Delft University of Technology

Delft Center for Systems and Control

Technical report 14-013

Estimation of the generalized traffic average speed based on microscopic measurements: Addendum*

A. Jamshidnejad and B. De Schutter

October 2014

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

* This report can also be downloaded via https://pub.bartdeschutter.org/abs/14_013.html

Estimation of the generalized traffic average speed based on microscopic measurements: Addendum

A. Jamshidnejad** and B. De Schutter

Delft Center for Systems and Control
Delft University of Technology

Abstract

This addendum contains the extended proof of the tight upper and lower bounds (41) and (42) of the manuscript "Estimation of the generalized traffic average speed based on microscopic measurements" by A. Jamshidnejad and B. De Schutter, Oct. 2014.

A Proof

Suppose that we have the following data from the loop detector:

$$\begin{aligned} \mathbf{V}_A &= \{v_{A,i} \mid i = 1, 2, \dots, n_A\}, \\ v_{A,\min} &= \min_{i=1,\dots,n_A} (v_{A,i}), \\ v_{A,\max} &= \max_{i=1,\dots,n_A} (v_{A,i}), \end{aligned} \tag{A.1}$$

$$h_A = \frac{1}{n_A} \sum_{i=1}^{n_A} h_{A,i-1} \tag{A.2}$$

where this data is according to the sampling window A with length L_A and width T_A. Moreover, the parameters m_A and M_A could be calculated by (24) and (25). Then from (32) and (35) we obtain the lower and upper bounds for the TSMS, so that we can write:

$$v^{\text{lower}}(A) \leq \text{TSMS}(A) \leq v^{\text{upper}}(A) \tag{A.3}$$

Now we construct a new sampling window B (see Figure 1) with all speed data the same as that of A, but with the headway a different constant, i.e.,

$$h_B = h_A + \Delta h \tag{A.4}$$

and

$$L_B = L_A, \qquad T_B = n_A h_B \tag{A.5}$$

and we want to find Δh such that:

$$v^{\text{lower}}(B) \leq \text{TSMS}(A) \leq v^{\text{upper}}(B) \tag{A.6}$$

Note that $n_A = n_B$.

From Figure 1 we see that for $\Delta h > 0$ some of the vehicles that are located in the second set for A, might be located in the first set for B. Let W be the number of vehicles from the second set of A that are in the first set for B. Then we have:

$$m_B = m_A - W \tag{A.7}$$

**Email: a.jamshidnejad@tudelft.nl

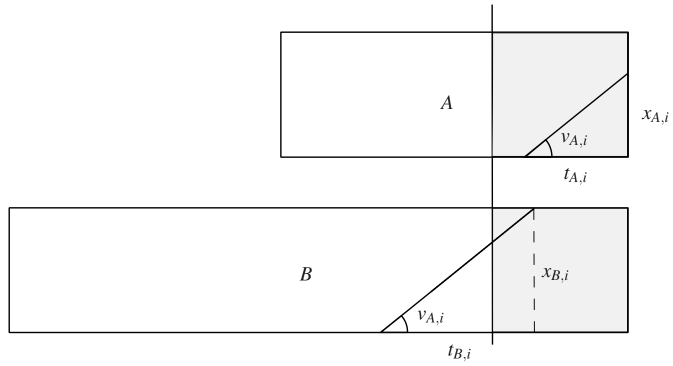

Figure 1: Sampling windows A and B with the same data sets (i.e., speed values, number of vehicles, and length of the window), but different time headways (we have $\alpha_i > 1$ and $\alpha_i x_i = L_A$)

We start with considering the upper bound of B to find the conditions under which this upper bound is also an upper bound for $\text{TSMS}(A)$. First from (35) for the factor multiplied by $H_{A,1\to n_A-m_A+1}$, i.e.,

$$f(m_A) = \frac{n_A - \frac{M_A}{2} + 1}{(n_A - m_A + 1) + \frac{(M_A - 1)}{2m_A}(2m_A - M_A)}, \qquad \text{with:} \quad M_A = \frac{v_{A,\min}}{v_{A,\max}} m_A + 1 \tag{A.8}$$

we can easily show that ($\alpha = \frac{v_{A,\min}}{v_{A,\max}}$):

$$\begin{aligned} \frac{\partial f(m_A)}{\partial m_A} &= \frac{(2\alpha^2 - 6\alpha + 4)n_A + 2\alpha^2 - 4\alpha + 2}{4n_A^2 + (f_1(\alpha)m_A - 4\alpha + 8)n_A + f_2(\alpha)m_A^2 + f_3(\alpha)m_A + \alpha^2 - 4\alpha + 4} \geq 0 \\ f_1(\alpha) &= -4\alpha^2 + 8\alpha - 8 \\ f_2(\alpha) &= \alpha^4 - 4\alpha^3 + 8\alpha^2 - 8\alpha + 4 \\ f_3(\alpha) &= 2\alpha - 8\alpha^2 + 12\alpha - 8 \end{aligned} \tag{A.9}$$

