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Optimal Trajectory Planning for Trains under a Moving Block Signaling System

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Abstract— **The optimal trajectory planning problem for trains under a moving block signaling system is considered. This optimal trajectory planning problem is significant for punctuality, energy consumption, passenger comfort, etc. In a moving block signaling system, the minimum distance between two successive trains is the instantaneous braking distance required by the following train plus a safety margin. The constraints caused by the moving block signaling system are described as nonlinear inequalities, which can be transformed into linear inequalities using piecewise affine approximations. The optimal trajectory planning problem is subsequently recast as a mixed integer linear programming problem, which can be solved efficiently by existing solvers. A case study is used to demonstrate the performance of the proposed approach.**

I. INTRODUCTION

The energy efficiency of railway systems is more and more significant due to the rising energy price and environmental concerns. Furthermore, the energy consumption is one of the major expenses components in the operational costs in railway systems, e.g. in China they are about 13-16% of the annual operation and maintenance costs of urban railway systems [1]. Even a small improvement in the energy consumption can make the railway operators save a lot of money.

The research on the optimal control of train operations began in the 1960s and since then various methods have been proposed for the problem. These methods can be grouped into two main categories: analytical solutions and numerical optimization. For analytical solutions, the maximum principle is applied and it results in four optimal regimes (maximum traction, cruising, coasting, and maximum braking) [2]–[4]. It is difficult to obtain the analytical solution if more realistic conditions are considered, which introduce more complex nonlinear terms into the model and the constraints [5]. Numerical optimization approaches are applied more and more to the train optimal control problem due to the comparable high computing power available nowadays. A number of advanced techniques such as fuzzy and genetic algorithms have been proposed to calculate the optimal reference trajectory. But in these approaches, the optimal solution is not always guaranteed. Therefore, in [6] a mixed integer linear programming (MILP) approach has been proposed to solve the optimal trajectory problem. The resulting MILP problem can be solved efficiently using existing commercial and free solvers that guarantee finding the global optimum of the MILP problem.

However, the approaches mentioned above ignore the impact caused by the signaling systems, e.g. a fixed-block signaling system or a moving block signaling (MBS) system. Especially in busy railway systems, the train's operation must be adjusted by the signaling system. Lu and Feng [7] consider the constraints caused by the leading train in a four-aspected fixed block signaling system when optimizing the trajectory of the following train. More specifically, a parallel genetic algorithm is used to optimize the trajectories for the leading train and the following train and it results a lower energy consumption [7]. In addition, Gu et al. [8] applied nonlinear programming method to optimize the trajectory for the following train. Two situations of the leading train, i.e. running and stopped, are studied and the corresponding strategies are proposed for the following train. Ding et al. [1] took the constraints caused by the MBS system into account and developed an energy-efficient multi-train control simulator to calculate the optimal trajectories for the locomotives with discrete levels of control. However, the control sequences in their simulator are determined by a predefined logic, which is not necessarily optimal. Therefore, in this paper we will extended our MILP approach of [6] to solve the trajectory planning for two trains under an MBS system since the MILP problem can be solved efficiently.

This paper is structured as follows. Section II summarizes the train model and the MILP approach for a single train based on [6]. Section III introduces the constraints for the following train caused by the leading train under the MBS system. Section IV shows how to include these constraints into the MILP formulation. Two approaches are proposed here. In the first approach the leading train's trajectory is assumed to be fixed or just given, and the optimal control problem is solved for the following train only with the constraints caused by the MBS system. The second approach consists in optimizing the trajectories of the leading train and the following train simultaneously. Section V illustrates the calculation of the optimal trajectories using the data from Beijing Yizhuang subway line.

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II. Train model and the MILP approach

The literature on train optimal control usually uses the mass-point model of train [9]. The motion of a train can be described by [4]:

$$m\rho\frac{\mathrm{d}v}{\mathrm{d}t}=u(t)-R_{\mathrm{b}}(v)-R_{\mathrm{l}}(s,v), \tag{1}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}=v, \tag{2}$$

where m is the mass of the train, ρ is a factor to consider the rotating mass [10], v is the velocity of the train, s is the position of the train, u is the control variable, i.e. the traction or braking force, which is bounded by the maximum traction force $u_{\max}$ and the maximum braking force $u_{\min}$, $u_{\min}\le u\le u_{\max}$. According to the Strahl formula [11] the basic resistance $R_{\mathrm{b}}(v)$ can be described as

