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Stabilization and robust H_∞ control for sector-bounded switched nonlinear systems ⋆

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Abstract

This paper presents stability analysis and robust H_∞ control for a particular class of switched systems characterized by nonlinear functions that belong to sector sets with arbitrary boundaries. The sector boundaries can have positive and/or negative slopes, and therefore, we cover the most general case in our approach. Using the special structure of the system but without making additional assumptions (e.g. on the derivative of the nonlinear functions), and by proposing new multiple Lyapunov function candidates, we formulate stability conditions and a control design procedure in the form of matrix inequalities. The proposed Lyapunov functions are more general than the quadratic functions previously proposed in the literature, as they incorporate the nonlinearities of the system and hence, lead to less conservative stability conditions. The stabilizing switching controllers are designed through a bi-level optimization problem that can be efficiently solved using a combination of a convex optimization algorithm and a line search method. The proposed optimization problem is achieved using a special loop transformation to normalize the arbitrary sector bounds and by other linear matrix inequalities (LMI) techniques.

Key words: Switched nonlinear systems; Stability; H_∞ control; Linear matrix inequalities.

1 Introduction

Switched systems arise in cases where several dynamical models are required to represent a system due to e.g. uncertainty in parameters, or specific applications that utilize switching between controllers in order to achieve a higher performance (Liberzon, 2003; Hu et al., 2008; Aleksandrov et al., 2011; Geromel et al., 2013).

In this work, we study a special case of switched systems comprising a set of nonlinear subsystems. In each subsystem, the evolution of states is governed by linear combinations of nonlinear state-dependent functions. Furthermore, a switching signal determines the active subsystem at each time instant. The nonlinear functions are assumed to belong to sector sets with arbitrary (positive or negative, and possibly asymmetric) slopes for the sector boundaries. Thus, in the non-switched case, we cover more general cases of nonlinear functions compared e.g. to the Lure' type systems studied by Castelan et al. (2008); Gonzaga et al. (2012) and to the nonlinear systems that admit diagonal-type Lyapunov functions investigated by Kazkurewicz and Bhaya (1999); Aleksandrov et al. (2011).

Our paper contains 3 main contributions with respect to the state-of-the-art: 1) inclusion of sector bounds with arbitrary slopes for nonlinear functions (moreover, the nonlinear functions are no longer required to have an unbounded integral), 2) stability analysis for this class of switched systems under arbitrary switching using a less conservative approach based on multiple Lyapunov functions and the concept of average dwell time, 3) stabilization and robust disturbance attenuation of these systems using a bi-level convex optimization problem.

For stability analysis under arbitrary switching, we propose a family of Lyapunov functions that incorporate both quadratic functions of states and also the integrals of nonlinearities in the subsystems. Since the proposed Lyapunov candidate functions are general and include the nonlinear dynamics, this choice in general will lead to less conservative stability conditions compared to e.g. the choice of quadratic functions (see Castelan et al. (2008); da Silva et al.

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(2013) for a specific non-switched case). Based on the concept of average dwell-time (Hespanha, 2004), which allows fast switching occasionally, we formulate feasibility problem based on matrix inequalities that are nonlinear in a single scalar variable, in order to find a lower bound for the average dwell time to ensure asymptotic stability.

Next, we investigate the stabilization problem for the given class of switched systems in case of unstable modes and disturbances. Combining the proposed Lyapunov functions and their derivatives in order to obtain a single expression that can be used to design controllers is challenging. This is because the Lyapunov functions include the integrals of nonlinear functions while in the time derivative of the Lyapunov functions, the nonlinear functions appear explicitly. Using a transformation to normalize the sector bounds and congruence transformations in order to re-arrange the matrix inequalities into linear ones, design conditions are formulated in the form of a bi-level optimization problem with high-level problem non-convex only in a single scalar variable, and a convex low-level problem. Hence, the overall problem can be efficiently solved using a line search method along with LMIs feasibility checking.

In the following, we first present the problem statement and next, the stability conditions for the system under arbitrary switching. The design of robust stabilizing controllers is presented next and finally, the performance of the proposed scheme is shown using an example.