Therefore, by reducing m_A to m_B (i.e., by increasing h_A to h_B), the factor (A.8) becomes smaller and could produce a tighter upper bound (note that the equality occurs for $\alpha = 1$, i.e., for uniform speeds).

Now we should find the extreme/worst case where $\text{TSMS}(A)$ might violate $v^{\text{upper}}(B)$. Thus, we will try to strengthen $\text{TSMS}(A)$ and to weaken $v^{\text{upper}}(B)$ at the same time in order to produce the worst possible case. For the given A and B, the case where

$$v_{A,j} = v_{A,\max}, \qquad \text{for} \qquad j = n_A - m_A + W, \ldots, n_A \tag{A.10}$$

makes $\text{TSMS}(A)$ stronger with respect to $v^{\text{upper}}(B)$, because the speed values of the vehicles in (A.10) would not appear in (35) for $v^{\text{upper}}(B)$, but will strengthen $\text{TSMS}(A)$.

Now we consider the vehicles $v_{A,i}, i = n_A - m_A, \ldots, n_A - m_A + W - 1$ where these vehicles will appear in both the harmonic mean of the first set of B and in $\text{TSMS}(A)$. We first introduce the following two lemmas:

Lemma 1 *For fixed values of* $v_{A,j},\ j = 1, 2, \ldots, n_A - m_A + 1$*, suppose that we have*

$$\begin{aligned} &\text{H}\left(\{v_{A,j} | j = 1, 2, \ldots, n_A - m_A + 1\} \cup \{v_{A,i} | i = n_A - m_A + 2, \ldots, n_A - m_A + W - 1\}\right) < \\ &\quad \text{TSMS}\left(\{v_{A,j} | j = 1, 2, \ldots, n_A - m_A + 1\} \cup \{v_{A,i} | i = n_A - m_A + 2, \ldots, n_A - m_A + W - 1\}\right) \end{aligned} \tag{A.11}$$

Then, the difference between the two following values

$$\text{H}\left(\{v_{A,j} | j = 1, 2, \ldots, n_A - m_A + 1\} \cup \{v_{A,i} | i = n_A - m_A + 2, \ldots, n_A - m_A + W - 1\}\right)$$

and

$$\text{TSMS}\left(\{v_{A,j}|j=1,2,\ldots,n_A-m_A+1\}\cup\{v_{A,i}|i=n_A-m_A+2,\ldots,n_A-m_A+W-1\}\right)$$

becomes maximum, if we have:

$$v_{A,i}<v_{A,j}, \qquad i=n_A-m_A+2,\ldots,n_A-m_A+W-1 \tag{A.12}$$

where $\text{H}(\cdot)$ *stands for the harmonic mean.*

Proof: We denote the traveled distance and the travel time of vehicle $v_{A,k}$ for $k=1,\ldots,n_A-m_A+1$ by respectively x_k and t_k, and also the traveled distance and the travel time of $v_{A,i}$, $i=n_A-m_A+2,\ldots,n_A-m_A+W-1$ by x_i and t_i. Furthermore, to ease the notations we use "$j\in 1^{\text{st}}$ set of A" to indicate $j=1,2,\ldots,n_A-m_A+1$. Then we will have:

$$\text{TSMS}\left(\{v_{A,j}|j\in 1^{\text{st}}\text{ set of }A\}\cup v_{A,i}\right)=\frac{x_1+\ldots+x_{n_A-m_A+1}+x_i}{t_1+\ldots+t_{n_A-m_A+1}+t_i} \tag{A.13}$$

$$\text{H}\left(\{v_{A,j}|j\in 1^{\text{st}}\text{ set of }A\}\cup v_{A,i}\right)=\frac{n_A-m_A+2}{\frac{t_1}{x_1}+\ldots+\frac{t_{n_A-m_A+1}}{x_{n_A-m_A+1}}+\frac{t_i}{x_i}} \tag{A.14}$$