$$R_{\mathrm{b}}(v)=m(a_1+a_2v^2),$$

where a_1 and a_2 depend on the train characteristics and the wind speed. The line resistance $R_{\mathrm{l}}(s,v)$ caused by track slope, curves, and tunnels can be described by [12]

$$R_{\mathrm{l}}(s,v)=mg\sin\alpha(s)+f_{\mathrm{c}}(r(s))+f_{\mathrm{t}}(l_{\mathrm{t}}(s),v), \tag{3}$$

where g is the gravitational acceleration, $\alpha(s)$, $r(s)$, and $l_{\mathrm{t}}(s)$ are the slope, the radius of the curve, and the length of the tunnel at position s along the track, respectively. The curve resistance $f_{\mathrm{c}}(\cdot)$ and the tunnel resistance $f_{\mathrm{t}}(\cdot)$ are given by empirical formulas (see [12] for details).

Franke et al. [9] choose kinetic energy per mass unit $\tilde{E}=0.5v^2$ and time t as states, and the position s as the independent variable. The reference trajectory planning problem for trains is formulated as [6]:

$$J=\int_{s_{\mathrm{start}}}^{s_{\mathrm{end}}}\left(u(s)+\lambda\cdot\left|\frac{\mathrm{d}u(s)}{\mathrm{d}s}\right|\right)\mathrm{d}s \tag{4}$$

subject to the model (1) and (2), the constraints

$$u_{\min}\le u(s)\le u_{\max}, \tag{5}$$

$$0<\tilde{E}(s)\le\tilde{E}_{\max}(s), \tag{6}$$

and the boundary conditions,

$$\tilde{E}(s_{\mathrm{start}})=\tilde{E}_{\mathrm{start}},\ \ \tilde{E}(s_{\mathrm{end}})=\tilde{E}_{\mathrm{end}}, \tag{7}$$

$$t(s_{\mathrm{start}})=0,\ \ t(s_{\mathrm{end}})=T, \tag{8}$$

where the objective function J is a weighted sum of the energy consumption and the riding comfort of the train operation; $\tilde{E}_{\max}(s)$ is equal to $0.5V_{\max}^2(s)$ and $V_{\max}(s)$ is the maximum allowable velocity, which depends on the train characteristics and line conditions, and as such it is usually a piecewise constant function of the coordinate s [3], [4]; s_{start}, $\tilde{E}(s_{\mathrm{start}})$, and $t(s_{\mathrm{start}})$ are the position, the kinetic energy per mass, and the time at the beginning of the route; s_{end}, $\tilde{E}(s_{\mathrm{end}})$, and t_{end} are the position, the kinetic energy per mass, and the time at the end of the route; the scheduled running time T is given by the timetable or the rescheduling process. It is assumed that the unit kinetic energy $\tilde{E}(s)>0$, which means the train's speed is always strictly larger than zero, i.e. the train travels nonstop [3].

In [6], an operation of a train is described via a continuous-time mass-point model, which is discretized in space. The position horizon $[s_{\mathrm{start}},s_{\mathrm{end}}]$ is split into N intervals and it is assumed that the track and train parameters as well as the traction or the breaking force are constant in each interval $[s_k,s_{k+1}]$ with length $\Delta s_k=s_{k+1}-s_k$, for $k=1,2,\ldots,N$. The discrete-space model is then transcribed into a piecewise affine (PWA) model by approximating the nonlinear terms through PWA functions. Furthermore, by applying some transformation properties [13], the PWA model is formulated as the following mixed logical dynamic model:

$$\begin{aligned}x(k+1)=&A_kx(k)+B_ku(k)+C_{1,k}\delta(k)+C_{2,k}\delta(k+1)\\&+D_{1,k}z(k)+D_{2,k}z(k+1)+e_k,\end{aligned} \tag{9}$$

$$\begin{aligned}&R_{1,k}\delta(k)+R_{2,k}\delta(k+1)+R_{3,k}z(k)+R_{4,k}z(k+1)\\&\le R_{5,k}u(k)+R_{6,k}x(k)+R_{7,k},\end{aligned} \tag{10}$$

where the states $x(k)$ involves kinetic energy per mass $E(k)$ and time $t(k)$, i.e. $x(k)=\begin{bmatrix}E(k) & t(k)\end{bmatrix}^T$, $u(k)$ is the control variable, i.e. the traction or braking force, which is bounded by the maximum traction force $u_{\max}$ and the maximum braking force $u_{\min}$, $\delta(k)$ and $z(k)$ are a vector of logical variables and a real-valued vector of auxiliary variables respectively, which are introduced by the transformation properties of [13].

