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# Robust $H_\infty$ control for switched nonlinear systems with application to high-level urban traffic control

Mohammad Hajiahmadi, Bart De Schutter and Hans Hellendoorn

**Abstract**—This paper presents robust switching control strategies for switched nonlinear systems with constraints on the control inputs. First, a model transformation is proposed in a way that the constraint on the continuous control inputs is relaxed. Next, the effect of disturbance is taken into account and the  $L_2$ -gain analysis and  $H_\infty$  control design problem for switched nonlinear systems are formulated and proved. Furthermore, the obtained control laws are utilized for urban traffic networks modeled on a high-level using macroscopic fundamental diagram representation. The flow transferred between urban regions along with the timing plans for each region are controlled using continuous and switching controllers. The control objective is translated into a stability and disturbance attenuation problem for the urban network represented as a switched nonlinear system. The uncertain trip demands are considered as norm-bounded disturbance inputs. One major advantage of the proposed scheme is that the parameters of the feedback switching law are obtained offline. Hence, real-time control is possible with this scheme. The achieved results show great performance of the proposed approach in handling uncertain demand profiles.

## I. INTRODUCTION

Switched systems are a class of hybrid systems that consist of a set of subsystems and a switching signal selecting the active subsystems. Switched systems arise in cases in which several dynamical system models are required to model a system due to e.g. uncertainty in parameters, or specific applications that utilize switching between a set of controllers in order to achieve a higher performance. Stability analysis of switched systems has attracted attention recently [1], [2]. Stabilization and control synthesis for switched linear systems have been widely studied using common and/or multiple Lyapunov function methods and for time and/or state dependent switching [3], [4], while stability of switched nonlinear systems have been investigated for particular classes of systems [5]–[7].

Moreover, the disturbance attenuation problem for switched systems has attracted attention of researchers only in recent years. For the particular cases of switched nonlinear systems, the  $H_\infty$  control problem is proposed based on the Hamilton-Jacobi inequalities for nonlinear systems [8]–[10]. As an example, in [10] a nonlinear switched system is considered that is affine both in control input and disturbance input. The model contains a set of nonlinear subsystems each controlled with an unconstrained continuous control input. In this paper, we study the stabilization problem for switched nonlinear systems that

are affine in the control and disturbance inputs. The aim is to extend the current results of stabilization and  $H_\infty$  control for the constrained control case.

The motivation for this research is based on a practical hybrid model developed for large-scale urban traffic network control [11]. In this model two types of controllers are defined; perimeter control for limiting the flow of vehicles traveling between urban regions and discrete control for switching between the signal timing plans of urban areas. The model is developed based on the existence of macroscopic fundamental diagrams (MFD) for the urban areas [12]. The MFD provides a relationship between the accumulation of vehicles and the network trip completion rate. In fact, the MFD representation makes it possible to efficiently model a large-scale urban network at an aggregate high-level and to subsequently develop control strategies that are less computationally complex.

More specifically, we aim at designing a new control scheme for urban networks represented by the hybrid model developed in [11] but without having exact knowledge about the traffic demands and at the same time with less computational effort. Basically, we consider the model in [11] as a switched nonlinear system. The main objective is to reduce the total delay. In this paper, minimization of the total delay in the network using perimeter and switching timing plans controllers is treated as an stabilization and disturbance attenuation problem. Since there are constraints on the perimeter control inputs, we propose a model transformation to relax them. The trip demands in the network are considered as exogenous disturbance signals. The main requirement of the proposed approach is that the disturbance is norm bounded and belongs to the class of square integrable functions. This assumption is valid for finite time intervals (e.g. the peak hours) in which the trip demands inside the urban network are bounded and have a finite average. Another major advantage of the proposed control scheme is that the design procedure is implemented off-line and the feedback control laws do not require on-line computations like in [11].

The paper is organized as follows. The problem formulation along with a model transformation is presented in Section II. The stability problem in the absence of disturbance is discussed in Section III. In Section IV the disturbance is taken into account and the  $L_2$ -gain is defined for the switched nonlinear model. Further,  $H_\infty$  control via switching between modes is proposed. In the case study section, the aggregate hybrid traffic model is presented. Next, the  $H_\infty$  controller is designed for a two-region urban city case, and performance of the proposed method is evaluated. Finally, Section VI contains the concluding remarks and ideas for further research.