by a few computations and simplifications, we obtain:

$$\begin{aligned}
&\text{H}\left(\{v_{A,j}|j\in 1^{\text{st}}\text{ set of }A\}\cup v_{A,i}\right)<\text{TSMS}\left(\{v_{A,j}|j\in 1^{\text{st}}\text{ set of }A\}\cup v_{A,i}\right)\\
&\Leftrightarrow\\
&\boxed{\begin{aligned}&(x_1-x_i)\frac{(x_1t_i-x_it_1)}{x_i}x_2\ldots x_{n_A-m_A+1}+\\ &\vdots\\ &+(x_{n_A-m_A+1}-x_i)\frac{(x_{n_A-m_A+1}t_i-x_it_{n_A-m_A+1})}{x_i}x_1\ldots x_{n_A-m_A}\end{aligned}}\\
&+(x_2-x_1)(x_2t_1-x_1t_2)x_3\ldots x_{n_A-m_A+1}+\\
&\vdots\\
&+(x_{n_A-m_A+1}-x_{n_A-m_A})(x_{n_A-m_A+1}t_{n_A-m_A}-x_{n_A-m_A}t_{n_A-m_A+1})x_1\ldots x_{n_A-m_A-1}>0
\end{aligned} \tag{A.15}$$

Since the following holds:

$$x_1=\ldots=x_{n_A-m_A+1}=L_A \quad , \quad x_i\le L_A$$

hence,

$$x_j-x_i\ge 0 \qquad \text{for} \qquad j\in 1^{\text{st}}\text{ set of }A \tag{A.16}$$

Therefore, if we have:

$$\frac{x_jt_i}{x_i}-t_j>0 \tag{A.17}$$

which is equivalent to:

$$v_{A,j}>v_{A,i} \tag{A.18}$$

the boxed terms in (A.15) will definitely be positive. From (A.17) to make the boxed terms maximum, we should have:

$$v_{A,i}=v_{A,\min}$$

The same reasoning holds when more than one vehicle $v_{A,i}$ is considered, i.e., if we consider

$$v_{A,i}=v_{A,\min}, \qquad \text{for} \qquad i=n_A-m_A+2,\ldots,n_A-m_A+W-1 \tag{A.19}$$

then (A.11) will be satisfied and the difference between the two terms given by Lemma 1 will be maximum. □

Lemma 2 *For $N, a \geq 0$ and $D, b > 0$ we will have:* $\frac{a}{b} \geq \frac{N}{D} \Leftrightarrow \frac{N+a}{D+b} \geq \frac{N}{D}$.

As explained before, we are looking for the extreme case. Hence we suppose to have the conditions given by Lemma 1, which based on Lemma 2 and (A.10) indicate that:

$$\mathrm{TSMS}(A) \geq \mathrm{TSMS}\left(\{v_{A,j} | j \in 1^{\mathrm{st}} \text{ set of } A\} \cup v_{A,i}\right) \tag{A.20}$$

and hence,

$$\mathrm{TSMS}(A) \geq \mathrm{H}\left(\{v_{A,j} | j \in 1^{\mathrm{st}} \text{ set of } A\} \cup v_{A,i}\right) \tag{A.21}$$

Thus the worst case occurs if we have both (A.10) and (A.19).

Now we can write $\mathrm{TSMS}(A)$ and $v^{\mathrm{upper}}(B)$ for the worst case, as follows. First for $\mathrm{TSMS}(A)$:

$$\mathrm{TSMS}(A) = \frac{\sum_{A,x} + v_{A,\min}\left[W\frac{L_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A\right] + v_{A,\max}\left[(m_A - W - 1)\frac{L_A}{v_{A,\max}} - \frac{(M_A-1)(M_A-2)}{2}h_A\right]}{\sum_{A,t} + W\frac{L_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A + (m_A - W - 1)\frac{L_A}{v_{A,\max}} - \frac{(M_A-1)(M_A-2)}{2}h_A} \tag{A.22}$$

where

$$\sum_{A,x} = \sum_{j=1}^{n_A - m_A + 1} v_{A,j} t_{A,j} \quad , \quad \sum_{A,t} = \sum_{j=1}^{n_A - m_A + 1} t_{A,j}$$

are the traveled distance and the travel time of the first set of vehicles in A. The second term of the denominator corresponds to the travel time of the first W vehicles located in the second set of A, for which we have:

$$\begin{aligned}\sum_{j=n_A-m_A+1}^{n_A-m_A+W} t_{A,j} &= \left(\frac{L_A}{v_{A,\min}} - h_A\right) + \left(\frac{L_A}{v_{A,\min}} - 2h_A\right) + \ldots + \left(\frac{L_A}{v_{A,\min}} - Wh_A\right) \\ &= W\frac{L_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A\end{aligned}$$

and the second term of the numerator corresponds to the traveled distance of the first W vehicles located in the second set of A.