The objective of the trajectory planning problem is considered as the energy consumption of the train operation without regenerative braking, which could be described as

$$J=\sum_{k=1}^{N}\max(0,u(k))\Delta s_k. \tag{11}$$

As shown in [6], the optimal control problem can be recast as the following mixed integer linear programming (MILP) problem by introducing a new variable $\omega(k)$ to deal with the function $\max(0,u(k))$ in the objective function (11):

$$\min_{\tilde{V}}\quad C_J^T\tilde{V}, \tag{12}$$

subject to

$$F_1\tilde{V}\le F_2x(1)+f_3 \tag{13}$$

$$F_4\tilde{V}=F_5x(1)+f_6 \tag{14}$$

where $C_J=\begin{bmatrix}0 & \cdots & 0 & \Delta s_1 & \cdots & \Delta s_N\end{bmatrix}^T$, $\tilde{V}=\begin{bmatrix}\tilde{u}^T & \tilde{\delta}^T & \tilde{z}^T & \tilde{\omega}^T\end{bmatrix}^T$, $\tilde{u}=\begin{bmatrix}u^T(1) & u^T(2) & \ldots & u^T(N)\end{bmatrix}^T$, $\tilde{\delta}=\begin{bmatrix}\delta^T(1) & \delta^T(2) & \ldots & \delta^T(N+1)\end{bmatrix}^T$, $\tilde{z}=\begin{bmatrix}z^T(1) & z^T(2) & \ldots & z^T(N+1)\end{bmatrix}^T$, $\tilde{\omega}=\begin{bmatrix}\omega^T(1) & \omega^T(2) & \ldots & \omega^T(N)\end{bmatrix}^T$, for properly defined F_1, F_2, f_3, F_4, F_5, and f_6. The MILP problem (12)-(14) can be solved by several existing commercial and free solvers, such as CPLEX, Xpress-MP, GLPK.

III. Moving block signaling system constraints

A. Constraints caused by moving block signaling system

A moving block signaling (MBS) system relies on the continuous bidirectional communication links between trains and zone controllers. A zone controller calculates the limit-of-movement-authority for every train in the zone it controls and make sure that trains will be running with a safe distance with respect to other trains. More specifically, the limit-of-movement-authority indicates the tail of the preceding train with a safety margin included, i.e. the maximum position that a train is allowed to move to. In addition, the limit-of-movement-authority of the following train moves forward continuously as the leading train travels. There exists four moving block signaling schemes theoretically [14]: moving space block signaling, moving time block signaling, pure moving block signaling, and relative moving block signaling. Takeuchi et al. [15] evaluated the first three schemes and compared them with the fixed block signaling scheme based on two basic criteria, namely steady state performance and perturbed performance. It is concluded that the pure moving block signaling scheme gives the best performance. Furthermore, the pure moving block signaling scheme is the basis of all currently implemented systems [15], such as the European train control system levels 2 and 3 [16] and the positive train control system in the US [17]. The pure moving block signaling scheme adjusts the minimum instantaneous distance between two successive trains using the speed of the following train according to the following equation:

$$s^{\mathrm{L}}(t)-s^{\mathrm{F}}(t)\geq(v^{\mathrm{F}}(t))^2/(2a_{\mathrm{b}}^{\mathrm{F}})+S_{\mathrm{SM}}, \tag{15}$$

where $s^{\mathrm{L}}(t)$ and $s^{\mathrm{F}}(t)$ are the positions of the noses of the leading train and the following train at time t, $v^{\mathrm{F}}(t)$ is the speed of the following train, $a_{\mathrm{b}}^{\mathrm{F}}$ is the normal braking rate, i.e. 0.75 times the maximum deceleration, and S_{SM} is the safety margin distance. The minimum distance between two successive trains is basically the instantaneous braking distance required by the following train plus a safety margin. Therefore, even if the leading train comes to a sudden halt, a collision can be avoided by using the minimum distance. Therefore, the MBS system in this paper uses the pure moving block signaling scheme. However, the approach proposed could be extended to other moving block signaling schemes.

