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Linear model predictive control based on approximate optimal control inputs and constraint tightening

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Abstract— **In this paper we propose a model predictive control scheme for discrete-time linear time-invariant systems based on inexact numerical optimization algorithms. We assume that the solution of the associated quadratic program produced by some numerical algorithm is possibly neither optimal nor feasible, but the algorithm is able to provide estimates on primal suboptimality and primal feasibility violation. By tightening the complicating constraints we can ensure the primal feasibility of the approximate solutions generated by the algorithm. Finally, we derive a control strategy that has the following properties: the constraints on the states and inputs are satisfied, asymptotic stability of the closed-loop system is guaranteed, and the number of iterations needed for a desired level of suboptimality can be determined.**

I. INTRODUCTION

Model predictive control (MPC) has become a popular advanced control technology due to its ability to handle hard input and state constraints. MPC was first implemented in slow systems such as industrial processes [16], but due to the increase of computing power and data transmission capabilities of modern digital devices it has been extensively studied also in the context of controlling fast embedded systems and distributed control of network systems.

Recently there has been a growing interest in developing faster MPC schemes for embedded systems, by improving the computational efficiency and providing worst-case computational complexity certificates for the applied solution methods, making these schemes implementable on hardware with limited computational power [7]–[9], [12], [13], [17]. In large network system settings many decomposition methods have been proposed for the synthesis of distributed MPC schemes as well [3], [10]–[12], [14], [20], [22].

Typical requirements for the practical implementation of real-time MPC include certification of the worst-case execution time, reduced memory usage, simple numerical iterations that can be easily implemented on cheap and/or certifiable hardware and software, and distributed computations. Classical approaches meeting these requirements such as explicit MPC [1], or methods based on interior point algorithms [17] can fail due to the large dimension of the problems or complex iterations that involve matrix inversion. An alternative is provided by dual first order methods [12]–[14], [18]. Although these methods are characterized by simple computations and offer tight worst-case bounds on the required number of iterations, they can ensure feasibility only at optimality. In order to avoid this drawback, new dual methods based on constraint tightening have been proposed in [5], [6], [12], [19]. The authors in [6], [19] present suboptimal stable MPC schemes able to ensure also feasibility of the primal variables using a constraint tightening approach. In these schemes the tightening is applied to both the state and the input constraints, while the parameters measuring the suboptimality and the degree of tightening are fixed for all initial states of the MPC scheme. In [5], stability and feasibility of an MPC scheme is also ensured using a constraint tightening approach and suboptimality results are based on dual subgradient analysis. In the present paper we extend the main results from [12] on stability and feasibility of a suboptimal MPC scheme, where we now assume that the suboptimal control inputs are computed using a generic inexact numerical optimization algorithm.

Contribution: The main contribution of this paper is to propose a suboptimal MPC scheme that ensures both feasibility and stability with only a limited number of optimization iterations. Since in many MPC schemes an approximate solution of the optimization problem that has to be solved at each time instant might not be feasible, we solve approximately an auxiliary problem obtained by tightening the constraints of the original one. We show that the approximate solution of the tightened problem is also a suboptimal feasible solution for our original optimization problem and thus we obtain an MPC scheme that ensures feasibility, suboptimality, and stability for the closed loop system. Compared to the recent papers [6], [19], in our approach the tightening is applied only to the state constraints while the parameters measuring the suboptimality and the tightening are chosen adaptively, i.e., depending on the initial state of the MPC scheme. This leads to a more flexible and potentially less conservative approach. The proposed MPC scheme can accommodate any QP solver in order to find an approximate solution of the problem. Further, in order to establish a bound on the number of iterations required to find a desired solution, we also provide a dual fast gradient method taken from [12], which is at least one order of magnitude faster than the one in [5].

Paper outline: The paper is organized as follows. In Section II we formulate the linear MPC problem and we also introduce a tightened problem that helps us to recover feasibility. In Section III we prove the feasibility, suboptimality, and stability of the proposed MPC scheme. In Section IV we present an inexact dual fast gradient algorithm for finding the approximate solution required by our MPC scheme. Finally, in Section V we apply our scheme to a ball on plate system.

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Notation: We work in the space $\mathbb{R}^n$ composed by column vectors. For $u, v \in \mathbb{R}^n$ we denote the standard Euclidean inner product $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$, the Euclidean norm $\|u\| = \sqrt{\langle u, u \rangle}$, and the projection onto non-negative orthant $\mathbb{R}^n_+$ as $[u]_+$. We use the same notations $\langle \cdot, \cdot \rangle$, $\|\cdot\|$, and $[\cdot]_+$ for spaces of different dimension.

II. MPC PROBLEMS FOR LINEAR SYSTEMS

We consider discrete-time systems, defined by the following linear difference equations:

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$ represent the state and the input of the system at time t, respectively. We also impose local state and input constraints:

$$x(t) \in X, \quad u(t) \in U \quad \forall t \geq 0, \tag{2}$$

where $X \subseteq \mathbb{R}^{n_x}$ is a polyhedral set and $U \subseteq \mathbb{R}^{n_u}$ is a simple convex set, i.e., the projection on this set can be computed efficiently. Moreover, we assume that both X and U contain the origin in their interior. For the system (1) we consider a quadratic convex stage cost:

$$\ell(x(t), u(t)) = \frac{1}{2} \|x(t)\|_Q^2 + \frac{1}{2} \|u(t)\|_R^2,$$

where $\|x\|_Q^2 = x^T Q x$. The following assumption is valid throughout the paper:

Assumption 2.1: The pair (A, B) is stabilizable, $Q \succeq 0$, $R \succ 0$ and there exists a matrix $C \in \mathbb{R}^{m \times n_x}$ such that $Q = C^T C$ and the pair (A, C) is detectable.