The third term of the denominator corresponds to the travel time of the last $m_A - W - 1$ vehicles located in the second set of A, for which we have:

$$\begin{aligned}\sum_{j=n_A-m_A+W+1}^{n_A} t_{A,j} &= (m_A - W - 1 - (M_A - 2))\frac{L_A}{v_{A,\max}} + \\ &(M_A - 2)\left[\left(\frac{L_A}{v_{A,\max}} - h_A\right) + \left(\frac{L_A}{v_{A,\max}} - 2h_A\right) + \ldots + h_A\right] \\ &= (m_A - W - 1)\frac{L_A}{v_{A,\max}} - \frac{(M_A-1)(M_A-2)}{2}h_A\end{aligned} \tag{A.23}$$

and the third term of the numerator corresponds to the traveled distance of the last $m_A - W - 1$ vehicles located in the second set of A.

Now we consider the upper bound for the sampling window B. Here we know that the first W vehicles in the second set of A are located in the first set of B. Extension of the headway (or the width of the sampling window) might also have an effect on the number of vehicles in the second subset of B (i.e., the last $M_B - 2$ vehicles in the second set). Therefore, in general we consider W' number of vehicles that were in the second subset of A are now out of this subset for B. Consequently we will have:

$$M_B = M_A - W'$$

For the upper bound for B from (34) we can write:

$$v^{\mathrm{upper}}(B) = \frac{\sum_x + v_{A,\min}\frac{WL_A}{v_{A,\min}} + v_{A,\max}\left[\frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}(h_A - \Delta h_A)\right]}{\sum_{A,t} + \frac{WL_A}{v_{A,\min}} + \frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}(h_A - \Delta h)} \tag{A.24}$$

where the following summations in the numerator and denominator of $v^{\text{upper}}(B)$ represent the traveled distance and the travel time of the first set of the sampling window B respectively:

$$\begin{aligned}&\text{Traveled distance of the } 1^{\text{st}} \text{ set of } B:\\ &\textstyle\sum_{A,x} + v_{A,\min}\dfrac{WL_A}{v_{A,\min}}\\ &\text{Travel time of the } 1^{\text{st}} \text{ set of } B:\\ &\textstyle\sum_{A,t} + \dfrac{WL_A}{v_{A,\min}}\end{aligned} \tag{A.25}$$

Also the following two terms are the traveled distance and the travel time of the second set of the sampling window B:

$$\begin{aligned}&\text{Traveled distance of the } 2^{\text{nd}} \text{ set of } B:\\ &v_{A,\max}\left[\frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}(h_A - \Delta h)\right]\\ &\text{Travel time of the } 2^{\text{nd}} \text{ set of } B:\\ &\frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}(h_A - \Delta h)\end{aligned}$$

Now we apply Lemma 3 (see Appendix B) to see what conditions are needed for (A.57) to be applicable to (A.22) and (A.24). Equations (A.22) and (A.24) could be rewritten as:

$$\text{TSMS}(A) = \frac{\text{TSMS}\left(\{v_{A,j} | j \in 1^{\text{st}} \text{ set of } A\}\right) \sum_{A,t} + c_1' \cdot v_{A,\max} + c_2' \cdot v_{A,\min}}{\sum_{A,t} + c_1' + c_2'} \tag{A.26}$$

where

$$c_1' = \frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - 1)(M_A - 2)}{2}h_A \tag{A.27}$$

and

$$c_2' = \frac{WL_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A \tag{A.28}$$

normalizing the coefficients with respect to $\sum_{A,t}$,

$$\text{TSMS}(A) = \frac{\text{TSMS}\left(\{v_{A,j} | j \in 1^{\text{st}} \text{ set of } A\}\right) + \frac{c_1'}{\sum_{A,t}} \cdot v_{A,\max} + \frac{c_2'}{\sum_{A,t}} \cdot v_{A,\min}}{1 + \underbrace{\frac{c_1'}{\sum_{A,t}}}_{w_1'} + \underbrace{\frac{c_2'}{\sum_{A,t}}}_{w_2'}} \tag{A.29}$$