In practice, the minimum distance in the MBS system is larger than that of (15) because the driver or of the automatic train control system need time to react situations. Furthermore, the train length has to be considered too. Therefore, the minimum distance of a practical MBS system is modified as [15]

$$s^{\mathrm{L}}(t)-s^{\mathrm{F}}(t)\geq L_{\mathrm{r}}^{\mathrm{F}}+(v^{\mathrm{F}}(t))^2/(2a_{\mathrm{b}}^{\mathrm{F}})+S_{\mathrm{SM}}+L_{\mathrm{t}}^{\mathrm{L}}, \tag{16}$$

where $L_{\mathrm{r}}^{\mathrm{F}}$ is the distance that the following train will travel during the reaction time $t_{\mathrm{r}}^{\mathrm{F}}$ of the driver and/or train equipment of the following train, and $L_{\mathrm{t}}^{\mathrm{L}}$ is the length of the leading train. The value of the reaction time could be obtained from experience. The minimum distance between two successive trains (16) is equivalent to the minimum time difference of two successive trains

$$t^{\mathrm{F}}(s)-t^{\mathrm{L}}(s)\geq t_{\mathrm{r}}^{\mathrm{F}}+t_{\mathrm{b}}^{\mathrm{F}}(s)+t_{\mathrm{safe}}^{\mathrm{F}}(s), \tag{17}$$

where $t^{\mathrm{L}}(s)$ and $t^{\mathrm{F}}(s)$ are the time instants at which of the front of respectively the leading train and the following train pass position s. The braking time of the following train $t_{\mathrm{b}}^{\mathrm{F}}(s)$ and the time margin $t_{\mathrm{safe}}^{\mathrm{F}}(s)$ caused by the safe margin distance and the train length are defined as

$$t_{\mathrm{b}}^{\mathrm{F}}(s)=v^{\mathrm{F}}(s)/a_{\mathrm{b}}^{\mathrm{F}}, \tag{18}$$

$$t_{\mathrm{safe}}^{\mathrm{F}}(s)=(S_{\mathrm{SM}}+L_{\mathrm{t}}^{\mathrm{L}})/v^{\mathrm{F}}(s), \tag{19}$$

where $v^{\mathrm{F}}(s)$ is the speed of the following train at position s.

In order to ensure that near stations a train's operation is not impeded by the signaling system, i.e. a train's operation is not then affected by the (in principle slowly moving or stopped) train in front, the minimum headway is introduced. This is the minimum time separation between successive trains at train stations, and it is defined as [15]

$$H_{\min}=t_{\mathrm{d}}^{\mathrm{L}}+t_{\mathrm{in-out}}=t_{\mathrm{d}}^{\mathrm{L}}+t_{\mathrm{r}}^{\mathrm{F}}+t_{\mathrm{b,max}}^{\mathrm{F}}+t_{\mathrm{safe}}^{\mathrm{L}}, \tag{20}$$

with the run-in/run-out time $t_{\mathrm{in-out}}=t_{\mathrm{r}}^{\mathrm{F}}+t_{\mathrm{b,max}}^{\mathrm{F}}+t_{\mathrm{safe}}^{\mathrm{L}}$, where $t_{\mathrm{b,max}}^{\mathrm{F}}$ is the time it takes the following train to come to a full stop when it is running at its maximum speed, i.e. $t_{\mathrm{b,max}}^{\mathrm{F}}=v_{\max}^{\mathrm{F}}/a_{\mathrm{b}}$, $t_{\mathrm{d}}^{\mathrm{L}}$ is the station dwell time of the leading train, and the run-out time $t_{\mathrm{safe}}^{\mathrm{L}}$ is the time that the leading train needs to completely clear the secure section (i.e. a special section to protect the leading train), if present, and including a safety margin, i.e. $t_{\mathrm{safe}}^{\mathrm{L}}=\sqrt{2(S_{\mathrm{SM}}+L_{\mathrm{t}}^{\mathrm{L}}+L_{\mathrm{s}})/a_{\mathrm{acc}}^{\mathrm{L}}}$. The acceleration of the leading train $a_{\mathrm{acc}}^{\mathrm{L}}$ is usually considered as a constant value for the minimum headway calculation.