Based on Assumption 2.1 we denote with $K \in \mathbb{R}^{n_u \times n_x}$ the gain associated with the infinite horizon linear quadratic regulator (LQR) defined by the matrices A, B, Q and R and with P the solution of the algebraic Riccati equation associated with the LQR problem. We also introduce a terminal cost:

$$V^{\mathrm{f}}(x) = \frac{1}{2} \|x\|_P,$$

and a terminal polyhedral set X^{f} which we assume to be a μ-contractive set for the closed-loop system $x(t+1) = (A+BK)x(t)$, not necessarily maximal, with $\mu < 1$ (see e.g., [21] for a detailed discussion), i.e.:

$$\forall x \in X^{\mathrm{f}} \Rightarrow x \in X, Kx \in U \text{ and } (A+BK)x \in \mu X^{\mathrm{f}}. \tag{3}$$

For a prediction horizon of length N, the MPC problem for (1), with a given initial state $x \in X_N$, where X_N denotes a region of attraction (see e.g. [21] for more details), can be formulated as:

$$\begin{aligned} V^*(x) = \min_{x(t), u(t)} & \sum_{t=0}^{N-1} \ell(x(t), u(t)) + V^{\mathrm{f}}(x(N)) \\ \text{s.t: } & x(t+1) = Ax(t) + Bu(t), \ x(0) = x \\ & x(t) \in X, \ u(t) \in U, \ x(N) \in X^{\mathrm{f}} \quad \forall t = 0, \ldots, N-1. \end{aligned} \tag{4}$$

For the input trajectory of the system we use the notation:

$$\mathbf{u} = \left[u(0)^T \cdots u(N-1)^T \right]^T \in \mathbb{R}^n.$$

By eliminating the states from the dynamics (1), the MPC problem (4) can be expressed as a quadratic convex optimization problem:

$$\begin{aligned} V^*(x) = \min_{\mathbf{u} \in \mathbf{U}} V_N(x, \mathbf{u}) \quad & \left(= \frac{1}{2} \mathbf{u}^T \mathbf{Q} \mathbf{u} + (\mathbf{W}x + \mathbf{w})^T \mathbf{u} \right) \\ \text{s.t: } \quad \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \leq 0, \end{aligned} \tag{$\mathbf{QP}(x)$}$$

where $\mathbf{Q}$ is positive definite due to the assumption that R is positive definite, the convex set $\mathbf{U}$ is the Cartesian product of the sets U for N times and the inequalities $\mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \leq 0$ are obtained by eliminating the states from the constraints $x(t) \in X$ and $x(N) \in X^{\mathrm{f}}$. Here we consider $\mathbf{G} \in \mathbb{R}^{p \times n}$. Note that if the set U is simple, then the Cartesian product set $\mathbf{U}$ is also a simple set. In MPC, at each time instant, given the initial state x, we need to solve the optimization problem (4) or equivalently optimization problem ($\mathbf{QP}(x)$).

We denote by $\mathbf{u}^*(x)$ the unique optimal solution of ($\mathbf{QP}(x)$). Further, we denote by $\mathbf{u}^{\mathrm{f}}(x)$ the LQR solution:

$$\mathbf{u}^{\mathrm{f}}(x) = [(Kx(0))^T \cdots (Kx(N-1))^T]^T,$$

where $x(0) = x$ and $x(t+1) = (A+BK)x(t)$ for all $t = 0, \ldots, N-1$.

In many practical situations, e.g. when we have fast dynamics and hard real-time computational requirements, or when we need to perform distributed computations, finding the solution $\mathbf{u}^*(x)$ of ($\mathbf{QP}(x)$) is difficult. Thus, we assume that we have available an algorithm that can deliver in a computationally predictable way an approximate ϵ-solution $\bar{\mathbf{u}}(x) = \mathrm{Alg}((\mathbf{QP}(x)), \epsilon)$ such that:

$$\bar{\mathbf{u}}(x) = \mathbf{u}^{\mathrm{f}}(x) \text{ if } x \in X^{\mathrm{f}} \tag{5}$$

or

$$\begin{aligned} & \bar{\mathbf{u}}(x) \in \mathbf{U}, \ \|[\mathbf{G}\bar{\mathbf{u}}(x) + \mathbf{E}x + \mathbf{g}]_+\| \leq \epsilon \text{ and} \\ & |V_N(\bar{\mathbf{u}}(x)) - V^*(x)| \leq \epsilon \text{ otherwise.} \end{aligned} \tag{6}$$

We note that in this setting, the approximate solution $\bar{\mathbf{u}}(x)$ is indeed suboptimal for the MPC scheme but it may also be infeasible since the constraints $\mathbf{G}\bar{\mathbf{u}}(x) + \mathbf{E}x + \mathbf{g} \leq 0$ might not be satisfied. In many applications, such as the MPC problem, the constraints typically represent physical limitations of actuators, or safety limits and operating conditions of the controlled plant. Thus, ensuring the feasibility of the primal variables, i.e., $\mathbf{u} \in \mathbf{U}$ and $\mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \leq 0$, becomes a critical requirement. We will see further how we can modify the original problem ($\mathbf{QP}(x)$) in order to find an approximate optimal solution that is also feasible.