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## II. PROBLEM STATEMENT

Consider the following switched nonlinear system:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t)) \cdot u(t) + p_{\sigma(t)}(x(t)) \cdot \omega(t), \quad (1)$$

where  $x(t) \in \mathbb{R}^{n_x}$  is the state,  $u(t) \in \mathbb{R}^{n_u}$  is the control input, and  $\omega(t) \in \mathbb{R}^{n_\omega}$  is the disturbance input. The switching signal is denoted by  $\sigma(t)$  and is assumed to be piecewise constant. The variable  $\sigma(t)$  takes values from a pre-defined index set  $\{1, \dots, N\}$ , and for each value that  $\sigma(t)$  assumes, the state space model (1) is governed by a different set of vector functions  $f_i(\cdot)$ ,  $g_i(\cdot)$ , and  $p_i(\cdot)$  from the following sets:

$$f_{\sigma(t)} \in \{f_1, \dots, f_N\} \quad (2)$$

$$g_{\sigma(t)} \in \{g_1, \dots, g_N\} \quad (3)$$

$$p_{\sigma(t)} \in \{p_1, \dots, p_N\} \quad (4)$$

The vector functions  $f_i$ ,  $g_i$ , and  $p_i$  are continuous functions of states such that  $f_i(0) = 0$ ,  $g_i(0) = 0$ , and  $p_i(0) = 0$ . Moreover, the control input  $u$  is constrained as follows:

$$u(t) \in [0, 1]^{n_u} \quad (5)$$

The aim is to design a state feedback control together with a switching rule in order to stabilize the system and to reduce the effects of disturbances. However, the constraint (5) on the control input limits the design freedom. Therefore, in the following we reformulate the model in order to simplify the stabilization problem.

As mentioned before the control input  $u(t)$  is constrained in  $[0, 1]^{n_u}$ . However, for specific applications the sensitivity to small variations of the control input is relatively low and therefore a finite set of values is enough for controlling the system (For instance, in the urban traffic control, the control input is the ratio of green and red phases in traffic lights. Thus, only a few values in the interval  $[0, 1]$  could be selected and implemented in reality). To be more precise, we can assume that  $u(t)$  is quantized and hence it can be rewritten as:

$$u(t) = u_0 \cdot \left( \sum_{l=0}^r 2^l \cdot \delta_l(t) \right), \quad (6)$$

with  $u_0 \in \mathbb{R}$  a constant and  $\delta_l(t) \in \{0, 1\}^{n_u}$ . The set of possible input values is then finite and its cardinality is  $2^{r+1}$ , while the difference between two consecutive values is determined by  $u_0$ .

Using (6), the system in (1) can be reformulated as:

$$\dot{x}(t) = f'_{\sigma'(t)}(x(t)) + p_{\sigma'(t)}(x(t)) \omega(t), \quad x(0) = x_0, \quad (7)$$

where  $f'_{\sigma'(t)} \in \{f'_1, \dots, f'_{N'}\}$ . By quantizing the control input  $u$  as in (6) new modes are introduced and therefore we denote the total number of modes by  $N'$  with a new set of vector functions  $\{f'_1, \dots, f'_{N'}\}$  that are determined using the functions  $f_i$  and  $g_i$  and the values that the quantized input  $u$  can take.

The current formulation helps to have a concise design procedure as we reflect the effects of the continuous control input  $u$  in the switching signal  $\sigma'$  and hence, we have to deal only with one type of control input (switching).

## III. STABILIZATION IN THE ABSENCE OF DISTURBANCE

In this section, the stability problem is formulated for the system (7) in the absence of disturbances. The resulting model is

$$\dot{x}(t) = f'_{\sigma'(t)}(x(t)), \quad x(0) = x_0, \quad (8)$$

with  $f'_{\sigma'(t)} \in \{f'_1, \dots, f'_{N'}\}$  and  $N'$  the total number of modes including the ones introduced by the quantization of  $u$ .

It is assumed that the state vector  $x(t)$  is available for feedback for all  $t \geq 0$ , and the aim is to determine a piecewise constant function  $r(\cdot) : \mathbb{R}^{n_x} \rightarrow \{1, \dots, N'\}$ , such that the switching law

$$\sigma'(t) = r(x(t)) \quad (9)$$

guarantees that the equilibrium  $x = 0$  is globally asymptotically stable for (8). It should be noted that we do not assume that any of the vector fields in the set  $\{f'_1, \dots, f'_{N'}\}$  is either locally or globally asymptotically stable. The candidate Lyapunov function  $\vartheta(\cdot)$  is constructed as follows:

$$\vartheta(x) := \min_{i=1, \dots, N'} V_i(x), \quad (10)$$

where  $V_1, \dots, V_{N'}$  are differentiable, positive definite, and radially unbounded functions of  $x$ . However, this function might not be differentiable everywhere even if the functions  $V_i$  are all differentiable. To overcome this issue, the notion of Metzler matrices is used [13], [14]. A Metzler matrix is a matrix in which all the off-diagonal components are non-negative. For our goal, we limit the attention to the class of Metzler matrices denoted by  $\mathcal{M}$  and containing all matrices  $M \in \mathbb{R}^{N' \times N'}$  with elements  $\mu_{ij}$ , such that:

$$\mu_{ij} \geq 0 \quad \forall i \neq j, \quad \sum_{i=1}^{N'} \mu_{ij} = 0, \quad \forall j \quad (11)$$