2 Problem statement and background

Consider the following switched nonlinear system:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}f\big(x(t)\big) + H_{\sigma(t)}\omega(t), \tag{1}$$
$$u(t) = K_{\sigma(t)}x(t) + F_{\sigma(t)}f(x), \tag{2}$$
$$y(t) = C_{\sigma(t)}g\big(x(t)\big), \tag{3}$$

with $x = (x_1, \dots, x_n)^{\mathrm{T}}$ the state vector, $u \in \mathbb{R}^{n_u}$ the control input, $\omega \in \mathbb{R}^{n_\omega}$ the exogenous input, $y \in \mathbb{R}^{n_y}$ the output, and $f : \mathbb{R}^n \to \mathbb{R}^n : x_i \to f_i(x_i)$, $g : \mathbb{R}^n \to \mathbb{R}^n : x_i \to g_i(x_i)$ nonlinear vector functions. Moreover, the switching signal σ is defined as a piecewise constant function, $\sigma(\cdot) : [0, +\infty) \to \{1, \dots, N\}$.
Assumption 1. *The scalar functions f_i are continuous and belong to the class $\mathcal{S}_{\mathrm{c1}}$ defined as follows:*

$$\mathcal{S}_{\mathrm{c1}} = \{\phi : \mathbb{R} \to \mathbb{R} |\ \big(\phi(\zeta) - \alpha\zeta\big)\big(\phi(\zeta) - \beta\zeta\big) \le 0, \phi(0) = 0,\ \forall\zeta \in \mathbb{R},\ \alpha, \beta \in \mathbb{R}, \alpha < \beta\}, \tag{4}$$

Note that the functions f_i are not required to lie only in the 1st and the 3rd quadrant as in Gonzaga et al. (2012), nor to have unbounded integrals as in Kazkurewicz and Bhaya (1999); Aleksandrov et al. (2011).
Assumption 2. *The scalar functions g_i are continuous and belong to the class $\mathcal{S}_{\mathrm{c2}}$ defined as follows:*

$$\mathcal{S}_{\mathrm{c2}} = \{\psi : \mathbb{R} \to \mathbb{R} |\ \exists\delta \text{ such that } |\psi(\zeta)| \le \delta|\zeta|,\ \forall\zeta \in \mathbb{R}\}, \tag{5}$$

In fact, $\mathcal{S}_{\mathrm{c2}}$ is a special case of the class $\mathcal{S}_{\mathrm{c1}}$ and functions that belong to the class $\mathcal{S}_{\mathrm{c2}}$ are bounded within a symmetric convex double cone with the origin as apex. Moreover, the nonlinear functions in system (1) can also be considered as state-dependent disturbances. Therefore, specific applications in which this type of disturbances affect the system, can be treated with our proposed analysis and control tools.

For the non-switched case of system (1), Ionsian and Susya (1991) proved that a Lyapunov function formulated as:

$$V(x) = x^{\mathrm{T}}Px + \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(\xi)\mathrm{d}\xi, \tag{6}$$

always exists provided that A is Hurwitz, $x_i f_i(x_i) \ge 0$, $\forall i$ and E has nonnegative off-diagonal elements. However, stability of a composed switched system cannot be concluded from the stability of subsystems Liberzon (2003). Stability under arbitrary switching for system (1) with $A_i = 0$ using a common Lyapunov function of the form (6) but without the quadratic term is proposed by Aleksandrov et al. (2011). However, extension of the results obtained by Aleksandrov et al. (2011) to our more general model (1)–(3) and more important, to the robust stabilization problem is not possible. This is mainly because we need to combine and compare the Lyapunov functions and their derivatives in order to compose a stabilizing control law and this is not feasible with the current formulation of the

Lyapunov function (6) (due to the integral of the nonlinearities). One solution is to use only quadratic functions of states. However, this choice would increase the conservatism in the stability analysis. Therefore, in the following, we use a different Lyapunov function that still contains the nonlinearities in the model and meanwhile, is extendable for the design of stabilizing switching laws. In the first stage, we propose a less conservative approach (compared to the common Lyapunov function method) for stability under arbitrary switching, using the concept of dwell time.

3 Stability analysis under arbitrary switching with dwell time constraint

For the switched system (1) with $u(t), \omega(t) = 0 \ \forall t$, the following set of Lyapunov functions is proposed:

$$V_\ell(x) = x^{\mathrm{T}} P_\ell x + 2\sum_{i=1}^{n} \lambda_i^{(\ell)} \int_0^{x_i} f_i(\xi)\mathrm{d}\xi, \qquad \ell = 1, \ldots, N. \tag{7}$$

With this choice, the functions f_i are not required to have an unbounded integral (in contrast to the Lyapunov function in Aleksandrov et al. (2011)). The following theorem provides sufficient conditions for exponential stability of (1) using the concepts of multiple Lyapunov functions (Colaneri et al., 2008) and the average dwell time (Hespanha, 2004). Note that asymptotic stability of all subsystems is a necessary condition here.