A. Tightening the coupling constraints

In our proposed approach, instead of solving the original problem ($\mathbf{QP}(x)$), we consider a tightened version (similarly to [5], [6], [12], [19]). We introduce the following tightened problem associated with the original problem ($\mathbf{QP}(x)$):

$$\begin{aligned} V^*_{\epsilon_c}(x) = \min_{\mathbf{u} \in \mathbf{U}} \ V_N(x, \mathbf{u}) \quad & \left(= \frac{1}{2} \mathbf{u}^T \mathbf{Q} \mathbf{u} + (\mathbf{W}x + \mathbf{w})^T \mathbf{u} \right) \\ \text{s.t: } \quad \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} + \epsilon_c \mathbf{e} \leq 0, \end{aligned} \tag{$\mathbf{QP}_{\epsilon_c}(x)$}$$

where $\mathbf{e}$ denotes the vector with all entries 1. We state first the following assumption:

Assumption 2.2: There exists a strictly feasible vector $\tilde{\mathbf{u}}$ for problem ($\mathbf{QP}(x)$), i.e. exists $\tilde{\mathbf{u}} \in \mathbf{U}$ and $\mathbf{G}\tilde{\mathbf{u}}+\mathbf{E}x+\mathbf{g} < 0$. Then, we choose $\epsilon_{\rm c}$ to satisfy e.g. the following inequality:

$$0 < \epsilon_{\rm c} \leq \frac{1}{2} \min_{j=1,\ldots,p} \{-\left(\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g}\right)_j\}, \qquad (7)$$

with $\tilde{\mathbf{u}}$ being a strictly feasible vector for ($\mathbf{QP}(x)$) according to Assumption 2.2.

Remark 2.3 Note that with this choice for $\epsilon_{\rm c}$, input sequence $\tilde{\mathbf{u}}$ is also a strictly feasible vector for problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$), so that Assumption 2.2 still holds for this problem.

It is important to note that in our approach we apply the tightening only to the state constraints. Thus, our approach is less restrictive than the approach in [6], [19] where the tightening procedure is applied to both state and input constraints. It is straightforward to establish that both problems ($\mathbf{QP}(x)$) and ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$) are convex quadratic programs. Thus, without loss of generality we assume that for finding an ϵ-solution of ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$) we can invoke the same algorithm as for finding $\bar{\mathbf{u}}(x)$, i.e., $\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x) = \mathrm{Alg}((\mathbf{QP}_{\epsilon_{\rm c}}(x)), \epsilon)$.

If $x \in X^{\rm f}$ we do not need to introduce the tightened problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$) since in this case we have:

$$\bar{\mathbf{u}}(x) = \mathbf{u}^*(x) = \mathbf{u}^{\rm f}(x), \qquad (8)$$

which is feasible and optimal for problem ($\mathbf{QP}(x)$). In this case, it is also known that the value of the cost function is equal to the value of the terminal cost, i.e.:

$$V_N(x, \bar{\mathbf{u}}(x)) = V^*(x) = V^{\rm f}(x). \qquad (9)$$

We will discuss further how we can recover the feasibility, suboptimality and stability in the case $x \notin X^{\rm f}$.

III. Feasibility, stability, and suboptimality of the MPC scheme

As mentioned before, at each time step of the MPC scheme, given the initial state x, instead of applying the algorithm $\mathrm{Alg}((\mathbf{QP}(x)), \epsilon)$ to provide an ϵ-solution $\bar{\mathbf{u}}(x)$ of the optimization problem ($\mathbf{QP}(x)$) we apply $\mathrm{Alg}((\mathbf{QP}_{\epsilon_{\rm c}}(x)), \epsilon)$ for finding an ϵ-solution $\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x)$ of the tightened problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$). However, since we are interested in obtaining an approximate primal solution that may be suboptimal for the original problem ($\mathbf{QP}(x)$) but certainly primal feasible, we need to find first a relation between optimal values $V^*_{\epsilon_{\rm c}}(x)$ and $V^*(x)$. In order to find such a relation, let us denote by $\lambda^*_{\epsilon_{\rm c}}(x)$ an optimal Lagrange multiplier associated with the inequality constraints in problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$). Then, the following upper bound on $\|\lambda^*_{\epsilon_{\rm c}}(x)\|$ can be established (for a detailed discussion see [12]):

Lemma 3.1: Let $\lambda^*_{\epsilon_{\rm c}}(x)$ denote an optimal Lagrange multiplier associated with the inequality constraints $\mathbf{G}\mathbf{u}+\mathbf{E}x+\mathbf{g}+\epsilon_{\rm c}\mathbf{e} \leq 0$ in ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$). Then, the following upper bound on $\|\lambda^*_{\epsilon_{\rm c}}(x)\|$ can be established:

$$\|\lambda^*_{\epsilon_{\rm c}}(x)\| \leq 2\mathcal{R}_{\rm d},$$

with

$$\mathcal{R}_{\rm d} = \frac{V_N(x, \tilde{\mathbf{u}}) - d(\tilde{\lambda})}{\min\limits_{j=1,\ldots,p} \{-\left(\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g}\right)_j\}},$$

where $\tilde{\mathbf{u}}$ denotes a strictly feasible vector for problem ($\mathbf{QP}(x)$) (see Assumption 2.2), $d(\cdot)$ denotes the dual function of ($\mathbf{QP}(x)$) with respect to the inequality constraints $\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g} \leq 0$ and $\tilde{\lambda} \in \mathbb{R}^p_+$.