B. Considering the constraints of moving block signaling system into MILP approach

Recall that the mixed logical dynamic model of the train's operation is discrete in space with N space intervals with grid points s_k, $k=0,1,\ldots,N$ as shown in Section II. Here, we discretize the constraint (17) caused by the MBS system at the grid points s_k as

$$t^{\mathrm{F}}(k)\geq t^{\mathrm{L}}(k)+t_{\mathrm{r}}^{\mathrm{F}}+t_{\mathrm{b}}^{\mathrm{F}}(k)+t_{\mathrm{safe}}^{\mathrm{F}}(k),\quad \text{for}\quad k=1,2,\ldots,N-1, \tag{21}$$

$$t^{\mathrm{F}}(k)\geq t^{\mathrm{L}}(k)+t_{\mathrm{r}}^{\mathrm{F}}+t_{\mathrm{b,max}}^{\mathrm{F}}+t_{\mathrm{d}}^{\mathrm{L}}+t_{\mathrm{safe}}^{\mathrm{L}},\quad \text{for}\quad k=N. \tag{22}$$

In addition, some intermediate constraints are introduced to ensure that the points between the grid points also satisfy the constraints caused by the MBS system. According to (17), we obtain the following constraint for each $s\in[s_k,s_{k+1}]$ as:

$$t^{\mathrm{F}}(s)-t_{\mathrm{r}}^{\mathrm{F}}-t_{\mathrm{b}}^{\mathrm{F}}(s)-t_{\mathrm{safe}}^{\mathrm{F}}(s)\geq t^{\mathrm{L}}(s), \tag{23}$$

If we assume the left-hand side of (23) to be an affine function in the interval $[s_k,s_{k+1}]$, then we can add the following constraints:

$$\begin{aligned}(1-\alpha)(t^{\mathrm{F}}(k)-t_{\mathrm{r}}^{\mathrm{F}}-t_{\mathrm{b}}^{\mathrm{F}}(k)-t_{\mathrm{safe}}^{\mathrm{F}}(k))+\alpha(t^{\mathrm{F}}(k+1)-t_{\mathrm{r}}^{\mathrm{F}}\\ -t_{\mathrm{b}}^{\mathrm{F}}(k+1)-t_{\mathrm{safe}}^{\mathrm{F}}(k+1))\geq t^{\mathrm{L}}(s+\alpha\Delta s_k),\end{aligned} \tag{24}$$

for some real values α in a finite subset $S_\alpha \in [0,1)$, e.g. $S_\alpha = \{0.1, 0.2, \ldots, 0.9\}$, where $t^{\mathrm{L}}(s+\alpha\Delta s_k)$ is known if the optimal trajectory of the leading train is fixed. Note that for $\alpha = 0$ and $\alpha = 1$ (21) is retrieved (except if $k = N-1$). However, if the leading train's trajectory is not known beforehand, then we need to optimize both trajectories simultaneously. In this case, the term $t^{\mathrm{L}}(s+\alpha\Delta s_k)$ is unknown. If we assume the right-hand side of (23) is also an affine function, i.e. $t^{\mathrm{L}}(s+\alpha\Delta s_k) = (1-\alpha)t^{\mathrm{L}}(k) + \alpha t^{\mathrm{L}}(k+1)$, then it is sufficient to check (23) in the points k and $k+1$ (i.e., for $\alpha = 0$ and $\alpha = 1$), since due to linearity (23) will then also automatically be satisfied for all intermediary points.

Note that the constraints (21), (22), and (24) are linear in $t^{\mathrm{L}}(k)$, $t^{\mathrm{L}}(k+1)$, $t^{\mathrm{F}}(k)$. However, they are nonlinear in $v^{\mathrm{F}}(k)$ and $v^{\mathrm{F}}(k+1)$ since the time safe margin (19) is a nonlinear function of the following train's velocity $v^{\mathrm{F}}(k)$. Furthermore, the kinetic energy per mass $E^{\mathrm{F}}(k)$ is one of the states instead of $v^{\mathrm{F}}(k)$ with $E^{\mathrm{F}}(k) = 0.5(v^{\mathrm{F}}(k))^2$ (cf. Section II). Therefore, both the braking time $t_{\mathrm{b}}^{\mathrm{F}}(k)$ and the safe time margin $t_{\mathrm{safe}}^{\mathrm{F}}(k)$ are nonlinear functions of $E^{\mathrm{F}}(k)$, where

$$t_{\mathrm{b}}^{\mathrm{F}}(k) = \frac{1}{a_{\mathrm{b}}^{\mathrm{F}}}\sqrt{2E^{\mathrm{F}}(k)} \tag{25}$$

and

$$t_{\mathrm{safe}}^{\mathrm{F}}(k) = (S_{\mathrm{SM}} + L_{\mathrm{t}}^{\mathrm{L}})\frac{1}{\sqrt{2E^{\mathrm{F}}(k)}}. \tag{26}$$