The following theorem provides the design procedure for the stabilizing switching rule [5]:

*Theorem 1:* Assume there exist functions  $V_1, \dots, V_{N'}$ , which are all differentiable, positive definite, radially unbounded, and zero at zero. Furthermore, assume there exists matrix  $M \in \mathcal{M}$  with elements  $\mu_{ij}$  that satisfies the Lyapunov-Metzler inequalities

$$\frac{\partial V_i(x)}{\partial x} f'_i(x) + \sum_{j=1}^{N'} \mu_{ji} V_j(x) < 0, \quad i \in \{1, \dots, N'\}, \quad (12)$$

for all  $x \neq 0$ . Then the switching rule (9) with

$$r(x(t)) = \arg \min_{i=1, \dots, N'} V_i(x(t)), \quad (13)$$

makes the equilibrium point  $x = 0$  of (8) globally asymptotically stable [5].

*Proof:* The Lyapunov function (10) is piecewise differentiable, which means that it is not differentiable for all  $x \in \mathbb{R}^{n_x}$ . Therefore, we need to define the following derivative (see [5], [15]):

$$\mathbf{D}(\vartheta(x(t))) = \limsup_{\Delta t \rightarrow 0^+} \frac{\vartheta(x(t + \Delta t)) - \vartheta(x(t))}{\Delta t} \quad (14)$$

Assume that at an arbitrary  $t \geq 0$ , the state switching control is given by  $\sigma(t) = r(x(t)) = i$  for some  $i \in I(x(t)) = \{i : \vartheta(x) = V_i(x)\}$ . Hence, from (14) and (8), we have (using Theorem 1 in [16]):

$$\mathbf{D}(\vartheta(x(t))) = \min_{i \in I(x(t))} \frac{\partial V_i}{\partial x} f'_i \leq \frac{\partial V_i}{\partial x} f'_i, \quad (15)$$

Since (12) is valid for  $M \in \mathcal{M}$  and  $V_j \geq V_i$  for all  $j \in \{1, \dots, N'\} \setminus \{i\}$ , using the fact that  $i \in I(x(t))$  and by rewriting the Lyapunov-Metzler inequality (12) as follows:

$$\frac{\partial V_i}{\partial x} f'_i < - \sum_{j=1}^{N'} \mu_{ji} V_j, \quad i \in \{1, \dots, N'\}, \quad (16)$$

one can obtain:

$$\begin{aligned} \mathbf{D}(\vartheta(x(t))) &\leq \frac{\partial V_i}{\partial x} f'_i < - \sum_{j=1}^{N'} \mu_{ji} V_j \\ &\leq - \left( \sum_{j=1}^{N'} \mu_{ji} \right) V_i = 0, \quad \text{for all } x \neq 0 \end{aligned} \quad (17)$$

Thus, the switching law (13) makes the equilibrium point  $x = 0$  of the switched nonlinear system (8) globally asymptotically stable. ■

In order to design the switching law (13), one would need to find appropriate positive definite functions  $V_i$  and a Metzler matrix that satisfy the Lyapunov-Metzler inequalities (12) for all  $x \neq 0$ . Unfortunately, this is a tedious task in general since it includes determination of positive definite functions. Fortunately, the choice of quadratic functions works for many cases (e.g. for our case study). Nevertheless, finding the coefficients of the quadratic functions  $V_i$  along with the elements of the Metzler matrix constitutes a nonlinear feasibility optimization problem. In some cases, we can recast this problem as a Bilinear Matrix Inequalities (BMI) problem [17] and thus, take advantage of the existing solvers for BMIs. But the general case would be a multi-parametric optimization problem. Nonetheless, one can use the following alternative approach. By gridding the domain of the state  $x$ , one can formulate the Lyapunov-Metzler inequalities for each vertex of the grid. Depending on the characteristics of the system under study and the objectives, one can make grids with different levels of accuracy in a uniform or non-uniform way. Next, the remaining task is to find solutions for the parameters of  $V_i$  and the Metzler matrix in order to satisfy all Lyapunov-Metzler inequalities for all grid points. This is a nonlinear optimization problem in which the feasibility of all nonlinear inequality constraints has to be checked. Of course, there might exist multiple solutions for this problem but any feasible solution would work for finding the stabilizing switching law.