Proof: First, let us denote by $\mathcal{L}(x, \mathbf{u}, \lambda)$ and $\mathcal{L}_{\epsilon_{\rm c}}(x, \mathbf{u}, \lambda)$ the partial Lagrangian with respect to the inequality constraints in problem ($\mathbf{QP}(x)$) and ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$), respectively. Using Lemma 1 in [15] we can write:

$$\begin{aligned}
\|\lambda^*_{\epsilon_{\rm c}}(x)\| &\leq \frac{V_N(x, \tilde{\mathbf{u}}) - \min_{\mathbf{u}\in\mathbf{U}} \mathcal{L}_{\epsilon_{\rm c}}(x, \mathbf{u}, \tilde{\lambda})}{\min\limits_{j=1,\ldots,p} \{-\left(\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g} + \epsilon_{\rm c}\right)_j\}} \\
&= \frac{\left[V_N(x, \tilde{\mathbf{u}}) - \min_{\mathbf{u}\in\mathbf{U}} \mathcal{L}(x, \mathbf{u}, \tilde{\lambda})\right] - \left\langle \tilde{\lambda}, \epsilon_{\rm c}\mathbf{e} \right\rangle}{\min\limits_{j=1,\ldots,p} \{-\left(\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g}\right)_j\} - \epsilon_{\rm c}} \\
&\leq 2\mathcal{R}_{\rm d} \qquad \forall x \in X_N, \qquad (10)
\end{aligned}$$

where in the last inequality we used (7) and the fact that both $\tilde{\lambda}$ and $\epsilon_{\rm c}$ are nonnegative. ■

Note that for computing the bound $\mathcal{R}_{\rm d}$ we have the freedom of choosing $\tilde{\lambda} \in \mathbb{R}^p$. Thus, we can obtain a small enough bound on the norm of Lagrange multipliers $\lambda^*_{\epsilon_{\rm c}}(x)$.

Taking now into account that $\{\mathbf{u} : \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} + \epsilon_{\rm c}\mathbf{e} \leq 0\} \subseteq \{\mathbf{u} : \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \leq 0\}$, we have on the one hand:

$$V^*_{\epsilon_{\rm c}}(x) \geq V^*(x). \qquad (11)$$

On the other hand, from the the dual formulation of the tightened problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$) we have:

$$\begin{aligned}
V^*_{\epsilon_{\rm c}}(x) &= \min_{\mathbf{u}\in\mathbf{U}} V_N(x, \mathbf{u}) + \left\langle \lambda^*_{\epsilon_{\rm c}}(x), \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} + \epsilon_{\rm c}\mathbf{e} \right\rangle \\
&\leq \max_{\lambda\geq 0} \min_{\mathbf{u}\in\mathbf{U}} V_N(x, \mathbf{u}) + \langle \lambda, \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \rangle + \sqrt{p}\epsilon_{\rm c}\|\lambda^*_{\epsilon_{\rm c}}(x)\| \\
&\leq V^*(x) + 2\sqrt{p}\mathcal{R}_{\rm d}\epsilon_{\rm c}. \qquad (12)
\end{aligned}$$

We will see further how we can use relations (11) and (12) to recover the primal suboptimality for the original problem ($\mathbf{QP}(x)$) from the suboptimality of the tightened problem ($\mathbf{QP}_{\epsilon_{\rm c}}(x)$). We assume now that $\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x) = \mathrm{Alg}((\mathbf{QP}_{\epsilon_{\rm c}}(x)), \epsilon)$ with the accuracy ϵ satisfying:

$$\epsilon \leq \frac{1}{4} \min_{j=1,\ldots,p} \{-\left(\mathbf{G}\tilde{\mathbf{u}} + \mathbf{E}x + \mathbf{g}\right)_j\}, \qquad (13)$$

and the tightening parameter is chosen e.g. as:

$$\epsilon_{\rm c} = 2\epsilon, \qquad (14)$$

which satisfies (7). Thus, from (6) we have:

$$\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x) \in \mathbf{U}, \ \|[\mathbf{G}\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x) + \mathbf{E}x + \mathbf{g} + \epsilon_{\rm c}\mathbf{e}]_+\| \leq \epsilon \ \text{ and } \ |V_N(x, \bar{\mathbf{u}}_{\epsilon_{\rm c}}(x)) - V^*_{\epsilon_{\rm c}}(x)| \leq \epsilon.$$

