.

2) Solving for a given set of basis functions: The following feasibility program can be used to find an optimal n-th order leader function for a particular set of basis functions $\{b_i\}_{i=1}^{n}$:

$$\text{To find: } a_i \in \mathbb{R}^{n_{\mathrm{L}}}, i = 1, \ldots, n, \tag{7}$$

$$\text{s.t. } \mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}}) > \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \quad \forall u_{\mathrm{F}} \in \Omega_{\mathrm{F}} \setminus \{u_{\mathrm{F}}^{\mathrm{d}}\}, \tag{8}$$

$$u_{\mathrm{L}}^{\mathrm{d}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}^{\mathrm{d}}), \tag{9}$$

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}) \in \Omega_{\mathrm{L}} \; \forall u_{\mathrm{F}} \in \Omega_{\mathrm{F}}, \tag{10}$$

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}) = \sum_{i=1}^{n} a_i \odot b_i(u_{\mathrm{F}}), i = 1, \ldots, n. \tag{11}$$

In order to be able to efficiently solve the program (7)–(11), the constraints (8) and (10), which are complicating due to the sets of variables for which they need to be satisfied, can be substituted by a lower-level optimization problem, for which we begin by introducing its main components.

First, the distances d_1 and ω refer to the signed (shortest) distances between a point $(u_{\mathrm{L}}, u_{\mathrm{F}})$ and the boundary of the set Λ_{d}, and between a point u_{L} and the boundary of the decision space Ω_{L}, respectively. These Euclidean distances are allocated a negative sign if it concerns an exterior point of the set Λ_{d} and Ω_{L}, respectively. Further, d_2 denotes the Euclidean distance between a point $(u_{\mathrm{L}}, u_{\mathrm{F}})$ and $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, i.e.:

$$d_1((u_{\mathrm{L}}, u_{\mathrm{F}}), \Lambda_{\mathrm{d}}) = \mathcal{S}_{(u_{\mathrm{L}}, u_{\mathrm{F}})} \cdot \min_{(u_{\mathrm{L}}^{\mathrm{b}}, u_{\mathrm{F}}^{\mathrm{b}}) \in \mathrm{bd}(\Lambda_{\mathrm{d}})} \|(u_{\mathrm{L}}^{\mathrm{b}}, u_{\mathrm{L}}^{\mathrm{b}}) - (u_{\mathrm{L}}, u_{\mathrm{F}})\|_2, \tag{12}$$

$$\mathcal{S}_{(u_{\mathrm{L}}, u_{\mathrm{F}})} = \begin{cases} - \text{ if } \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) \geq 0 \\ + \text{ if } \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) \leq 0 \end{cases}, \tag{13}$$

$$d_2((u_{\mathrm{L}}, u_{\mathrm{F}}), (u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})) = \|(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) - (u_{\mathrm{L}}, u_{\mathrm{F}})\|_2, \tag{14}$$

$$\omega\left((u_{\mathrm{L}}, u_{\mathrm{F}}), \Omega_{\mathrm{L}})\right) = \mathcal{S}_{u_{\mathrm{L}}} \cdot \min_{u_{\mathrm{L}}^{\mathrm{b}} \in \mathrm{bd}(\Omega_{\mathrm{L}})} \|u_{\mathrm{L}}^{\mathrm{b}} - u_{\mathrm{L}}\|_2, \tag{15}$$

$$\mathcal{S}_{u_{\mathrm{L}}} = \begin{cases} - \text{ if } u_{\mathrm{L}} \notin \Omega_{\mathrm{L}} \\ + \text{ if } u_{\mathrm{L}} \in \Omega_{\mathrm{L}} \end{cases}, \tag{16}$$

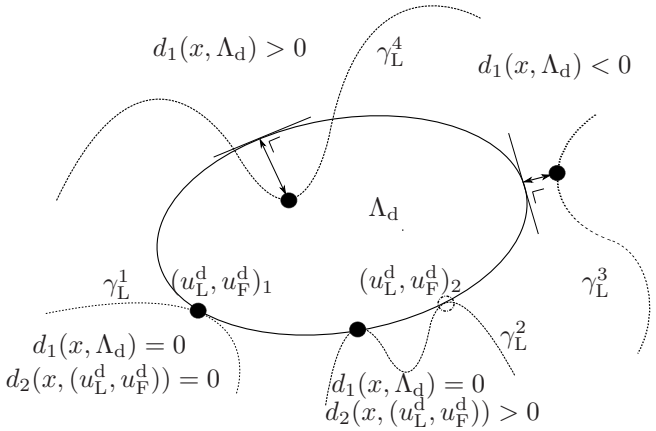


Fig. 2. Illustration of distance measures w.r.t the sublevel set Λ_{d}. The symbol x indicated on the different curves is used to represent the black points $(u_{\mathrm{L}}, u_{\mathrm{F}})$.