#### IV. DISTURBANCE ATTENUATION VIA STATE SWITCHING CONTROL

In this section, we present an approach to tackle the disturbance attenuation problem mentioned in Section II. The model of the system under control is as follows:

$$\dot{x}(t) = f'_{\sigma'(t)}(x(t)) + p_{\sigma'(t)}(x(t))\omega(t), \quad x(0) = x_0, \quad (18)$$

$$y(t) = h_{\sigma'(t)}(x(t)), \quad (19)$$

with  $y(t) \in \mathbb{R}^{n_y}$  the output vector and  $h_{i'}(x)$ ,  $i' \in \{1, \dots, N'\}$ , continuous vector functions with  $h_{i'}(0) = 0$ . Moreover, we assume that the disturbance vector  $\omega$  belongs to the space of square integrable functions on  $[0, T]$ ,  $\forall T \geq 0$ , as follows:

$$\|\omega\|_{L_2[0,T]} = \left( \int_0^T \omega^T(t)\omega(t)dt \right)^{1/2} < \infty, \quad (20)$$

##### A. $L_2$ -gain

The system (18) has an  $L_2$ -gain  $\gamma > 0$  under some switching law  $\sigma'$  if  $\|y\|_{L_2[0,T]} \leq \gamma \|\omega\|_{L_2[0,T]}$  for all nonzero  $\omega \in L_2[0, T]$  ( $0 \leq T < \infty$ ) and for initial condition  $x(0) = 0$ . It follows that:

$$\begin{aligned} \|y\|_{L_2[0,T]} \leq \gamma \|\omega\|_{L_2[0,T]} &\Rightarrow \int_0^T (\|y(t)\|^2 - \gamma^2 \|\omega(t)\|^2) dt \leq 0 \\ &\Rightarrow \int_0^T (\|h_{\sigma'(t)}(x(t))\|^2 - \gamma^2 \|\omega(t)\|^2) dt \leq 0, \end{aligned} \quad (21)$$

for any  $T > 0$  when  $x(0) = 0$ . The aim is to design a switching strategy  $\sigma'$  such that system (18) has  $L_2$ -gain  $\gamma$  or equivalently, to have an  $H_\infty$  disturbance attenuation level  $\gamma$ .

##### B. Achieving $L_2$ -gain via $H_\infty$ Control

The approach for  $H_\infty$  control of switched nonlinear systems proposed in [10] is not applicable for control of (1), as the input  $u$  is constrained in the box  $[0, 1]^{n_u}$ . Nevertheless, we transformed the model using quantization of the input variable and obtained the model in (7). For this model, the following problem is defined. Assume that a constant  $\gamma > 0$  is given, the goal is to design a switching law  $\sigma'$ , such that the origin of the closed-loop system is globally asymptotically stable when  $\omega(t) = 0, \forall t \geq 0$ , and the overall  $L_2$ -gain from  $\omega$  to  $y = h_{\sigma'(t)}(x)$  on any finite time interval  $[0, T]$  is less than or equal to  $\gamma$ , i.e. :

$$\int_0^T (\gamma^2 \|\omega(t)\|^2 - \|h_{\sigma'(t)}(x(t))\|^2) dt \geq 0 \quad (22)$$

The following theorem provides the design procedure for the switching law (inspired by the linear case in [18]).

*Theorem 2:* Consider the switched system (7). Assume that there exist positive definite, differentiable, and radially unbounded functions  $V_i$ ,  $i \in \{1, \dots, N'\}$  and a Metzler matrix  $M$  with elements  $\mu_{ij}$ , such that the following Lyapunov-Metzler inequalities are satisfied:

$$\frac{\partial V_i}{\partial x} f'_i + \frac{1}{2\gamma^2} \frac{\partial V_i}{\partial x} p_i p_i^T \frac{\partial V_i}{\partial x} + \frac{1}{2} h_i^T h_i + \sum_{j=1}^{N'} \mu_{ji} V_j < 0, \quad (23)$$

for  $i = 1, \dots, N'$ . Then, the system (18) under the switching law

$$\sigma'(t) = r(x(t)) = \arg \min_{i=1, \dots, N'} V_i(x(t)), \quad (24)$$

has  $L_2$ -gain  $\gamma$ . Subsequently, in case  $\omega \equiv 0$ , the system is asymptotically stable.