Theorem 3. *Consider the system (1) with Assumption 1. Suppose there exist positive matrices $\Lambda_\ell = \mathrm{diag}\{\lambda_i^{(\ell)}\}$, symmetric matrices P_ℓ, positive diagonal matrices $\mathcal{T}_\ell$, for $\ell = 1, \ldots, N$, and a positive scalar ε, such that:*

$$\begin{bmatrix} P_\ell A_\ell + A_\ell^{\mathrm{T}} P_\ell + \varepsilon(P_\ell + \Lambda_\ell \mathcal{D}_\beta) - \mathcal{T}_\ell \mathcal{D}_\alpha \mathcal{D}_\beta & \star \\ E_\ell^{\mathrm{T}} P_\ell + \Lambda_\ell A_\ell + \frac{1}{2}\mathcal{T}_\ell(\mathcal{D}_\alpha + \mathcal{D}_\beta) & \Lambda_\ell E_\ell + E_\ell^{\mathrm{T}}\Lambda_\ell - \mathcal{T}_\ell \end{bmatrix} < 0, \tag{8}$$

$$P_\ell + \Lambda_\ell \mathcal{D}_\alpha > 0, \qquad \forall \ell \in \{1, \ldots, N\}, \tag{9}$$

with $\mathcal{D}_\alpha = \mathrm{diag}\{\alpha_i\}$, $\mathcal{D}_\beta = \mathrm{diag}\{\beta_i\}$. System (1) with $u, \omega \equiv 0$ is globally exponentially stable under arbitrary switching, if the average dwell time of consecutive switching instants for any arbitrary interval (t_0, t) is bounded by:

$$T_{\mathrm{D}}(t_0, t) \geq \frac{1}{\varepsilon} \log\left(\max_{j,\ell \in \{1,\ldots,N\}} \frac{b_{\max,j}}{a_{\min,\ell}} \right), \qquad \forall t > t_0, \tag{10}$$

where $a_{\min,\ell}$ denotes the smallest singular value of $(P_\ell + \Lambda_\ell \mathcal{D}_\alpha)$ and $b_{\max,j}$ is the largest singular value of $(P_j + \Lambda_j \mathcal{D}_\beta)$.

Proof. Since $\lambda_i^{(\ell)} > 0$, we have:

$$\sum_{i=1}^{n} \lambda_i^{(\ell)} \int_0^{x_i} \alpha_i \xi \mathrm{d}\xi \leq \sum_{i=1}^{n} \lambda_i^{(\ell)} \int_0^{x_i} f_i(\xi)\mathrm{d}\xi \leq \sum_{i=1}^{n} \lambda_i^{(\ell)} \int_0^{x_i} \beta_i \xi \mathrm{d}\xi \tag{11}$$

Therefore, for each Lyapunov function V_ℓ, the following inequalities hold:

$$a_{\min,\ell}\|x\|^2 \leq x^{\mathrm{T}}(P_\ell + \Lambda_\ell \mathcal{D}_\alpha)x \leq V_\ell(x) \leq x^{\mathrm{T}}(P_\ell + \Lambda_\ell \mathcal{D}_\beta)x \leq b_{\max,\ell}\|x\|^2 \tag{12}$$

The sector condition (4), for the function f_i, can be written in the following quadratic form:

$$\begin{bmatrix} x_i \\ f_i(x_i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \alpha_i \beta_i & -\frac{\alpha_i + \beta_i}{2} \\ -\frac{\alpha_i + \beta_i}{2} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ f_i(x_i) \end{bmatrix} \leq 0, \tag{13}$$

Using the S-procedure (Boyd et al., 1994), if (8) and (13) hold, we obtain:

$$\underbrace{\begin{bmatrix} x \\ f(x) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_\ell A_\ell + A_\ell^{\mathrm{T}} P_\ell & P_\ell E_\ell + A_\ell^{\mathrm{T}}\Lambda_\ell \\ E_\ell^{\mathrm{T}} P_\ell + \Lambda_\ell A_\ell & \Lambda_\ell E_\ell + E_\ell^{\mathrm{T}}\Lambda_\ell \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix}}_{\dot{V}_\ell(x)} < -\varepsilon x^{\mathrm{T}}(P_\ell + \Lambda_\ell \mathcal{D}_\beta)x < -\varepsilon V_\ell(x) \tag{14}$$