Similarly,

$$v^{\text{upper}}(B) = \frac{\text{TSMS}\left(\{v_{A,j} | j \in 1^{\text{st}} \text{ set of } A\}\right) + \frac{c_1}{\sum_{A,t}} \cdot v_{A,\max} + \frac{c_2}{\sum_{A,t}} \cdot v_{A,\min}}{1 + \underbrace{\frac{c_1}{\sum_{A,t}}}_{w_1} + \underbrace{\frac{c_2}{\sum_{A,t}}}_{w_2}} \tag{A.30}$$

with

$$c_1 = \frac{(m_A - W - 1)L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}(h_A - \Delta h_A) \tag{A.31}$$

and

$$c_2 = \frac{WL_A}{v_{A,\min}} \tag{A.32}$$

Therefore, case C2 of the lemma 3 holds for (A.29) and (A.30) if we have:

$$\frac{\overbrace{\boxed{(m_A - W - 1)\frac{L_A}{v_{A,\max}} - \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}h_A + (n_A - m_A + 1)(M_A - 1)h_A}}^{N'} + \frac{(M_A - W' - 1)(M_A - W' - 2)}{2}\Delta h}{\overbrace{\boxed{W\frac{L_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A}}^{D} + \frac{W(W+1)}{2}h_A}$$

$$\geq \frac{\overbrace{\boxed{(m_A - W - 1)\frac{L_A}{v_{A,\max}} - \frac{(M_A - 1)(M_A - 2)}{2}h_A + (n_A - m_A + 1)(M_A - 1)h_A}}^{N}}{\overbrace{\boxed{W\frac{L_A}{v_{A,\min}} - \frac{W(W+1)}{2}h_A}}^{D}} \tag{A.33}$$

Note that from lemma 3 the extreme case where the inequality given by (A.57) might be violated takes place for $X = X_{\max}$ or equivalently

$$\text{TSMS}\left(\{v_{A,j} | j \in 1^{\text{st}} \text{ set of } A\}\right) = v_{A,\max} \Rightarrow \sum\nolimits_{A,t} = (n_A - m_A + 1)(M_A - 1)h_A$$

Next we use Lemma 2, where for (A.33),

$$a = \frac{(M_A - W' - 1)(M - W' - 2)}{2}\Delta h + \frac{W'(2M_A - W' - 3)}{2}h_A$$
$$b = \frac{W(W+1)}{2}h$$

Since $W' < M_A - 2$, then $W' = M_A - 3$ makes a minimum. Now, if the following holds:

$$\frac{\frac{(M_A - (M_A - 3) - 1)(M_A - (M_A - 3) - 2)}{2}\Delta h + \frac{(M_A - 3)(2M_A - (M_A - 3) - 3)}{2}h_A}{\frac{\cancel{W}(W+1)}{2}\cancel{h_A}} \geq \frac{(m_A - W - 1)(M_A - 1)h_A - \frac{(M_A - 1)(M_A - 2)}{2}h_A + (n_A - m_A + 1)(M_A - 1)h_A}{\cancel{W}m_A\cancel{h_A} - \frac{\cancel{W}(W+1)}{2}\cancel{h_A}} \tag{A.34}$$

which can be simplified to the following expression:

$$\Delta h \geq \frac{n_A M_A W - M_A W^2 - n_A W^2 - n_A W - M_A^2 m_A - \frac{1}{2}M_A W}{2m_A - W - 1}h_A \tag{A.35}$$

In addition we have:

$$(m_A - W)(h_A + \Delta h) = m_A h_A \Rightarrow \Delta h_A = \frac{W}{m_A - W}h_A \tag{A.36}$$

Substituting (A.36) in (A.35) we will finally get:

$$\boxed{W \geq \frac{n_A(M_A - 1)}{n_A + M_A}} \tag{A.37}$$

and if we also have:

$$(m_A - W - 1)(M_A - 1)h_A - \frac{(M_A - 1)(M_A - 2)}{2}h_A + (n_A - m_A + 1)(M_A - 1)h_A \geq 0 \Leftrightarrow$$
$$W \leq n_A - \frac{M_A}{2} + 1 \tag{A.38}$$

then based on lemma 2 (A.24) will be an upper bound for (A.22).