Before proceeding with the proof, we emphasize again that the switching signal is assumed to be piecewise constant. In other words, one can define a switching sequence as  $\{(t_k, r(x(t_k)))\}_{k=1}^\infty$  with  $r(x(t_k)) \in \{1, \dots, N'\}$ , while the switching rule remains unchanged in the interval  $[t_k, t_{k+1})$ .

*Proof:* Assume that the switching sequence in the interval  $[0, T]$  is defined as:

$$\left\{ (t_k, r(x(t_k))) \mid r(x(t_k)) \in \{1, \dots, N'\}, k = 1, 2, \dots, l \right\} \quad (25)$$

with  $t_1 = 0$  and  $t_l \leq T$ . Under the switching law (13) in each time interval  $[t_k, t_{k+1})$  we have:

$$\begin{aligned} \frac{\partial V_i}{\partial x} f'_i + \frac{1}{2\gamma^2} \frac{\partial V_i}{\partial x} p_i p_i^T \frac{\partial^T V_i}{\partial x} + \frac{1}{2} h_i^T h_i < - \sum_{j=1}^{N'} \mu_{ji} V_j \\ \leq \left( - \sum_{j=1}^{N'} \mu_{ji} \right) V_i = 0 \end{aligned} \quad (26)$$

Now following a similar procedure as in [8], [18], we define

$$J = \int_0^T \left( \frac{1}{2} \|h_{\sigma'(t)}(x(t))\|^2 - \frac{\gamma^2}{2} \|\omega(t)\|^2 + \mathbf{D}(\vartheta(x(t))) \right) dt \quad (27)$$

According to the definition of  $\mathbf{D}(\vartheta(x))$  in (15) and taking into account the switching sequence (25), we obtain

$$\begin{aligned} J \leq & \sum_{k=1}^{l-1} \int_{t_k}^{t_{k+1}} \left( \frac{1}{2} \|h_{r(x(t_k))}(x)\|^2 - \frac{\gamma^2}{2} \|\omega\|^2 + \dot{V}_{r(x(t_k))}(x) \right) dt \\ & + \int_{t_l}^T \left( \frac{1}{2} \|h_{r(x(t_l))}(x)\|^2 - \frac{\gamma^2}{2} \|\omega\|^2 + \dot{V}_{r(x(t_l))}(x) \right) dt \end{aligned} \quad (28)$$

The derivative  $\dot{V}_{r(x(t_k))}$  is:

$$\begin{aligned} \dot{V}_{r(x(t_k))}(x(t)) & = \frac{\partial V_{r(x(t_k))}(x(t))}{\partial x} \cdot (f'_{r(x(t_k))}(x(t)) + p_{r(x(t_k))}(x(t)) \cdot \omega(t)) \end{aligned} \quad (29)$$

Substitution of (29) in (28) along with adding and subtracting the term  $\frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_k))}}{\partial x} p_{r(x(t_k))} p_{r(x(t_k))}^T \frac{\partial^T V_{r(x(t_k))}}{\partial x}$  and completing the squares yield (the arguments of the functions are dropped for reducing the complexity):

$$\begin{aligned} & \sum_{k=1}^{l-1} \int_{t_k}^{t_{k+1}} \left( \frac{\partial V_{r(x(t_k))}}{\partial x} f'_{r(x(t_k))} + \frac{1}{2} \|h_{r(x(t_k))}\|^2 \right. \\ & \quad \left. + \frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_k))}}{\partial x} p_{r(x(t_k))} p_{r(x(t_k))}^T \frac{\partial^T V_{r(x(t_k))}}{\partial x} \right. \\ & \quad \left. - \left\| \frac{\gamma}{\sqrt{2}} \omega - \frac{1}{\sqrt{2}\gamma} \frac{\partial V_{r(x(t_k))}}{\partial x} p_{r(x(t_k))} \right\|^2 \right) dt \\ & + \int_{t_l}^T \left( \frac{\partial V_{r(x(t_l))}}{\partial x} f'_{r(x(t_l))} + \frac{1}{2} \|h_{r(x(t_l))}\|^2 \right. \\ & \quad \left. + \frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_l))}}{\partial x} p_{r(x(t_l))} p_{r(x(t_l))}^T \frac{\partial^T V_{r(x(t_l))}}{\partial x} \right. \\ & \quad \left. - \left\| \frac{\gamma}{\sqrt{2}} \omega - \frac{1}{\sqrt{2}\gamma} \frac{\partial V_{r(x(t_l))}}{\partial x} p_{r(x(t_l))} \right\|^2 \right) dt \end{aligned} \quad (30)$$

Referring to (26), we can conclude that (30) is smaller or equal to zero. Hence,

$$J = \int_0^T \left( \frac{1}{2} \|h_{\sigma'(t)}\|^2 - \frac{\gamma^2}{2} \|\omega\|^2 + \mathbf{D}(\vartheta) \right) dt \leq 0 \quad (31)$$