Moreover, using (12), it can be easily shown that:

$$V_\ell(x) \leq \eta V_j(x), \quad \forall x \in \mathbb{R}^n, \quad \forall j, \ell \in \{1, \ldots, N\}, \tag{15}$$

with:

$$\eta = \max_{j,\ell \in \{1,\ldots,N\}} (b_{\max,j}/a_{\min,\ell}) \tag{16}$$

Now from (14) we obtain:

$$V_{\sigma(t_k)}(x(t)) \leq e^{-\varepsilon(t-t_k)} V_{\sigma(t_k)}(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \tag{17}$$

with t_k the k-th switching time instant. Using (15) and by iteration, we get:

$$V_{\sigma(t_k)}(x(t)) \leq e^{-\varepsilon(t-t_k)} V_{\sigma(t_k)}(x(t_k)) \leq e^{-\varepsilon(t-t_k)} \eta V_{\sigma(t_{k-1})}(x(t_k)) \leq \cdots \leq \eta^{N_\sigma(t,t_0)} e^{-\varepsilon(t-t_0)} V_{\sigma(t_0)}(x(t_0)), \tag{18}$$

where $N_\sigma(t, t_0)$ denotes the number of switchings in (t_0, t). Substituting

$$N_\sigma(t, t_0) \leq (t - t_0)/T_{\mathrm{D}}, \tag{19}$$

with T_{D} the average dwell time between successive switching time instants, results in:

$$V_{\sigma(t_k)}(x(t)) \leq e^{-(\varepsilon - \frac{\log \eta}{T_{\mathrm{D}}})(t-t_0)} V_{\sigma(t_0)}(x(t_0)) \tag{20}$$

Hence, using (20) and (12), we obtain:

$$\|x(t)\| \leq \frac{\max_{\ell \in \{1,\ldots,N\}} b_{\max,\ell}}{\min_{j \in \{1,\ldots,N\}} a_{\min,j}} e^{-\frac{1}{2}(\varepsilon - \frac{\log \eta}{T_{\mathrm{D}}})(t-t_0)} \|x(t_0)\| \tag{21}$$

Therefore, system (1) is globally exponentially stable for any switching pattern with $T_{\mathrm{D}}(t_0, t)$ satisfying (10). □

Remark 4. *If there exist common P and Λ matrices that satisfy the inequalities (8), it is easy to show that the bound (10) on the average dwell-time reduces to $T_{\mathrm{D}} \geq 0$, since (15) will hold for $\eta = 1$.*

Since (8) will be an LMI if the scalar variable ε is fixed, one can utilize an LMI optimization algorithm along with a line search method to find a feasible solution for (8)-(9).

4 Design of robust stabilizing switching laws

In this section, we synthesize switching laws together with the control input u in order to stabilize the switched nonlinear system and moreover, to minimize the effects of disturbances on the output of the system. Therefore, from now on we assume that none of the subsystems of (1) is locally or globally asymptotically stable.

We assume that the disturbance ω belongs to the L_2 space. The goal is to design a switching law σ of the form:

$$\sigma(t) = r(x(t)), \tag{22}$$

with $r(\cdot) : \mathbb{R}^n \to \{1, \ldots, N\}$ a piecewise constant function, such that the closed-loop system is globally asymptotically stable when $\omega(t) = 0, \forall t \geq 0$, and moreover, the desired L_2-gain γ from ω to $y = C_\sigma g(x)$ on any finite time interval $[0, T]$ is achieved, i.e. $\|y\|_{L_2[0,T]} \leq \gamma \|\omega\|_{L_2[0,T]}$. To this aim, a Lyapunov-like function is proposed as follows:

$$\mathcal{V}(x) = \min_{\ell=1,\ldots,N} V_\ell(x), \tag{23}$$

with V_ℓ selected as (7) with identical Λ_ℓ. Further, we define the class of Metzler matrices $\mathcal{M}$, with elements:

$$\mu_{ij} \geq 0 \ \forall i \neq j, \quad \sum_{i=1}^{N} \mu_{ij} = 0 \ \forall j, \tag{24}$$

(Geromel and Colaneri, 2006). Moreover, we limit our search for Metzler matrices to the cases in which the diagonal elements μ_{ii} are all equal (as is also done in Geromel and Deaecto (2009); Deaecto and Geromel (2010)).

In order to reach control design conditions in the form of linear matrix inequalities, we propose a transformation that brings the nonlinear functions f_i in (1) into the sector $[0, 1]$. The transformed system, with nonlinear functions $\bar{f}_i$ bounded in the sector $[0, 1]$, has the following structure:

$$\dot{x}(t) = \bar{A}_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) + \bar{E}_{\sigma(t)} \bar{f}\big(x(t)\big) + H_{\sigma(t)} \omega(t), \tag{25}$$

$$u(t) = \bar{K}_{\sigma(t)} x(t) + \bar{F}_{\sigma(t)} f(x), \tag{26}$$

$$y(t) = C_{\sigma(t)} g\big(x(t)\big), \tag{27}$$

with the following system matrices:

$$\begin{aligned} \bar{A}_{\sigma(t)} &= A_{\sigma(t)} + E_{\sigma(t)} \mathcal{D}_\alpha, \quad \bar{E}_{\sigma(t)} = E_{\sigma(t)} \Gamma, \\ \bar{K}_{\sigma(t)} &= K_{\sigma(t)} + F_{\sigma(t)} \mathcal{D}_\alpha, \quad \bar{F}_{\sigma(t)} = F_{\sigma(t)} \Gamma, \end{aligned} \tag{28}$$

where $\Gamma = \mathrm{diag}\{\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n\}$, and $\bar{f}_i$ defined as:

$$\bar{f}_i(x_i) = \big(f_i(x_i) - \alpha_i x_i\big) / (\beta_i - \alpha_i). \tag{29}$$