Further, using (14) we can write:

$$\left\|\left[\mathbf{G}\bar{\mathbf{u}}_{\epsilon_{\rm c}}(x) + \mathbf{E}x + \mathbf{g} + \epsilon_{\rm c}\mathbf{e}\right]_+\right\| \leq \epsilon = \frac{\epsilon_{\rm c}}{2} < \epsilon_{\rm c},$$

which implies that for all $j = 1, \dots, p$ we have: $[\mathbf{G}_j \bar{\mathbf{u}}_{\epsilon_c}(x) + \mathbf{E}_j x + \mathbf{g}_j + \epsilon_c]_+ < \epsilon_c$. Since $\mathbf{G}_j \bar{\mathbf{u}}_{\epsilon_c}(x) + \mathbf{E}_j x + \mathbf{g}_j + \epsilon_c \leq [\mathbf{G}_j \bar{\mathbf{u}}_{\epsilon_c}(x) + \mathbf{E}_j x + \mathbf{g}_j + \epsilon_c]_+$ we can conclude that $\mathbf{G}\bar{\mathbf{u}}_{\epsilon_c}(x) + \mathbf{E}x + \mathbf{g} < 0$ and therefore the feasibility of $\bar{\mathbf{u}}_{\epsilon_c}(x)$ for problem ($\mathbf{QP}(x)$) is guaranteed.

Further, since $\bar{\mathbf{u}}_{\epsilon_c}(x)$ is feasible we have on the one hand that $0 \leq V_N(x, \bar{\mathbf{u}}_{\epsilon_c}(x)) - V^*(x)$. On the other hand, using (12) we can write:

$$0 \leq V_N(x, \bar{\mathbf{u}}_{\epsilon_c}(x)) - V^*(x) \leq (1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}})\epsilon, \qquad (15)$$

and thus $\bar{\mathbf{u}}_{\epsilon_c}(x)$ is a feasible approximate solution of the original problem ($\mathbf{QP}(x)$).

We are interested now in proving stability of the proposed MPC scheme. For this purpose, we introduce first the following notation for the feasible suboptimal solution $\bar{\mathbf{u}}_{\epsilon_c}(x)$:

$$\bar{\mathbf{u}}_{\epsilon_c}(x) = \left[(\bar{\mathbf{u}}^0_{\epsilon_c}(x))^T \cdots (\bar{\mathbf{u}}^{N-1}_{\epsilon_c}(x))^T\right]^T.$$

Using this notation, the next state in our MPC scheme is then given by:

$$x^+ = Ax + B\bar{\mathbf{u}}^0_{\epsilon_c}(x). \qquad (16)$$

For the tightened problem with initial state x^+, we will also use the notations ϵ_c^+ and $\mathcal{R}_{\mathrm{d}}^+$ for the tightening parameter and the upper bound given in Lemma 3.1, respectively. The following result helps us to construct a strictly feasible vector $\tilde{\mathbf{u}}^+$ for the tightened problem ($\mathbf{QP}_{\epsilon_c^+}(x^+)$).

Lemma 3.2: Let x^+ be computed according to (16) and $\bar{\mathbf{u}}_{\epsilon_c}(x)$ be an ϵ-solution of problem ($\mathbf{QP}_{\epsilon_c}(x)$). Then, a strictly feasible vector of problem ($\mathbf{QP}(x^+)$) is given by:

$$\tilde{\mathbf{u}}^+ = \left[(\bar{\mathbf{u}}^1_{\epsilon_c}(x))^T \cdots (\bar{\mathbf{u}}^{N-1}_{\epsilon_c}(x))^T (Kx(N))^T\right]^T \qquad (17)$$

Proof: First, let us note that $\bar{\mathbf{u}}_{\epsilon_c}(x) \in \mathbf{U}$, which together with (3) leads to $\tilde{\mathbf{u}}^+ \in \mathbf{U}$. Further, let us recall that the first $N-1$ block inequalities $\mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} \leq 0$ are obtained from the state constraints $x(t) \in X$ for all $t = 1, \dots, N-1$ while the last block is deduced from $x(N) \in X^{\mathrm{f}}$. It is straightforward to observe that the first $N-1$ blocks of $\mathbf{G}\tilde{\mathbf{u}}^+ + \mathbf{E}x^+ + \mathbf{g}$ coincide with the last $N-1$ blocks of $\mathbf{G}\bar{\mathbf{u}}_{\epsilon_c}(x) + \mathbf{E}x + \mathbf{g}$ and thus the first $N-1$ block inequalities are strictly satisfied. Also from the μ-contractive property of the set X^{f} (see (3)) we can also deduce that the last block inequality, obtained from $x(N) \in X^{\mathrm{f}}$ is also strictly satisfied. Thus, we can conclude that $\tilde{\mathbf{u}}^+$ is a strictly feasible vector of problem ($\mathbf{QP}(x^+)$). ∎

Therefore, in the MPC problem for the next state x^+ we update the strictly feasible vector as explained above, and thus, if ϵ_c^+ satisfies (7), according to Remark 2.3 we have that $\tilde{\mathbf{u}}^+$ is also strictly feasible for the tightened problem ($\mathbf{QP}_{\epsilon_c^+}(x^+)$). It is well-known in the MPC framework (see e.g. [12], [21]) that if Assumption 2.1 is satisfied and K and P are computed according to Section II, then the following relation holds:

$$V_N(x^+, \tilde{\mathbf{u}}^+) \leq V_N(x, \bar{\mathbf{u}}_{\epsilon_c}(x)) - \|x\|_Q^2 \quad \forall x \in X_N. \qquad (18)$$