The nonlinear functions $f_1(\cdot): E^{\mathrm{F}}(k) \to \sqrt{2E^{\mathrm{F}}(k)}$ and $f_2(\cdot): E^{\mathrm{F}}(k) \to \frac{1}{\sqrt{2E^{\mathrm{F}}(k)}}$ could be approximated by PWA functions as follows. There are various methods for approximating functions in a PWA way, see e.g., the overview by Azuma et al. [18]. In this paper, we first select the number of regions of the PWA function and optimize the interval lengths and parameters of the affine functions using least-squares optimization for (25). Then the same number of regions and same interval lengths are used for the approximation of (26). If we consider approximations with 2 affine subfunctions, the PWA approximations of functions $f_1(\cdot)$ and $f_2(\cdot)$ can be written as

$$f_{1,\mathrm{PWA}}(E^{\mathrm{F}}(k)) = \begin{cases} \alpha_1 E^{\mathrm{F}}(k) + \beta_1 & \text{for } E_{\min} \le E^{\mathrm{F}}(k) < E_1, \\ \alpha_2 E^{\mathrm{F}}(k) + \beta_2 & \text{for } E_1 \le E^{\mathrm{F}}(k) \le E_{\max}, \end{cases}$$

$$f_{2,\mathrm{PWA}}(E^{\mathrm{F}}(k)) = \begin{cases} \lambda_1 E^{\mathrm{F}}(k) + \mu_1 & \text{for } E_{\min} \le E^{\mathrm{F}}(k) < E_1, \\ \lambda_2 E^{\mathrm{F}}(k) + \mu_2 & \text{for } E_1 \le E^{\mathrm{F}}(k) \le E_{\max}, \end{cases}$$

with optimized parameters α_1, α_2, β_1, β_2, λ_1, λ_2, μ_1, μ_2, and E_1. For more details and extension of this transformation into PWA functions, see [18]. Now the constraint (21) can be approximated as the following linear constraint:

$$\begin{aligned} t^{\mathrm{F}}(k) \ge{} & t^{\mathrm{L}}(k) + t_{\mathrm{r}}^{\mathrm{F}} + \frac{1}{a_{\mathrm{b}}^{\mathrm{F}}}(\alpha_1 E^{\mathrm{F}}(k) + \beta_1) + (S_{\mathrm{SM}} + L_{\mathrm{t}}^{\mathrm{L}})(\lambda_1 E^{\mathrm{F}}(k) \\ & + \mu_1), \quad \text{if} \quad E_{\min} \le E^{\mathrm{F}}(k) < E_1 \\ t^{\mathrm{F}}(k) \ge{} & t^{\mathrm{L}}(k) + t_{\mathrm{r}}^{\mathrm{F}} + \frac{1}{a_{\mathrm{b}}^{\mathrm{F}}}(\alpha_2 E^{\mathrm{F}}(k) + \beta_2) + (S_{\mathrm{SM}} + L_{\mathrm{t}}^{\mathrm{L}})(\lambda_2 E^{\mathrm{F}}(k) \\ & + \mu_2), \quad \text{if} \quad E_1 \le E^{\mathrm{F}}(k) \le E_{\max} \end{aligned}$$

Similarly, the constraints (22) and (23) can also be written as linear constraints. These approximated linear constraints caused by the MBS system can be easily included in the MILP approach and we still get an MILP problem.

IV. Optimal control problem for two trains in the moving block signaling system

There exist two solution approaches for the optimal trajectory planning problem for two trains in the MBS system. One is a greedy approach, where we first schedule the leading train, and next schedule the following train based on the results of the leading train. The other solution approach consists in optimizing trajectories of the two trains simultaneously. Furthermore, the approaches proposed here can be extended to more than two trains.

A. Given the trajectory of the leading train

If the optimal trajectory of the leading train is fixed and known by the zone controller, then the optimal control problem for two trains is reduced to the trajectory planning problem for the following train, which is then similar to the one in [6]. However, the constraints (21), (22), and (24) caused by the leading train in MBS system should be included. These constraints are significant for the trajectory planing, especially when the capacity utilization of the railway network is high. In this situation, the headway between successive trains is shorter, and therefore, it is more likely that the leading train would affect the following train, i.e. the train's operation could be impeded by the MBS system.

The coefficient matrices in the mixed logical dynamic model (9) and (10) are determined by the following train. Since the trajectory of the leading train is known by the zone controller, then $t^{\mathrm{L}}(k)$ and $t^{\mathrm{L}}(s+\alpha\Delta s_k)$ are known for the trajectory planning problem for the following train.