where $\mathrm{bd}(\Lambda_{\mathrm{d}})$ can be alternatively represented by the set:

$$\{(u_{\mathrm{L}}, u_{\mathrm{F}}) : \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})\}.$$

Based on the above definitions, we can introduce the following auxiliary distance expressions for a particular leader function γ_{L} parameterized according (11):

$$d\left(\gamma_{\mathrm{L}}, \Lambda_{\mathrm{d}}\right) = \max_{(u_{\mathrm{L}}, u_{\mathrm{F}}) \in \gamma_{\mathrm{L}}} \Big(d_1((u_{\mathrm{L}}, u_{\mathrm{F}}), \Lambda_{\mathrm{d}}) + \alpha d_2\left((u_{\mathrm{L}}, u_{\mathrm{F}}), (u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})\right) \Big), \tag{17}$$

$$\omega\left(\gamma_{\mathrm{L}}, \Omega_{\mathrm{L}}\right) = \min_{(u_{\mathrm{L}}, u_{\mathrm{F}}) \in \gamma_{\mathrm{L}}} \omega\left((u_{\mathrm{L}}, u_{\mathrm{F}}), \Omega_{\mathrm{L}}\right), \tag{18}$$

with $0 < \alpha \ll 1$.

The distance expressions are illustrated in Fig. 2 by four independent situations. If for all points $(u_{\mathrm{L}}, u_{\mathrm{F}})$ on γ_{L} the distance $d(\gamma_{\mathrm{L}}, \Lambda_{\mathrm{d}}) \leq 0$, this means that the curve does not intersect with Λ_{d}, except in the point $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, which is already guaranteed to be a point on γ_{L} by (9) in the higher-level program. The addition of the distance from $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, multiplied by a small constant α, ensures that in case a particular point is outside Λ_{d} and far from $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, it should retain a sufficient distance to $\mathrm{bd}(\Lambda_{\mathrm{d}})$ in order to prevent deviations from the desired follower response, as elaborated upon in the Remark below. A similar approach is adopted to ensure that $\gamma_{\mathrm{L}}(\Omega_{\mathrm{F}}) \subseteq \Omega_{\mathrm{L}}$. If the minimum signed shortest distance $\omega((u_{\mathrm{L}}, u_{\mathrm{F}}), \Omega_{\mathrm{L}})$ for a point $(u_{\mathrm{L}}, u_{\mathrm{F}})$ on γ_{L} to Ω_{L} is nonnegative, all leader elements fall within the leader decision space, satisfying $\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$.

Finally, the program (7)–(11) can now be solved by solving (19)–(21) below, together with the constraints (9) and (11) from the original program. The computation of the distances (12)-(18) required in (19)–(21) can thus be interpreted as a lower-level problem.

$$\max_{\{a_i\}_{i=1}^n} d\left(\gamma_{\mathrm{L}}, \Lambda_{\mathrm{d}}\right) \tag{19}$$

$$\text{s.t. } d\left(\gamma_{\mathrm{L}}, \Lambda_{\mathrm{d}}\right) \leq 0, \tag{20}$$

$$\omega\left(\gamma_{\mathrm{L}}, \Omega_{\mathrm{L}}\right) \geq 0. \tag{21}$$

$$u_{\mathrm{L}}^{\mathrm{d}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}^{\mathrm{d}}), \tag{22}$$

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}) = \sum_{i=1}^{n} a_i \odot b_i(u_{\mathrm{F}}), i = 1, \ldots, n. \tag{23}$$

Remark: By maximizing $d\left(\gamma_{\mathrm{L}}, \Lambda_{\mathrm{d}}\right)$, curves that follow the boundary of Λ_{d} closely and that therefore could result in the unwanted selection of a follower decision variable value different from $u_{\mathrm{F}}^{\mathrm{d}}$, will be less likely o be selected. By adopting an objective function based on this criterion, we thus enter the area of sensitivity and robustness issues, where the sensitivity could be defined based on the (weighted) vicinity of $\gamma_{\mathrm{L}}(u_{\mathrm{F}})$ to $\mathrm{bd}(\Lambda_{\mathrm{d}})$, potentially with increasing weights for the variable u_{F} approaching $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$.