In order to prove the asymptotic stability of the MPC scheme for all $x \in X_N$ we use similar arguments as in [12], [21] by showing that $V_N(x, \bar{\mathbf{u}}_{\epsilon_c}(x))$ is a Lyapunov function for the closed-loop system:

$$\begin{aligned}
V_N(x^+, \bar{\mathbf{u}}_{\epsilon_c^+}(x^+)) &\overset{(15)}{\leq} V^*(x^+) + (1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}}^+)\epsilon^+ \\
&\leq V^*_{\epsilon_c}(x^+) + (1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}}^+)\epsilon^+ \\
&\leq V_N(x^+, \tilde{\mathbf{u}}^+) + (1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}}^+)\epsilon^+ \\
&\overset{(18)}{\leq} V_N(x, \bar{\mathbf{u}}_{\epsilon_c}(x)) - \|x\|_Q^2 + (1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}}^+)\epsilon^+,
\end{aligned}$$

From (13) and previous discussion we have that by choosing e.g.

$$\epsilon^+ \leq \min\left\{ \frac{1}{2(1 + 4\sqrt{p}\mathcal{R}_{\mathrm{d}}^+)}\|x\|_Q^2, \; \frac{1}{4}\min_{j=1,\dots,p}\{-\left(\mathbf{G}\tilde{\mathbf{u}}^+ + \mathbf{E}x^+ + \mathbf{g}\right)_j\}\right\}, \qquad (19)$$

we get asymptotic stability of the closed-loop system.

We can conclude that choosing the accuracy ϵ and the tightening parameter ϵ_c according to (19) and (14), respectively, the proposed MPC scheme generates a sequence of inputs that ensures feasibility, suboptimality, and stability. Also, we can observe from (19) and (14) that both ϵ and ϵ_c are chosen adaptively, i.e. depending on the initial state of each step of the MPC scheme. More specifically, if for instance the norm of the initial state is big enough, i.e. the system is far from the origin, then the accuracy required for an approximate solution can be less stringent. Thus, our approach is less restrictive than the approach in [6], [19] where the accuracy is fixed for all initial states. Conversely, if we are sufficiently close to the origin, i.e., $x \in X^{\mathrm{f}}$, we do not have to apply the algorithm for finding the ϵ-solution since the optimal solution is given by $\mathbf{u}^{\mathrm{f}}(x)$.

We present further an algorithmic framework for the proposed MPC scheme:

ALGORITHM (*MPC scheme with feasibility, suboptimality, and stability guarantees*).

Input: $A, B, Q, R, X, U, X^{\mathrm{f}}, N, x$.

Step 0: Set $t = 0$ and compute offline $K, P, \mathbf{Q}, \mathbf{W}, \mathbf{w}, \mathbf{G}, \mathbf{E}, \mathbf{g}$, an initial strictly feasible vector $\tilde{\mathbf{u}}$ for ($\mathbf{QP}(x)$), accuracy ϵ according to (13) and tightening parameter ϵ_c from (14).

Repeat:

Step 1: Measure current state x.

If $x \in X^{\mathrm{f}}$:

Step 1.1: Compute $u = Kx$.

Step 1.2: Implement control input u.

Step 1.3: $t \leftarrow t + 1$, go to Step 1.

Else

Step 1.4: Compute the ϵ-solution

$$\bar{\mathbf{u}}_{\epsilon_c}(x) = \mathrm{Alg}((\mathbf{QP}_{\epsilon_c}(x)), \epsilon).$$

Step 1.5: Compute $u = \bar{\mathbf{u}}^0_{\epsilon_c}(x)$.

Step 1.6: Update the strictly feasible vector $\tilde{\mathbf{u}} \leftarrow \tilde{\mathbf{u}}^+$ according to (17), accuracy $\epsilon \leftarrow \epsilon^+$ according to (19) and the tightening parameter ϵ_c using (14).

Step 1.7: Implement control input u.

Step 1.8: $t \leftarrow t+1$, go to Step 1.

Note that in Step 0 of the proposed scheme we can compute the initial strictly feasible vector $\tilde{\mathbf{u}}$ for $(\mathbf{QP}_{\epsilon_c}(x))$ by solving the following linear program offline:

$$\max_{\gamma \geq 0, \mathbf{u} \in \mathbf{U}} \gamma$$
$$\text{s.t.:} \ \ \mathbf{G}\mathbf{u} + \mathbf{E}x + \mathbf{g} + \gamma \leq 0.$$

IV. Dual fast gradient algorithm for solving convex problems

In this section we present briefly an inexact dual fast gradient method that can be applied for finding an ϵ-solution of the optimization problem $(\mathbf{QP}(x))$ or $(\mathbf{QP}_{\epsilon_c}(x))$. A full analysis of this algorithm can be found in [12]. Since the algorithm can be applied to a wider class of problems we introduce first the following convex optimization problem:

$$F^* = \min_{\mathbf{u} \in \mathbf{U}} \left\{ F(\mathbf{u}) : \quad \mathbf{G}\mathbf{u} + \mathbf{g} \leq 0 \right\}, \tag{20}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a σ_{F}-strongly convex function, $\mathbf{U} \subseteq \mathbb{R}^n$ is a simple convex set, as assumed in Section II, $\mathbf{G} \in \mathbb{R}^{p \times n}$, and $\mathbf{g} \in \mathbb{R}^p$. We can notice that since $\mathbf{G}\mathbf{u} + \mathbf{g} \leq 0$ (also called *complicating constraints*) is a general polyhedron, the projection on this set is hard to compute, but the set $\mathbf{U}$ is simple (e.g. hyperbox, Euclidean ball, $\mathbb{R}^n$, etc.), i.e. the projection on this set can be computed very efficiently. We note that problems $(\mathbf{QP}(x))$ and $(\mathbf{QP}_{\epsilon_c}(x))$ are particular cases of problem (20) with F being a convex quadratic function, and in this case $\sigma_{\text{F}} = \lambda_{\min}(\mathbf{Q})$.