B. Optimizing trajectories of two trains simultaneously

We could also optimize the trajectories of the leading train and the following train simultaneously. The models for these two trains are of the form (9)-(10). The optimal control problem of these two successive trains can be rewritten in the form of the MILP problem (12), (13), (14). However, the numbers of the state variables, binary variables, auxiliary variables, and constraints are doubled now compared to the case of Section IV-A. Therefore, the size of this optimal trajectory planning problem is much bigger than the problem for a single train and the computation time of the bigger problem will be much longer. However, since optimizing the trajectories of two trains at the same time is a global optimization problem for these two trains, the control performance will in general be better than the case that the leading train's is optimized first or just given.

V. Case Study

In order to demonstrate the performance of the MILP approach for optimal trajectory planning for two trains under the MBS system, the line data and train characteristics of the Beijing Yizhuang subway line are used as a test case study.

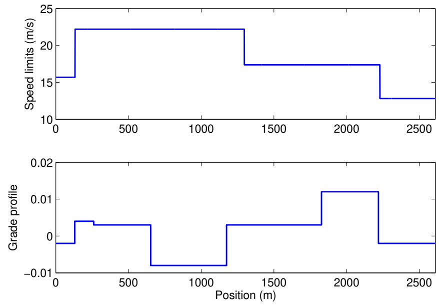

Fig. 1. Speed limits and grade profile between Ciqu and Yizhuang station

TABLE I
PARAMETERS OF TRAIN AND LINE PATH

Property	Symbol	Value
Train mass [kg]	m	$4.15 \cdot 10^5$
Basic resistance [N/kg]	R_b	$0.0142 + 4.1844 \cdot 10^{-5}v^2$
Mass factor	ρ	1.06
Maximum velocity [m/s]	V_max	22.2
Line length [m]	s_end	2610
Kinetic energy [J]	E_min	0.1
Traction force [N]	u_max	$3.15 \cdot 10^5$
Braking force (regular) [N]	u_min	$-3.32 \cdot 10^5$

In this paper, we only consider the last two stations in the Yizhuang subway line: Ciqu and Yizhuang. The track length between these two stations is 2610 m and the speed limits and grade profile are shown in Figure 1. The parameters of the train and the line path are listed in Table I. The rotating mass factor is often chosen as 1.06 in the literature [10] and therefore we also adopt this value. According to the assumption made in [6], the unit kinetic energy should be larger than zero. In this test case, the minimum kinetic energy is chosen as 0.1 J. The maximum traction force of the train in the Yizhuang line is a nonlinear function of the train's velocity and the maximum value of this function is 315 kN. The objective function of the optimal train control problem considered here is the energy consumption of the train operation without regenerative braking (cf. (11)), since the energy generated by regenerative braking scheme is consumed by electric resistance in the Yizhuang line.

The parameters for the MBS system constraints are listed in Table II. The length of the train is 90 m and the reaction time of the driver is 1 s. Based on the parameters given in Table II, the run-in/run-out time $t_{\mathrm{in-out}}$ in (20) is equal to 44.6 s and the minimum headway equals to 69.6 s. In the case study, the minimum headway is assumed as 70 s. In addition, we assume that there exist two extra constraints on the leading train from the scheduling process due to technical reasons: the speed after 1300 m is limited to 40 km/h, i.e. 11.1 m/s and the dwell time of the leading train at Yizhuang station is only 10 s. Note that these constraints only apply to the leading train, not to the following train. Furthermore, the running times for the leading train and the following train are 216 s and 194 s, respectively.

The length Δs_k for interval $[s_k, s_{k+1}]$ depends on the speed limits, gradient profile, and so on. In addition, if the number

TABLE II
PARAMETERS FOR THE CALCULATION OF THE MBS CONSTRAINTS

Property	Symbol	Value
Train length [m]	L_t	90
Safety margin [m]	S_SM	30
Secure section	L_s	60
Started acceleration [m/s^2]	$a^\mathrm{L}_\mathrm{acc}$	1
Braking deceleration [m/s^2]	a^F_b	0.9
Braking reaction time [s]	t_r	1
Station dwell time [s]	t_d	25

of the space intervals N is larger, then the accuracy will be better but the computation time of the MILP approach will be longer. According to the total length of the trip and the speed limits and grade profile in Figure 1, we choose 20 intervals, which implies that the length of each interval is 130.5 m, i.e. $\Delta s_k = 130.5$ m for $k = 1, 2, \ldots, 20$. That choice provides a good tradeoff of the computation time and the accuracy. In fact, we selected space intervals with a length of 500 m for the MILP approach in [25], where we compared the MILP approach with a state-of-the-art pseudospectral method with an average distance between collocation points of 50 m, and where we found that the MILP approach outperformed the pseudospectral method with 10% for the control performance and with two to three orders of magnitude for the computation speed.