In addition, we know that in (A.24), the vehicles in the second subset of B compensate for the reduction of $v^{\text{upper}}(B)$ due to the W last vehicles in the first set that move with $v_{A,\min}$. Therefore, we need to make sure that

$$h_B \leq (M_A - 1)h_A \tag{A.39}$$

Then for the extreme case we will have:

$$m_A - W \geq \frac{m_A h_A}{(M_A - 1)h_A} \Rightarrow W \leq m_A \frac{(M_A - 2)}{M_A - 1}$$

and finally we should have:

$$\boxed{W \leq \min\left\{ m_A \frac{M_A - 2}{M_A - 1}, n_A - \frac{M_A}{2} + 1 \right\}} \tag{A.40}$$

Finally, we need to select W such that both (A.37) and (A.40) are satisfied at the same time. Afterwards, from W we can obtain m_B using (40), and then from (24) we will have h_B. Hence, we can obtain M_B from (25). Then using m_B and M_B, $\text{TSMS}^{\text{upper}}(B)$ is calculated by (35), while we can make sure that the obtained value is an upper bound for $\text{TSMS}(A)$.

Now, we consider the lower bound of B and we seek for conditions under which this bound is also a lower bound for $\text{TSMS}(A)$. With a similar reasoning as we had for the upper bound, here the extreme case where $\text{TSMS}(A)$ might become equal to or less than $v^{\text{lower}}(B)$ is when we have:

$$v_{A,j} = v_{A,\min}, \quad \text{for} \qquad j = n_A - m_A + W, \ldots, n_A \tag{A.41}$$

Additionally, the worst case corresponds to the situation where all the W vehicles (that are located in the second set of A, but in the first set of B) move with $v_{A,\max}$, i.e.,

$$v_{A,j} = v_{A,\max}, \quad \text{for} \qquad j = n_A - m_A, \ldots, n_A - m_A + W - 1 \tag{A.42}$$

Finally, for $\text{TSMS}(A)$ and $v^{\text{lower}}(B)$ (from (29)) we will have:

$$\text{TSMS}(A) = \frac{\sum_{A,x} + v_{A,\max}\left[\dfrac{WL_A}{v_{A,\max}} - \dfrac{(W - m_A + M_A - 1)(W - m_A + M_A)h_A}{2}\right] + v_{A,\min}\dfrac{(m_A - W - 1)(m_A - W)h_A}{2}}{\sum_{A,t} + \dfrac{WL_A}{v_{A,\max}} - \dfrac{(W - m_A + M_A - 1)(W - m_A + M_A)h_A}{2} + \dfrac{(m_A - W - 1)(m_A - W)h_A}{2}} \tag{A.43}$$

$$v^{\text{lower}}(B) = \frac{\sum_{A,x} + v_{A,\max}\dfrac{WL_A}{v_{A,\max}} + v_{A,\min}\dfrac{(m_A - W - 1)(m_A - W)}{2}(h_A + \Delta h)}{\sum_{A,t} + \dfrac{WL_A}{v_{A,\max}} + \dfrac{(m_A - W - 1)(m_A - W)}{2}(h_A + \Delta h)} \tag{A.44}$$

The second term of the numerator of (A.43) corresponds to the first W vehicles in the second set of A. From (A.42) these move with $v_{A,\max}$. Hence, the travel time of the first W vehicles in the 2nd set of A is:

$$\underbrace{\frac{L_A}{v_{A,\max}}(m_A - M_A + 1)}_{\text{vehicles in set 2, but outside of subset 2 of } A} + \underbrace{\left[\left(\frac{L_A}{v_{A,\max}} - h_A\right) + \ldots + \left(\frac{L_A}{v_{A,\max}} - (W - m_A + M_A - 1)h_A\right)\right]}_{\text{vehicles in subset 2 of } A} \tag{A.45}$$

For (A.43) the third term of the numerator is corresponding to the last $m_A - W - 1$ vehicles in the second set of A, i.e., the vehicles given by (A.41). We know that the travel time of v_{A,n_A} is h_A, for v_{A,n_A-1} it is $2h_A$, and so on. Therefore:

$$\text{Travel time of the last } m_A - W - 1 \text{ vehicles in the 2nd set of } A\text{:}$$
$$h_A + 2h_A + \ldots + (m_A - W - 1)h_A = \frac{(m_A - W - 1)(m_A - W)h_A}{2} \tag{A.46}$$