Moreover, the Lyapunov function (7) has to be adapted to the transformed system. Therefore, we have:

$$\bar{V}_\ell(x) = x^{\mathrm{T}} \bar{P}_\ell x + 2 \sum_{i=1}^{n} \bar{\lambda}_i \int_0^{x_i} \bar{f}_i(\xi) \mathrm{d}\xi, \tag{30}$$

$$\bar{P}_\ell = P_\ell + \mathrm{diag}\Big\{\alpha_1 \lambda_1, \ldots, \alpha_n \lambda_n\Big\}, \tag{31}$$

$$\bar{\lambda}_i = \lambda_i (\beta_i - \alpha_i). \tag{32}$$

The following theorem provides the design tools for robust H_∞ control of the transformed system (25)–(27).
Theorem 5. *Suppose there exist positive definite matrices Q_ℓ and S_ℓ, positive diagonal matrices Z and U_ℓ, matrices W_ℓ, Y_ℓ, and scalar $\bar{\mu} < 0$, such that the following problem:*

$$\min_{Q_\ell, W_\ell, Y_\ell, S_\ell, U_\ell, Z, \rho, \bar{\mu}} \rho \tag{33}$$

subject to:

$$\begin{bmatrix} \bar{A}_\ell Q_\ell + Q_\ell \bar{A}_\ell^{\mathrm{T}} + B_\ell W_\ell + W_\ell^{\mathrm{T}} B_\ell^{\mathrm{T}} + \bar{\mu} Q_\ell & \star & \star & \star & \star \\ \bar{A}_\ell Q_\ell + B_\ell W_\ell + Y_\ell^{\mathrm{T}} B_\ell^{\mathrm{T}} + Z \bar{E}_\ell^{\mathrm{T}} + S_\ell & B_\ell Y_\ell + \bar{E}_\ell Z + Z \bar{E}_\ell^{\mathrm{T}} + Y_\ell^{\mathrm{T}} B_\ell^{\mathrm{T}} - U_\ell & \star & \star & \star \\ H_\ell^{\mathrm{T}} & H_\ell^{\mathrm{T}} & -\rho I & \star & \star \\ \|C_\ell\|_{\mathrm{F}} \Delta Q_\ell & 0 & 0 & -I & \star \\ -\bar{\mu} Q_\ell & 0 & 0 & 0 & \bar{\mu} Q_j \end{bmatrix} < 0, \tag{34}$$

$$\begin{bmatrix} Q_\ell & Q_\ell \\ Q_\ell & \mathcal{D}_{\alpha,(1)}^{-1} (\mathcal{D}_\beta - \mathcal{D}_\alpha) Z \end{bmatrix} > 0, \qquad \forall \ell, j \in \{1, \ldots, N\}, \ \ell \neq j. \tag{35}$$

has an optimal solution $\rho^* > 0$, then the switching rule:

$$\bar{\sigma}(t) = \bar{r}(x(t)) = \min \ \arg \min_{\ell = 1, \ldots, N} \bar{V}_\ell(x(t)), \tag{36}$$

[1] *with $\bar{P}_\ell = Q_\ell^{-1}$ and $\bar{\Lambda} = Z^{-1}$, along with the state feedback control:*

$$u(t) = \bar{K}_\ell x(t) + \bar{F}_\ell \bar{f}(x), \tag{37}$$

with $\bar{K}_\ell = W_\ell Q_\ell^{-1}$, $\bar{F}_\ell = Y_\ell Z^{-1}$, make the closed-loop switched system (25)–(27) globally asymptotically stable in the absence of disturbances, and further, guarantee an upper bound $\gamma^* = \sqrt{\rho^*}$ for L_2-gain.