Two cases will be considered here:

- Case A: the leading train's trajectory is given
  In this case, the trajectory of the leading train is optimized using the MILP proposed in [6] , which is shown as the solid line in Figure 2. The dashed line is the speed limit along the track. Due to the extra speed limit for the leading train, i.e. the speed limit is 11.1 m/s after 1300 m, the speed of the leading train obtained using the optimal control input is lower than 11.1 m/s. The trajectory planning problem of the following train with the MBS constraints is solved using the trajectory of the leading train given in Figure 2. The obtained optimal control input and the trajectory by applying these inputs to the nonlinear model for the following train is shown as the dash-dotted line in Figure 2. In the space interval $[2000, 2500]$, the following train is affected by the leading train, where the following train starts to slow down in order to satisfy the constraints caused by the MBS system. Later on, it accelerates again to satisfy the running time requirement. The energy consumption for the leading train and the following train are 124.8 MJ and 84.80 MJ respectively, as shown in Table III. The calculation time for the trajectory planning problem of the following train is 0.36 s.
- Case B: Optimizing the trajectories simultaneously
  When the trajectories of the leading train and the following train are optimized simultaneously, the optimal control inputs and the trajectories obtained by these inputs are shown in Figure 3, where the solid line are the trajectory and optimal input of the leading train and the dash dotted line are the results for the following

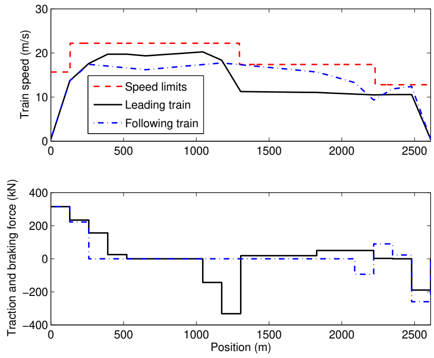

Fig. 2. The optimal trajectories and inputs with a headway 70 s when the trajectory of the leading train is given

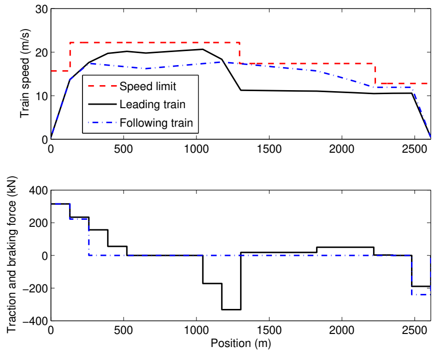

Fig. 3. The optimal trajectories and inputs with a headway 70 s when optimizing both trains simultaneously

train. The trajectory of the leading train is similar to that in Figure 2. However, the trajectory of the following train is smoother than that in Figure 2. In this case, the leading train does not affect the following train. The energy consumption for the leading train and the following train are 128.7 MJ and 70.11 MJ respectively. As we can see from Table III, the energy consumption of Case A is bigger than that of Case B. However, the computing time of the optimal control problem is 1.42 s, which is longer than that of Case A.

VI. Conclusions and Future Work

We have considered the optimal trajectory planning problem for trains under a moving block signaling (MBS) system. The nonlinear train model is formulated as a mixed logical dynamic model and the optimal control problem is recast as an MILP problem. The constraints caused by the MBS system can be approximated by piecewise affine approximations and then included into the MILP formulation. The simulation results show that the computation time of the optimal problem for the following train given the leading train's trajectory is shorter than that of two trains at the same time. However, when optimizing the trajectories of two trains simultaneously, the energy consumption is smaller.

When the number of trains taken into account increases, the size of the MILP problem will grow quickly and the computing time will be much longer. Therefore, developing efficient methods for solving large-scale trajectory planning problem for trains will be an important topic for future work.

TABLE III

Energy Consumption and Computation Time

Case	Leading train	Following train	Total consumption	Computing time
Case A	124.8 MJ	84.80 MJ	209.6 MJ	0.36 s
Case B	128.7 MJ	70.11 MJ	198.8 MJ	1.42 s

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