For (A.44) the second term corresponds to the W vehicles that are in the second set of A, but in the first set of B. The third term corresponds to the vehicles in the second set of B, where:

$$\begin{aligned}&\text{Travel time of the vehicles in the 2}^{\text{nd}}\text{ set of } B:\\&(h_A+\Delta h)+2(h_A+\Delta h)+\ldots+(m_A-W-1)(h_A+\Delta h)\end{aligned} \tag{A.47}$$

At the end, comparing (A.43) and (A.44) with the cases given in **Lemma A.1.**, case C3 is applicable here. Therefore, if we have:

$$\frac{\dfrac{WL_A}{v_{A,\max}}-\dfrac{(W-m_A+M_A-1)(W-m_A+M_A)}{2}h_A}{\dfrac{(m_A-W-1)(m_A-W)}{2}h_A+(n_A-m_A+1)m_Ah_A} \geq \frac{\dfrac{WL_A}{v_{A,\max}}-\dfrac{(W-m_A+M_A-1)(W-m_A+M_A)}{2}h_A+\dfrac{(W-m_A+M_A-1)(W-m_A+M_A)}{2}h_A}{\dfrac{(m_A-W-1)(m_A-W)}{2}h_A+(n_A-m_A+1)m_Ah_A+\dfrac{(m_A-W-1)(m_A-W)}{2}\Delta h} \tag{A.48}$$

Now we apply Lemma 2 (the contrapositive of the conditional statement). We obtain:

$$\frac{\dfrac{WL_A}{v_{A,\max}}-\dfrac{(W-m_A+M_A-1)(W-m_A+M_A)}{2}h_A}{\dfrac{(m_A-W-1)(m_A-W)}{2}h_A+(n_A-m_A+1)m_Ah_A} \geq \frac{\dfrac{(W-m_A+M_A-1)(W-m_A+M_A)}{2}h_A}{\dfrac{(m_A-W-1)(m_A-W)}{2}\Delta h} \tag{A.49}$$

which reduces to

$$\boxed{\Delta h \geq \frac{(W-m_A+M_A-1)(W-m_A+M_A)\left[(W+m_A)(W-m_A+1)+2m_A(n_A-W)\right]}{(m_A-W-1)(m_A-W)[-W^2+(2m-1)W+(M^2-2mM+M^2-M+m)]}h_A} \tag{A.50}$$

and if in addition to (A.50) we have:

$$2W(M_A-1)-(W-m_A+M_A-1)(W-m_A+M_A)\geq 0 \tag{A.51}$$

which reduces to

$$W \leq m_A+\sqrt{(2m_A-M_A)(M_A-1)}-\frac{1}{2} \tag{A.52}$$

then we can make sure that based on case C3 of Lemma 3:

$$\text{TSMS}^{\text{lower}}(B) \leq \text{TSMS}(A)$$

Then similar to what we explained for the upper bound before, we will have m_B and M_B and correspondingly $\text{TSMS}^{\text{lower}}(B)$ from (32), which could be used as a lower bound for $\text{TSMS}(A)$.
Note that a simple choice for W from (A.50) *and* (A.52) *is the following which indeed satisfies both conditions:*

$$\boxed{W \leq m_A-M_A+1} \tag{A.53}$$

B Weighted-average lemma

Lemma 3 *If for $w_1, w_1', w_2, w_2' \geq 0$ we have either of:*

C1.

$$w_1 > w_1' \quad \textit{and} \quad w_2 < w_2' \quad \textit{and} \quad \frac{w_1}{w_2} \geq \frac{w_1'}{w_2'}, \tag{A.54}$$

C2.

$$w_1 > w_1' \quad \textit{and} \quad w_2 > w_2' \quad \textit{and} \quad \frac{w_1+1}{w_2} \geq \frac{w_1'+1}{w_2'}, \tag{A.55}$$

C3.

$$w_1 < w_1' \quad \textit{and} \quad w_2 < w_2' \quad \textit{and} \quad \frac{w_1}{w_2+1} \geq \frac{w_1'}{w_2'+1}, \tag{A.56}$$

then we will have

$$\frac{X + w_1 X_{\max} + w_2 X_{\min}}{1 + w_1 + w_2} \geq \frac{X + w_1' X_{\max} + w_2' X_{\min}}{1 + w_1' + w_2'} \tag{A.57}$$