B. A Gridding Approach

In order to simplify the multi-level optimization problem explained in Section III-A.2 above that leads to an optimal leader function, a relaxed problem can be solved at a lower computational burden (depending on the accuracy of the approach and on the dimension of $\Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}}$), yet at the cost of a potentially suboptimal leader function. By adopting a gridding approach in which the follower decision space is approximated by a grid $\Omega_{\mathrm{F}}^{\mathrm{g}}$ of a desired accuracy and with $u_{\mathrm{F}}^{\mathrm{d}} \in \Omega_{\mathrm{F}}^{\mathrm{g}}$, the constraints (12)-(21) can be removed, leaving solely the feasibility program (7)-(11) in which (8) and (10) now form regular, i.e., easy-to-handle constraints. In other words, the expressions (8) and (10) can then be evaluated at each relevant grid point:

$$\mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}}) > \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \ \forall u_{\mathrm{F}} \in \Omega_{\mathrm{F}}^{\mathrm{g}} \backslash \{u_{\mathrm{F}}^{\mathrm{d}}\}, \tag{24}$$

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}) \in \Omega_{\mathrm{L}} \qquad \forall u_{\mathrm{F}} \in \Omega_{\mathrm{F}}^{\mathrm{g}}. \tag{25}$$

In particular, when a uniform grid is applied with a grid-size of precision δ, a point u_{F} can be at most $\epsilon := \sqrt{n_{\mathrm{F}} \cdot (\delta/2)^2}$ in Euclidean distance far from some grid point $u_{\mathrm{F}}^{\mathrm{g}} \in \Omega_{\mathrm{F}}^{\mathrm{g}}$. This implies that an undesirable intersection of a leader function γ_{L} with the sublevel set Λ_{d} can occur in an area of ϵ distance from $u_{\mathrm{F}}^{\mathrm{d}}$ in case the follower's decision is in the confined decision space $\Omega_{\mathrm{F}}^{\mathrm{g}}$. In order to yield a leader function that is certain to result in an equilibrium that is within ϵ distance from $u_{\mathrm{F}}^{\mathrm{d}}$ no part of Λ_{d} can be overlooked due to gridding that may result in an optimal follower decision that does not coincide with any grid point and that is far ($\gg \epsilon$) from the computed optimum. Instead of refining the gridding precision, this issue could be prevented by considering non-equidistant grid points on $\mathrm{bd}(\Lambda_{\mathrm{d}})$ or in particular points coinciding with the vertices of Λ_{d} in case it represents a polytope. Alternatively, grid points could be classified according to $\mathrm{conv}(\Lambda_{\mathrm{d}})$ instead of to Λ_{d} in (8). In this case, the computed leader functions are known to be at most ϵ in Euclidean distance far from an undesirable intersection with $\mathrm{conv}(\Lambda_{\mathrm{d}})$ hence with Λ_{d}. However, it should be noted that no feasible solution exists in this case unless $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \in \mathrm{bd}(\mathrm{conv}(\Lambda_{\mathrm{d}}))$.

C. Multidimensional Interpolation

A third and final approach proposed here to derive a nonlinear leader function is by using multivariate interpolation also called curve or (hyper)surface fitting by using piecewise polynomials (splines) [22], [23]. This method can provide an alternative to using basis functions especially in the case of a 2 or 3-dimensional decision space, where a spline approach benefits from the possibility to use graphical tools. Just as applies to the gridding approach in Section III-B, interpolation can also be useful in case the sublevel set Λ_{d} is not available analytically to the leader but instead a collection of data points is available.

Interpolation points could be systematically selected that are a given measure δ in Euclidean distance separated from the boundary of Λ_{d}. In the case that for a given set of data points, any interpolating spline yields an intersection with $\Lambda_{\mathrm{d}} \setminus \{(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})\}$, a new interpolation point should be added, further dividing the interval in which intersection occurred. Alternatively, in the case in which smoothness does not exhibit a desired property, one can immediately adopt an optimal, piecewise-affine function, where selection of points such that a linear interpolating function through these points does not intersect with Λ_{d} is facilitated by choosing interpolation points outside $\mathrm{conv}(\Lambda_{\mathrm{d}})$.

More information on deriving interpolating multivariate piecewise polynomials with different levels of continuity, can be found in [22], [23].

A Note on The Computational Complexity: The reverse Stackelberg game, defined in its most general form without the specification of a desired leader equilibrium, is a general version of the original Stackelberg game [8] in which the leader directly proposes the singleton $\{u_{\mathrm{L}}^{\mathrm{d}}\}$ to the follower instead of the mapping of Ω_{F} to the *set* Ω_{L}. By equivalence with the linear bilevel programming problem [24], (reverse) Stackelberg games can be proven to be strongly NP-hard.