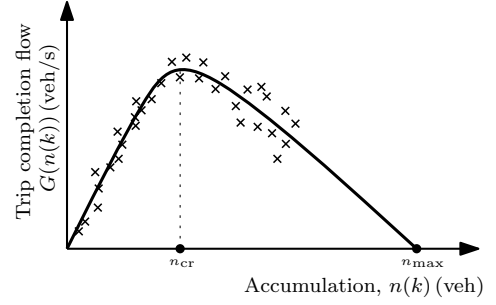


Fig. 1. A well-defined macroscopic fundamental diagram.

Note that  $V_i$  are positive definite functions with zero value at zero. Thus,

$$\int_0^T \left( \|h_{\sigma'(t)}\|^2 - \gamma^2 \|\omega\|^2 \right) dt \leq -2V_i(x(T)) \leq 0, \quad \forall i \quad (32)$$

Hence, the system has  $L_2$ -gain  $\gamma$ . Moreover, it is easy to show (by utilizing Lemma 3.2.6 in [8]) that the system is asymptotically stable when  $\omega \equiv 0$ . ■

Similar to the procedure explained in Section III, a feasibility problem has to be solved in order to find the parameters of the functions  $V_i$  along with  $\mu_{ij}$ . Moreover, the  $L_2$ -gain  $\gamma$  can be set either as an unknown parameter to be determined or a given constant. Basically, one can set a preliminary value for  $\gamma$  and solve the feasibility problem for the given  $\gamma$ . The procedure can be repeated with decreasing the value of  $\gamma$  until the problem becomes infeasible and no solution can be obtained for the parameters. By doing this a lower bound for the  $L_2$ -gain can be achieved.

In the next section, the obtained control design rules are implemented and evaluated for an urban network case study. As mentioned before, the network is represented by a high-level switched nonlinear model with perimeter control and switching between timing plans as control inputs.

## V. CASE STUDY

For urban network regions with homogeneously distributed congestion, the macroscopic fundamental diagram (MFD) (as depicted in Fig. 1) provides a unimodal, low-scatter relationship between network vehicle accumulation and network space-mean flow [19]. For an urban network divided into two regions; region 1, the periphery and region 2, the city center (as in Fig. 2), the hybrid MFD-based model is formulated as follows (based on the two-state model presented in [20]):

$$\dot{n}_1(t) = -G_{1,j}(n_1(t)) \cdot u(t) + \omega_{12}(t), \quad (33)$$

$$\dot{n}_2(t) = -G_{2,j}(n_2(t)) + G_{1,j}(n_1(t)) \cdot u(t) + \omega_{22}(t), \quad (34)$$

where  $n_i(t)$ ,  $i = 1, 2$ , is the accumulation in region  $i$  at time  $t$ . The trip completion flow  $G_{i,j}(n_i(t))$  (veh/s) is defined as the rate of vehicles reaching their destinations [21]. The timing plans for intersections inside each region can be altered. Consequently, instead of one MFD, a set of MFDs (each corresponds to a different timing plan) is defined. Therefore,  $G_{i,j}(n_i(t))$ , with  $j = 1, \dots, N_i$ , constitute the MFDs for region  $i$ .

The perimeter control  $u(t)$  may restrict vehicles to transfer between regions (in our case, the flow of vehicles is restricted

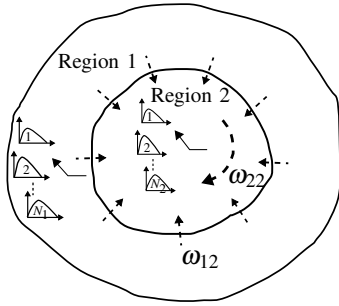


Fig. 2. Schematic two-region urban network.

from region 1, the periphery, to region 2, the city center). The perimeter control can be realized by e.g. coordinating green and red durations of signalized intersections placed on the border between two regions. We assume that the city center has two timing plans and therefore two MFDs ( $N_1 = 2$ ). Each MFD is modeled by a 3rd-order polynomial  $G_{2,j}(n_2) = 1/3600 \cdot (a_{2,j}n_2^3 + b_{2,j}n_2^2 + c_{2,j}n_2)$  with coefficients  $a_{2,1} = 1.4877 \cdot 10^{-7}$ ,  $b_{2,1} = -2.98 \cdot 10^{-3}$ ,  $c_{2,1} = 15.091$ ,  $a_{2,2} = 2.57 \cdot 10^{-7}$ ,  $b_{2,2} = -4.47 \cdot 10^{-3}$ ,  $c_{2,2} = 18.98$ . For the periphery, we assume that there exists only one timing plan and thus one MFD ( $N_2 = 1$ ). The MFD of periphery is denoted by  $G_1 = G_{1,1}$  and has  $a_{1,1} = a_{2,1}$ ,  $b_{1,1} = b_{2,1}$ ,  $c_{1,1} = c_{2,1}$  as its parameters.