Proof : First, we will consider the following definition:

$$D = (X_{\max} - X_{\min})\underbrace{(w_1 w_2' - w_1' w_2)}_{F_1} + (X_{\max} - X)\underbrace{(w_1 - w_1')}_{F_2} + (X - X_{\min})\underbrace{(w_2' - w_2)}_{F_3} \tag{A.58}$$

At this step we reduce the problem to the proof of $D \geq 0$. The equivalency of (A.57) and (A.58) being positive is as follows:

$$\begin{aligned}
&D \geq 0 \Leftrightarrow \\
&(X_{\max} - X_{\min})(w_1 w_2' - w_1' w_2) + (X_{\max} - X)(w_1 - w_1') + (X - X_{\min})(w_2' - w_2) \geq 0 \Leftrightarrow \\
&w_1 w_2' X_{\max} - w_1 w_2' X_{\min} - w_1' w_2 X_{\max} + w_1' w_2 X_{\min} + \\
&\qquad w_1 X_{\max} - w_1 X - w_1' X_{\max} + w_1' X + \\
&\qquad w_2' X - w_2' X_{\min} - w_2 X + w_2 X_{\min} \geq 0 \Leftrightarrow \\
&w_1 w_2' X_{\max} + w_1' w_2 X_{\min} + w_1 X_{\max} + w_1' X + w_2' X + w_2 X_{\min} + X + w_1' w_1 X_{\max} + w_2' w_2 X_{\min} \\
&\qquad \geq w_1' w_2 X_{\max} + w_1 X + w_1' X_{\max} + w_2' X_{\min} + w_2 X + X + w_1' w_1 X_{\max} + w_2' w_2 X_{\min} \Leftrightarrow \\
&(X + w_1 X_{\max} + w_2 X_{\min})(1 + w_1' + w_2') \geq (X + w_1' X_{\max} + w_2' X_{\min})(1 + w_1' + w_2') \\
&\xLeftrightarrow{\text{since } w_1, w_1', w_2, w_2' \geq 0} \frac{X + w_1 X_{\max} + w_2 X_{\min}}{1 + w_1 + w_2} \geq \frac{X + w_1' X_{\max} + w_2' X_{\min}}{1 + w_1' + w_2'}
\end{aligned} \tag{A.59}$$

All the terms $X_{\max} - X_{\min}$, $X_{\max} - X$, and $X - X_{\min}$ in (A.58) are non-negative. Now we study three cases:

1. Suppose that we have (A.54). Hence, we already know that F_1, F_2, and F_3 in (A.58) are positive. Then we will know definitely that D is non-negative.

2. Suppose that we have (A.55). Considering (A.58) the minimum value of D is obtained when $X = X_{\max}$ and hence the value that is multiplied by the negative factor F_3 adopts its maximum value. Therefore (A.58) will reduce to:

   $$D_{C2} = (X_{\max} - X_{\min})(w_1 w_2' - w_1' w_2 + w_2' - w_2) \tag{A.60}$$

   Then from (A.55) we could write:

   $$\begin{aligned}
   &\frac{w_1 + 1}{w_2} \geq \frac{w_1' + 1}{w_2'} \Leftrightarrow \\
   &(w_1 + 1) w_2' \geq (w_1' + 1) w_2 \Leftrightarrow w_1 w_2' - w_1' w_2 + w_2' - w_2 \geq 0
   \end{aligned} \tag{A.61}$$

   Hence D_{C2} will definitely be non-negative and so will D.

3. Suppose that we have (A.56). Then the minimum value of D is obtained when $X = X_{\min}$ and the value multiplied by the negative factor F_2 adopts its maximum value. Therefore (A.58) will reduce to:

   $$D_{C3} = (X_{\max} - X_{\min})(w_1 w_2' - w_1' w_2 + w_1 - w_1') \tag{A.62}$$

   Then from (A.56) we could write:

   $$\begin{aligned}
   &\frac{w_1}{w_2 + 1} \geq \frac{w_1'}{w_2' + 1} \Leftrightarrow \\
   &w_1 (w_2' + 1) \geq w_1' (w_2 + 1) \Leftrightarrow w_1 w_2' - w_1' w_2 + w_1 - w_1' \geq 0
   \end{aligned} \tag{A.63}$$

   Then D_{C3} will definitely be non-negative.

□