By first selecting a desired leader equilibrium $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ and by specifying a particular class of leaders functions, the problem becomes easier to handle. Nonetheless, the numerical complexity of the proposed methods should be evaluated. Here it should be noted that finding a suitable leader function based on the program (7)–(11) is easier to approach in the case that the sublevel set Λ_{d} represents a polytope, i.e., $\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}})$ is linear in u_{L} and u_{F}, while the decision spaces are convex. In general, the complexity of finding a suitable leader function depends strongly on the geometry of the sublevel set, the choice of the basis functions, and the order of the basis functions or splines. The computational time required to compute a leader function is evaluated in Section IV below for two different scenarios.

IV. Worked Example

In this section, the multilevel and gridding-based basis function approach as well as the interpolation method are ap-

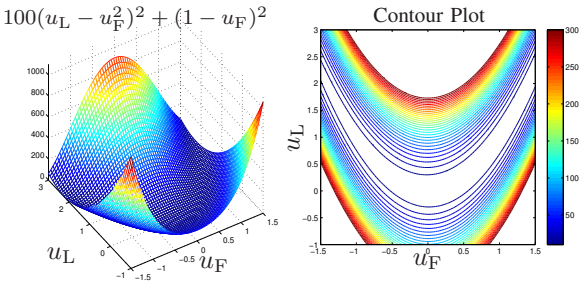


Fig. 3. The Rosenbrock function and several level curves.

plied to a sublevel set derived from the nonconvex (extended) Rosenbrock function that is adopted as a function structure for $\mathcal{J}_F(u_L, u_F)$, for both a 2- and 3-dimensional decision space. The nonconvex Rosenbrock function [25] is often used to illustrate the performance of optimization algorithms and can be written as follows in terms of decision variables of the reverse Stackelberg game:

$$f(u_L, u_F) = (1 - u_L)^2 + 100(u_F - u_L^2)^2, \tag{26}$$

as depicted in Fig. 3 together with several level curves. For higher dimensions we adopt the extended Rosenbrock function [26], i.e., in this particular case it can be written as:

$$\begin{aligned} f(u_{L,1}, u_{L,2}, u_F) &= (1 - u_F)^2 + 100(u_{L,1} - (u_F)^2)^2 \\ &+ (1 - u_{L,1})^2 + 100(u_{L,2} - (u_{L,1})^2)^2. \end{aligned} \tag{27}$$

We chose to adopt a cubic spline interpolator as well as a simple polynomial of the 5-th order for the basis function cases, implying the following leader functions for the 2D (28) and 3D (29) case:

$$\gamma_L(u_F) = \sum_{i=0}^{5} a_i \cdot (u_F)^i, \qquad u_L \in \mathbb{R}, u_F \in \mathbb{R}, \tag{28}$$

$$\gamma_L(u_F) = \begin{bmatrix} \sum_{i=0}^{5} a_i \cdot (u_F)^i \\ \sum_{j=6}^{11} a_j \cdot (u_F)^{j-6} \end{bmatrix}, \qquad u_L \in \mathbb{R}^2, u_F \in \mathbb{R}, \tag{29}$$

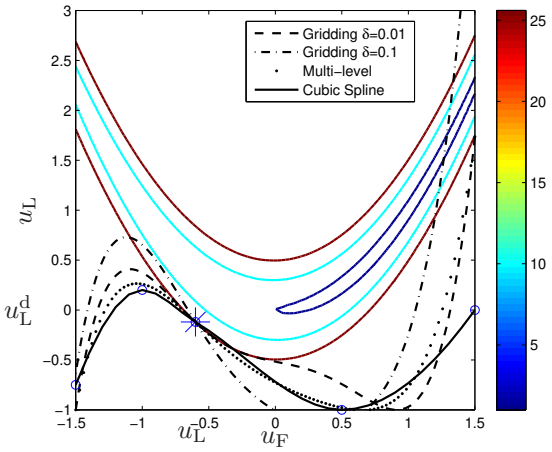
where the respective decision spaces are restricted to $u_F \in [-1.5, 1.5]$, $u_L, u_{L,1} \in [-1, 2]$, and $u_{L,2} \in [-0.75, 1.5]$.