As discussed before, the perimeter control input is limited in  $[0, 1]$  and therefore we use the quantization technique presented in Section II in order to achieve a complete switching system as follows:

$$\dot{n}_1(t) = -G'_{1,j'}(n_1(t)) + \omega_{12}(t), \quad (35)$$

$$\dot{n}_2(t) = -G'_{2,j'}(n_2(t)) + G'_{1,j'}(n_1(t)) + \omega_{22}(t), \quad (36)$$

where the perimeter control input can take values from the set  $\{0.1, 0.4, 0.7, 1\}$ . The number of modes is  $2 \cdot 4 = 8$  and therefore  $j' \in \{1, \dots, 8\}$ .

Here, we assume that the scenario simulates a morning peak in which a high trip demand  $\omega_{12}$  from the periphery (region 1) to the city center (region 2) exists while there is also a demand  $\omega_{22}$  for trips inside the center. To take into account the uncertainty around the demands, we add a zero mean white Gaussian noise with variance 0.1 (veh/s) to the base profiles as shown in Figure. 3 (a)-(b).

In order to determine the switching law  $\sigma$ , we use quadratic functions  $V_i(n_i) = 1/2(\alpha_i n_i^2 + \beta_i n_i^2)$ . Thus the switching rule is defined as:

$$\sigma(t) = r(n_i(t)) = \arg \min_{i \in \{1, \dots, 8\}} 1/2(\alpha_i n_1^2 + \beta_i n_2^2) \quad (37)$$

The parameters  $\alpha_i$  and  $\beta_i$  along with a feasible attenuation level  $\gamma$  are determined using (23) and the gridding technique described in Section III (the nonlinear feasibility problem is solved using the *fmincon* function inside the *Tomlab* toolbox of MATLAB). The obtained parameters are as follows:

$$(\alpha_i, \beta_i) \in \{(3.8014, 2.9193), (6.5982, 4.3430), (9.9993, 5.7571), (5.4335, 6.2613), (7.2388, 3.2234), (4.5741, 0.2113), (8.4626, 0.2899), (4.8048, 1.0877)\}$$

with  $\gamma = 0.8 \cdot 3600$ . The initial accumulations are  $n_1(0) = 6200$  (veh),  $n_2(0) = 5200$  (veh). The states are measured and

plugged into the switching law (37) in order to find the active subsystem (corresponding to a specific MFD and perimeter value). The closed-looped system is simulated for one hour and results are depicted in Fig. 3. In order to show the effectiveness of the proposed control strategy, results of the some simple control strategies are presented in Fig. 4. It can be observed that the switching  $H_\infty$  control is able to stabilize the system and also significantly reduce the effects of the trip demands (disturbances), while in almost all the simple control strategies either one or both regions end up in the gridlock situation (as the states grow unboundedly in the figures). Only in one case, when timing plan 2 is chosen for the center, the accumulations eventually decrease by the end of simulation time (Fig. 4-(f)).

It should be noted that the proposed control strategy is computationally efficient and can be implemented in real-time since the switching law (37) is computed in a very short time (with 16 multiplication, 8 addition and a minimum operation). This is a great advantage over other existing approaches like MPC which usually require online optimization. Furthermore, the  $L_2$ -gain of the controlled system is determined by setting the initial conditions to zero and by using (20) (the output of the system is defined as  $y = (n_1 \ n_2)^T$ ). The achieved gain  $\|y\|_{L_2}/\|\omega\|_{L_2}$  for the assumed demand profile is  $0.1691 \cdot 3600$ .

## VI. CONCLUSIONS AND FUTURE WORK

Stabilization and  $H_\infty$  control of switched nonlinear systems with constrained input was presented in this paper. The  $L_2$ -gain analysis for the switched nonlinear systems was formulated and the  $H_\infty$  control design procedure was presented in order to achieve a desirable level of disturbance attenuation. Furthermore, a model transformation was proposed in order to overcome the constraint on the control inputs. The proposed switching control schemes were theoretically proved. Furthermore, the results were utilized for high-level control of urban networks modeled by hybrid MFD representation and the obtained results showed significant performance of our approach in case of having uncertain demand profiles. Moreover, as mentioned before the Lyapunov functions required for the feedback switching law are determined off-line and thus the proposed method has a major advantage over the existing MPC schemes both for the real-time implementation and for handling the uncertain demand profiles.