Proof. For the transformed system, $\bar{\mathcal{V}}(x)$ is defined as in (23) based on $\bar{V}_\ell(x)$. Since $\bar{\mathcal{V}}$ is piecewise differentiable, we define the so-called Dini derivative (see Geromel and Colaneri (2006)):

$$\mathbf{D}^+\big(\bar{\mathcal{V}}(x(t))\big) = \lim_{\delta t \to 0^+} \sup \frac{\bar{\mathcal{V}}(x(t+\delta t)) - \bar{\mathcal{V}}(x(t))}{\delta t} \tag{38}$$

Assume that at an arbitrary time $t \geq 0$, the switching law is given by $\bar{\sigma}(t) = \bar{r}(x(t)) = \ell$ for some $\ell \in \mathcal{I}(x(t)) = \{\ell : \bar{\mathcal{V}}(x) = \bar{V}_\ell(x)\}$. Hence, from (38) and (1), we have (using Theorem 1 of Lasdon (1970)):

$$\mathbf{D}^+\big(\bar{\mathcal{V}}(x(t))\big) = \min_{i \in \mathcal{I}(x(t))} \Big[\frac{\partial \bar{V}_i}{\partial x}\Big(\bar{A}_\ell x + B_\ell u + \bar{E}_\ell f(x) + H_\ell \omega\Big)\Big] \leq \frac{\partial \bar{V}_\ell}{\partial x}\Big(\bar{A}_\ell x + B_\ell u + \bar{E}_\ell f(x) + H_\ell \omega\Big) \tag{39}$$

where ℓ denotes the index of the active subsystem determined from (36). We consider a Metzler matrix with equal diagonal elements, i.e. $\mu_{ii} = \bar{\mu},\ \bar{\mu} < 0$. This implies that

$$\bar{\mu}^{-1} \sum_{j=1, j\neq\ell}^{N} \mu_{j\ell} = 1 \tag{40}$$

Taking this into account, the Schur complement is performed to (34) with respect to the last row and column. We multiply the result by $\mu_{j\ell}$, sum up for all $j \neq \ell$ and then, multiply by $\bar{\mu}^{-1}$.

Now, we pre- and post-multiply the result by the matrix $\mathrm{diag}\{Q_\ell^{-1}, Z^{-1}, I, I\}$ with $Q_\ell^{-1} = \bar{P}_\ell$, $Z^{-1} = \bar{\Lambda}$, and next, we change the variables $W_\ell Q_\ell^{-1} = \bar{K}_\ell$, $Y_\ell Z^{-1} = \bar{F}_\ell$, $S_\ell Q_\ell^{-1} = 0.5\bar{\mathcal{T}}_\ell$, and finally $U_\ell Z^{-1} = \bar{\mathcal{T}}_\ell$. Furthermore, we define a new variable $\mathcal{T}_\ell$ for the positive diagonal matrix $\bar{\Lambda}\bar{\mathcal{T}}_\ell$. Applying again the Schur complement to the resulting matrix with respect to the fourth row and column, rearranging terms and pre-multiplying by $[x^{\mathrm{T}}, \bar{f}^{\mathrm{T}}(x), \omega^{\mathrm{T}}]$ and post-multiplying by its transpose yields:

$$\begin{aligned}
&\underbrace{\begin{bmatrix} x \\ \bar{f}(x) \\ \omega \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{P}_\ell(\bar{A}_\ell + B_\ell\bar{K}_\ell) + (\bar{A}_\ell + B_\ell\bar{K}_\ell)^{\mathrm{T}}\bar{P}_\ell & \star & \star \\ (\bar{E}_\ell + B_\ell\bar{F}_\ell)^{\mathrm{T}}\bar{P}_\ell + \bar{\Lambda}(\bar{A}_\ell + B_\ell\bar{K}_\ell) & \bar{\Lambda}(\bar{E}_\ell + B_\ell\bar{F}_\ell) + (\bar{E}_\ell + B_\ell\bar{F}_\ell)^{\mathrm{T}}\bar{\Lambda} & \star \\ H_\ell^{\mathrm{T}}\bar{P}_\ell & H_\ell^{\mathrm{T}}\bar{\Lambda} & 0 \end{bmatrix} \begin{bmatrix} x \\ \bar{f}(x) \\ \omega \end{bmatrix}}_{\frac{\partial \bar{V}_\ell}{\partial x}\left(\bar{A}_\ell x + B_\ell u + \bar{E}_\ell \bar{f}(x) + H_\ell \omega\right)} \\
&\quad < \begin{bmatrix} x \\ \bar{f}(x) \end{bmatrix} \begin{bmatrix} 0 & \star \\ -\frac{1}{2}\mathcal{T}_\ell & \mathcal{T}_\ell \end{bmatrix} \begin{bmatrix} x \\ \bar{f}(x) \end{bmatrix} - \sum_{j=1}^{N} \mu_{j\ell} x^{\mathrm{T}}\bar{P}_j x - \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}}\Delta^2 x + \rho\omega^{\mathrm{T}}\omega
\end{aligned} \tag{41}$$