By moving the complicating constraints into the cost via Lagrange multipliers we define the dual function:

$$d(\lambda) = \min_{\mathbf{u} \in \mathbf{U}} \mathcal{L}(\mathbf{u}, \lambda), \tag{21}$$

where $\mathcal{L}(\mathbf{u}, \lambda) = F(\mathbf{u}) + \langle \lambda, \mathbf{G}\mathbf{u} + \mathbf{g} \rangle$ denotes the partial Lagrangian w.r.t. the complicating constraints $\mathbf{G}\mathbf{u} + \mathbf{g} \leq 0$. We also denote by $\mathbf{u}(\lambda)$ the optimal solution of the *inner problem*:

$$\mathbf{u}(\lambda) = \arg\min_{\mathbf{u} \in \mathbf{U}} \mathcal{L}(\mathbf{u}, \lambda). \tag{22}$$

Since F is strongly convex, it can be proven that the gradient of the dual function $d(\lambda)$ is given by:

$$\nabla d(\lambda) = \mathbf{G}\mathbf{u}(\lambda) + \mathbf{g}$$

and it is Lipschitz continuous with constant $L_{\text{d}} = \frac{\|G\|^2}{\sigma_{\text{F}}}$ (see [12] for more general settings). If we assume that strong duality holds, we have for the *outer problem*:

$$F^* = \max_{\lambda \geq 0} d(\lambda), \tag{23}$$

for which we denote an optimal solution by λ^*. Since we cannot usually solve the inner optimization problem (22) exactly, but with some inner accuracy ϵ_{in} obtaining an approximate optimal solution $\bar{\mathbf{u}}(\lambda)$, we have to use inexact gradients and approximate values of the dual function d. Thus, we introduce the following two notions:

$$\bar{d}(\lambda) = \mathcal{L}(\bar{\mathbf{u}}(\lambda), \lambda) \text{ and } \bar{\nabla} d(\lambda) = \mathbf{G}\bar{\mathbf{u}}(\lambda) + \mathbf{g}.$$

If we assume that $\bar{\mathbf{u}}(\lambda)$ is computed such that the following inner ϵ_{in}-optimality holds:

$$\bar{\mathbf{u}}(\lambda) \in \mathbf{U}, \ \ \mathcal{L}(\bar{\mathbf{u}}(\lambda), \lambda) - \mathcal{L}(\mathbf{u}(\lambda), \lambda) \leq \frac{\epsilon_{\text{in}}}{2}, \tag{24}$$

then the next lemma provides bounds for function $d(\lambda)$ in terms of a linear and a quadratic model that use approximate information of the dual function and of its gradient.

Lemma 4.1: [4], [12] Let F be strongly convex and for a given λ let $\bar{\mathbf{u}}(\lambda)$ be computed such that (24) is satisfied. Then, the following inequalities are valid:

$$\begin{aligned} 0 \geq & d(\mu) - [\bar{d}(\lambda) + \langle \bar{\nabla} d(\lambda), \mu - \lambda \rangle] \\ & \geq -L_{\text{d}} \|\mu - \lambda\|^2 - \epsilon_{\text{in}} \quad \forall \mu \in \mathbb{R}^p_+. \end{aligned} \tag{25}$$

We give the following inexact dual fast gradient scheme:

Algorithm (IDFG)(λ^0)

Given $\lambda^0 \in \mathbb{R}^p_+$, for $k \geq 0$ compute:

1) $\bar{\mathbf{u}}^k \approx \arg\min_{\mathbf{u} \in \mathbf{U}} \mathcal{L}(\mathbf{u}, \lambda^k)$ such that (24) holds
2) $\hat{\lambda}^k = \left[\lambda^k + \frac{1}{2L_{\text{d}}} \bar{\nabla} d(\lambda^k) \right]_+$
3) $\lambda^{k+1} = \frac{k+1}{k+3} \hat{\lambda}^k + \frac{2}{k+3} \left[\lambda^0 + \frac{1}{2L_{\text{d}}} \sum_{s=0}^{k} \frac{s+1}{2} \bar{\nabla} d(\lambda^s) \right]_+$,

where we recall that $\bar{\nabla} d(\lambda^k) = \mathbf{G}\bar{\mathbf{u}}^k + \mathbf{g}$. If we define now the following average sequence for the primal variables:

$$\hat{\mathbf{u}}^k = \sum_{s=0}^{k} \frac{2(s+1)}{(k+1)(k+2)} \bar{\mathbf{u}}^s, \tag{26}$$

the next result gives an estimate on primal feasibility violation and primal suboptimality of the proposed algorithm.