Nevertheless, the current approach is based on the determination of positive definite functions satisfying nonlinear inequality constraints. Finding the appropriate function for the general case of switched nonlinear systems is hard while in the linear case the constraints can be recast as linear matrix inequalities. Therefore, we aim at simplifying our model in order to establish a switched affine model for the urban network.

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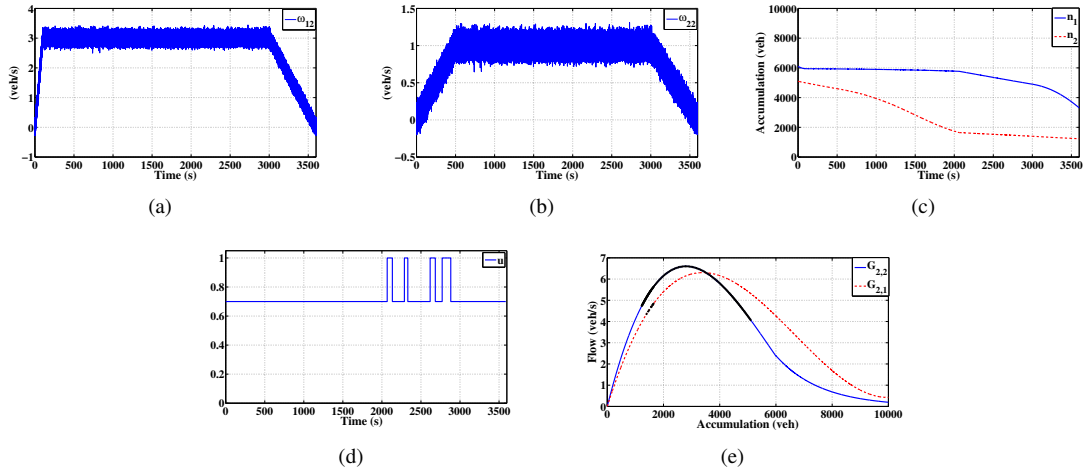


Fig. 3. Closed-loop simulation with switching control, (a) Trip demand from region 1 to region 2, (b) trip demands inside region 2, (c) accumulations, (d) perimeter control input converted from the switching law, (e) switching between MFDs of region 2 obtained from the switching law  $\sigma$ .

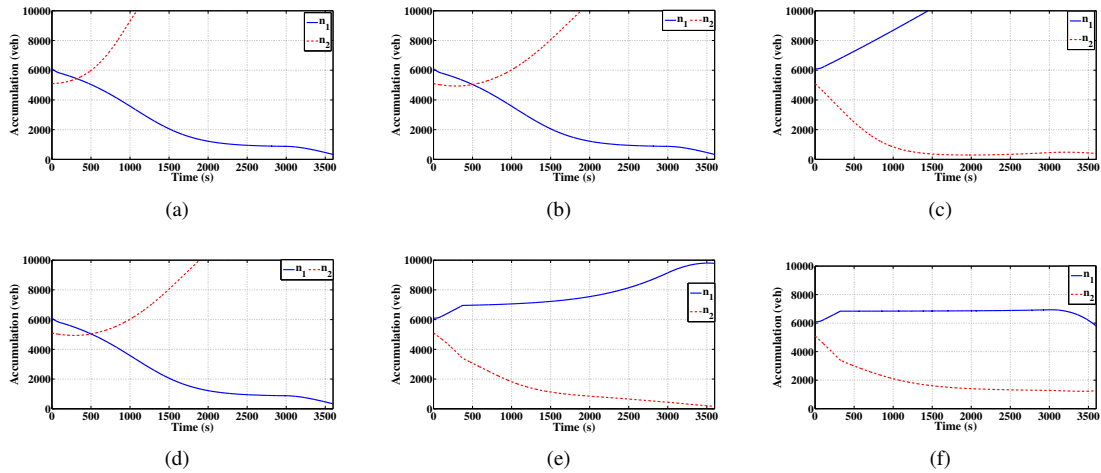


Fig. 4. Results for fixed-control strategies, (a)  $u = 1$  and timing plan 1 ( $G_{2,1}$ ), (b)  $u = 1$  and timing plan 2 ( $G_{2,2}$ ), (c)  $u = 0.1$  and timing plan 1, (d)  $u = 0.1$  and timing plan 2, (e)  $u = 0.1$  when  $n_2 > n_{cr}$ , otherwise  $u = 1$ , together with timing plan 1, (f) same strategy as in (e) but with timing plan 2.

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