Based on the sector condition (13) with $\alpha_i = 0, \beta_i = 1$, and since $\bar{V}_j \geq \bar{V}_\ell$ for all $j \in \{1, \ldots, N\} \setminus \{\ell\}$, we obtain:

$$\begin{aligned}
\mathbf{D}^+\big(\bar{\mathcal{V}}(x(t))\big) &\leq \frac{\partial \bar{V}_\ell}{\partial x}\Big(\bar{A}_\ell x + B_\ell u + \bar{E}_\ell f(x) + H_\ell \omega\Big) < -\sum_{j=1}^{N} \mu_{j\ell} x^{\mathrm{T}}\bar{P}_j x - \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}}\Delta^2 x + \rho\omega^{\mathrm{T}}\omega \\
&< -x^{\mathrm{T}}\bar{P}_\ell x \underbrace{\sum_{j=1}^{N} \mu_{j\ell}}_{=0} - \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}}\Delta^2 x + \rho\omega^{\mathrm{T}}\omega < -y^{\mathrm{T}}y + \rho\omega^{\mathrm{T}}\omega.
\end{aligned} \tag{42}$$

[1] Note that in (36), we take the minimum argument, in case of having multiple minimum V_ℓ (as a result of sliding motion).

The last inequality is concluded using the following:

$$y^{\mathrm{T}}y = \|y\|_2^2 = \|C_{\sigma(t)}g(x)\|_2^2 \leq \|C_{\sigma(t)}\|_{\mathrm{F}}^2 \cdot \|g(x)\|_2^2 \leq \|C_{\sigma(t)}\|_{\mathrm{F}}^2 x^{\mathrm{T}}\Delta^2 x, \tag{43}$$

where the first inequality is obtained using (2.3.7) of Golub and Van Loan (1989) and $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm. The positive-definiteness of the Lyapunov functions (7) defined for the original system should be preserved under the proposed transformation. Pre- and post-multiplying (35) by the diagonal matrix $\mathrm{diag}\{Q_\ell^{-1}, I\}$, and then performing the Schur complement to the result, we obtain:

$$\bar{P}_\ell - \mathcal{D}_{\alpha,(1)}(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} > 0, \tag{44}$$

since $Q_\ell^{-1} = \bar{P}_\ell$ and $Z^{-1} = \bar{\Lambda}$. We use the fact that $\mathcal{D}_\alpha = \mathrm{diag}\{\alpha_i\}$ can be written as subtraction of two positive definite diagonal matrices, i.e. $\mathcal{D}_\alpha = \mathcal{D}_{\alpha,(1)} - \mathcal{D}_{\alpha,(2)}$. Therefore, (44) guarantees

$$\bar{P}_\ell - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} > 0. \tag{45}$$

On the other hand, using (31)–(32), we get:

$$0 < \bar{P}_\ell - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} = P_\ell + \mathcal{D}_\alpha\Lambda - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}(\mathcal{D}_\beta - \mathcal{D}_\alpha)\Lambda = P_\ell \tag{46}$$

Finally, adding and subtracting the term $\mathcal{D}_\alpha\Lambda$ to the condition $Q_\ell^{-1} = \bar{P}_\ell > 0$, yields:

$$0 < \bar{P}_\ell - \mathcal{D}_\alpha\Lambda + \mathcal{D}_\alpha\Lambda = P_\ell + \mathcal{D}_\alpha\Lambda. \tag{47}$$

Condition (47) is similar to (9) which ensures positive-definiteness of Lyapunov functions (7). □

Remark 6. *If the variable $\bar{\mu}$ is fixed, the optimization problem (33)–(35) can be efficiently solved by any LMI solver. The optimal value of $\bar{\mu}$ corresponding to the minimum $\gamma = \sqrt{\rho}$ is obtained by a line search method together with LMIs feasibility checking. Note that the γ^* obtained from the optimization (33)–(35) is an upper bound for the L_2-gain of the system (25)–(27). Hence, the actual controlled system would outperform the estimated L_2-gain γ^*.*

Proposition 7. *The switching law $\sigma(t)$ determined by (36) with:*

$$\lambda_i = (Z^{-1})_{ii} \cdot (\beta_i - \alpha_i)^{-1}, \tag{48}$$

$$P_\ell = Q_\ell^{-1} - Z^{-1} \cdot \mathrm{diag}\Big\{\frac{\alpha_1}{\beta_1 - \alpha_1}, \dots, \frac{\alpha_n}{\beta_n - \alpha_n}\Big\}, \tag{49}$$

together with the state feedback control law (2) with gains:

$$F_\ell = \bar{F}_\ell\Gamma^{-1}, \quad K_\ell = \bar{K}_\ell - F_\ell\mathcal{D}_\alpha, \tag{50}$$

make the closed-loop switched system (1)–(3) globally asymptotically stable for $\omega \equiv 0$, and guarantees an upper bound for the L_2-gain γ^* (obtained from optimization (33)-(35)) for disturbance signals that belong to the $\mathcal{L}_2$ space. The proof follows directly from the relations between the original and the transformed system's matrices.