Theorem 4.2: [12, Section II.C] Let F be strongly convex, the sequences $\left(\bar{\mathbf{u}}^k, \hat{\lambda}^k, \lambda^k\right)_{k \geq 0}$ be generated by algorithm **(IDFG)** and $\hat{\mathbf{u}}^k$ be given by (26). Then, an estimate on primal feasibility violation for the original problem (20) is given by:

$$\|[\mathbf{G}\hat{\mathbf{u}}^k + \mathbf{g}]_+\| \leq v(k, \epsilon_{\text{in}}),$$

where $v(k, \epsilon_{\text{in}}) = \frac{16 L_{\text{d}} \|\lambda^*\|}{(k+1)^2} + \frac{8 L_{\text{d}} \|\lambda^0\|}{(k+1)^2} + 4\sqrt{L_{\text{d}} \frac{\epsilon_{\text{in}}}{k+1}}$. Moreover, an estimate on primal suboptimality is given by:

$$-\left(\|\lambda^*\| + \|\lambda^0\|\right) v(k, \epsilon_{\text{in}}) \leq F(\hat{\mathbf{u}}^k) - F^* \leq \frac{4 L_{\text{d}} \|\lambda^0\|^2}{(k+1)^2} + (k+1)\epsilon_{\text{in}}.$$

Proof: The proof can be found in [12]. ■

Since usually λ^* is unknown, we can use an upper bound $\mathcal{R}_{\text{d}}$ for $\|\lambda^*\|$, e.g. computed according to Lemma 3.1. For simplicity, we assume now that $\lambda^0 = 0$ and $\epsilon_{\text{in}} = 0$ (for the general case $\epsilon_{\text{in}} \neq 0$ see [12]). Then, after $k = \left\lfloor 4\sqrt{\frac{L_{\text{d}} \mathcal{R}_{\text{d}}}{\epsilon}} \right\rfloor$ iterations, we obtain the following estimates on feasibility violation and primal suboptimality:

$$\begin{aligned} & \|[\mathbf{G}\hat{\mathbf{u}}^k + \mathbf{g}]_+\| \leq \epsilon \\ & -\mathcal{R}_{\text{d}} \epsilon \leq F(\hat{\mathbf{u}}^k) - F^* \leq 0. \end{aligned}$$

Thus, if we redefine $\epsilon = \max\{\epsilon, \mathcal{R}_{\text{d}} \epsilon\}$ we can conclude that $\hat{\mathbf{u}}^k$ is an ϵ-solution for problem (20) satisfying (6). We can conclude that Algorithm **(IDFG)** can be used for finding an ϵ-solution of problem $(\mathbf{QP}_{\epsilon_c}(x))$.

V. Numerical simulations

In order to demonstrate the applicability of our proposed algorithm we implemented the MPC scheme on a ball on plate system [18]. We consider box constraints on the states and the inputs as in [18], while for the stage costs we use the matrices $Q = q_1 q_1^T$, where $q_1 = [2\ 1]^T$, $R = 1$, and we compute the terminal matrix P as the solution of the LQR problem.

In Figure 1, we plot the evolution of the states and inputs over a simulation horizon $N_{\text{sim}} = 400$ using a prediction horizon $N = 5$. For each initial state x of the MPC scheme we consider two scenarios. In the first one (continuous line) we apply our Algorithm **(IDFG)** for finding an ϵ-solution of problem ($\mathbf{QP}_{\epsilon_c}(x)$) with ϵ equal to the right-hand side term of (13) for the first step of the MPC scheme and equal to the right-hand side term of (19) for the rest of the steps. We also considered $\epsilon_c = 2\epsilon$. In the second scenario (dashed line) we solve exactly the optimization problem ($\mathbf{QP}(x)$). From Figure 1 we can observe that the states and inputs have very similar trajectories in both scenarios and the system is driven to the equilibrium point.

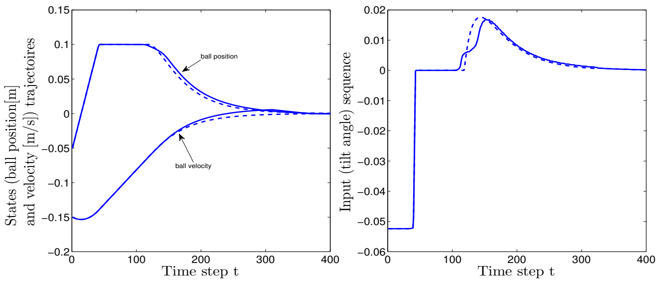

Fig. 1. The trajectories of the states and inputs for a simulation horizon $N_{\text{sim}} = 400$ and prediction horizon $N = 5$: Algorithm **(IDFG)** - continuous line, exact solver - dashed line.

VI. Conclusions

In this paper we have proposed a model predictive control scheme for discrete-time linear time-invariant systems based on the general framework of inexact numerical optimization algorithms. In the present paper we have developed our main results on stability and feasibility of a suboptimal MPC scheme using [12], and computed suboptimal control inputs using inexact dual gradient algorithms. We have derived a control strategy that has the following properties: the constraints on the states and inputs are satisfied, asymptotic stability of the closed-loop system is guaranteed and the number of iterations needed for a certain level of suboptimality can be determined.

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