A. Results

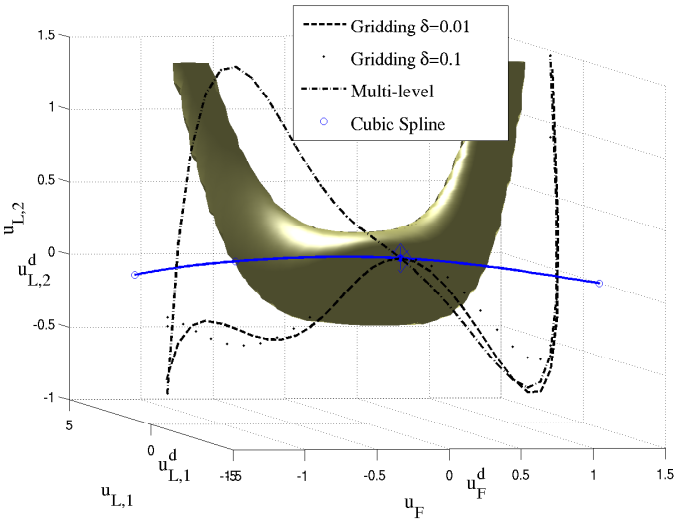
In Fig. 4 several leader functions are plotted for both the 2- and 3-dimensional case. It can be seen that the curves indeed remain within the constrained decision space Ω_L and outside $\Lambda_d \setminus \{(u_L^d, u_F^d)\}$, except in case gridding was applied with an insufficient precision ($\delta = 0.1$); there, the computed curves intersect with Λ_d.

In order to give an indication of the computational requirements, Table I shows the computation time[1] for deriving the leader functions for this particular (extended) Rosenbrock function. First, it is interesting to observe that while the

[1]These CPU times were obtained adopting the 64-bit Matlab 7.12.0 (R2011a) fmincon environment on a Linux PC with a 3GHz Intel Core Duo processor and 3.7Gb RAM.



(a) $u_F \in \mathbb{R}, u_L \in \mathbb{R}$



(b) $u_L \in \mathbb{R}^2, u_F \in \mathbb{R}$

Fig. 4. Nonlinear leader functions through (u_L^d, u_F^d) and outside the sublevel set Λ_d for $\mathcal{J}_F$.

curves in 3D have a larger number of coefficients, the computation time is in fact lower for all methods but the spline approach. This may be explained by the larger degree of freedom for possible curves staying outside Λ_d but within Ω_L, i.e., it easier to find a curve that satisfies these constraints in this particular 3D case. Further, it can be observed that increasing the gridding precision to $\delta = 0.001$ leads to a relatively large increase in time, while an optimal curve is already obtained for $\delta = 0.01$. Finally, the interpolating spline functions are significantly more time-efficient for the current cases. This can however be explained by the fact that the constraints (8) and (10) are not incorporated in this method in comparison to the other approaches, which causes the risk of a suboptimal solution based on the chosen data points. Moreover, only a small number of data points were required to obtain an optimal leader function in this case.

As a part of further research, a more elaborate computa-

TABLE I

SUMMARY OF COMPUTATIONAL RESULTS

	Method	CPU time
2-D	Basis Functions (multilevel)	5.2369 s
2-D	Gridding $\delta = 0.1$	1.3485 s
2-D	Gridding $\delta = 0.01$	1.7664 s
2-D	Gridding $\delta = 0.001$	6.7659 s
2-D	Interpolating Cubic Spline	0.0028 s (5 data points)
3-D	Basis Functions (multilevel)	1.5734 s
3-D	Gridding $\delta = 0.1$	1.1548 s
3-D	Gridding $\delta = 0.01$	1.6780 s
3-D	Gridding $\delta = 0.001$	9.5181 s
3-D	Interpolating Cubic Spline	0.1750 s (4 data points)

tional analysis should be conducted, investigating the effect of different levels of complexity, i.e., of basis functions and in particular of higher dimensional decision spaces with $n_{\mathrm{F}} > 1$, in which case the leader function embodies a (hyper)surface instead of a curve.

V. CONCLUSION

A systematic approach is provided for computing nonlinear leader functions that solve the single-leader single-follower reverse Stackelberg game to optimality. This hierarchical game can be applied as an optimization structure in multi-level control problems, yet thus far mostly leader functions of the affine structure have been investigated that are not able to solve the game in general. Three methods have been proposed: in the basis function approaches, suitable parameters are determined either through multi-level optimization, or on the basis of a gridding of the decision spaces. The third method relies on the derivation of interpolating splines. While all methods were able to compute an optimal nonlinear leader function, their computational efforts were different. As computational efficiency is a critical aspect of the implementation of a reverse Stackelberg game in a multi-level control framework, a more elaborate evaluation of higher-dimensional cases should also be considered.

ACKNOWLEDGEMENTS

Research supported by the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of Excellence, and by the European COST Action TU1102.

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