Remark 8. *Indeed, asymptotic stability of (1) under switching law (36) is ensured even if sliding mode behavior occurs. In case of a sliding mode, the result of $\arg\min_{\ell=1,\dots,N} V_\ell(x(t))$ might not be unique. However, using (42) (assume $u, \omega \equiv 0$ for simplicity), it can be shown that the time-derivative of the minimum Lyapunov function is strictly negative along the Filippov solution of the system (Similar to the approach in Remark 2 of Geromel and Colaneri (2006)). From the Lyapunov function (23) and the time-derivative (39), a switch from any $\ell \in \mathcal{I}(x)$ to some $j \in \mathcal{I}(x)$ is allowed only if:*

$$\frac{\partial V_j}{\partial x}(A_\ell x + E_\ell f(x)) \leq \frac{\partial V_\ell}{\partial x}(A_\ell x + E_\ell f(x)). \tag{51}$$

For the extended model of the switched system including the sliding motion, formulated as:

$$\dot{x}(t) = \Big(\sum_{\ell\in\mathcal{I}(x)} \theta_\ell A_\ell\Big)x + \Big(\sum_{\ell\in\mathcal{I}(x)} \theta_\ell E_\ell\Big)f(x), \quad 0 \leq \theta_\ell, \quad \sum_{\ell\in\mathcal{I}(x)} \theta_\ell = 1, \tag{52}$$

we have:

$$\frac{\partial V_j}{\partial x}\left[\Big(\sum_{\ell\in\mathcal{I}(x)}\theta_\ell A_\ell\Big)x+\Big(\sum_{\ell\in\mathcal{I}(x)}\theta_\ell E_\ell\Big)f(x)\right]=\sum_{\ell\in\mathcal{I}(x)}\theta_\ell\frac{\partial V_j}{\partial x}\Big(A_\ell x+E_\ell f(x)\Big)\le\sum_{\ell\in\mathcal{I}(x)}\theta_\ell\frac{\partial V_\ell}{\partial x}\Big(A_\ell x+E_\ell f(x)\Big)<0,$$

where the first inequality holds from (51) under sliding mode, and the last inequality is justified using the same reasoning as in (42). Hence, we conclude that the derivative of the positive definite function V_j is strictly negative along the trajectories of (52).

5 Example

To show the performance and also to emphasize that other methods in the literature are not able to cope with the more general case we discussed in this paper, we choose an example of the system (1)–(3) with the following functions and system matrices:

$$f_1(x_1)=\begin{cases}|\sin(x_1)| & -\pi<x_1<\pi\\ 0 & \text{otherwise}\end{cases},\qquad f_2(x_2)=\begin{cases}2x_2 & 0\le x_2\\ 0.5x_2 & 0>x_2\end{cases} \tag{53}$$

$$A_1=\begin{bmatrix}4 & 1\\ 2.3 & 3\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 1\\ 2 & 4\end{bmatrix},\ A_3=\begin{bmatrix}2 & -1\\ -7 & 4\end{bmatrix} \tag{54}$$

$$E_1=\begin{bmatrix}2 & 8\\ -3 & 1\end{bmatrix},\ E_2=\begin{bmatrix}-3 & 5\\ 0 & 2\end{bmatrix},\ E_3=\begin{bmatrix}4 & 8\\ -1 & 1\end{bmatrix} \tag{55}$$

and $H_\ell=I_{2\times2}$, $B_\ell=[1,1]^{\mathrm{T}}$, $C_\ell=[1,1]$. Moreover, $g_i(x_i)=x_i$ and the disturbance signals are taken as $\omega_1(t)=\omega_2(t)=100$, for $0\le t\le1$, and otherwise equal to zero. Note that none of the subsystems has Hurwitz matrices. Moreover, function f_1 does not have unbounded integral as it is a requirement for the methods in Kazkurewicz and Bhaya (1999) and Aleksandrov et al. (2011). Now following the design procedure in Theorem 5 with sector bounds $(\alpha_1,\beta_1)=(-1,1)$ and $(\alpha_2,\beta_2)=(0.5,2)$, the state feedback (2) along with the switching rule (36) make the equilibrium $x=0$ globally asymptotically stable in the absence of disturbance. Using a line search method and the convex optimization problem (33)–(35) (solved using the Yalmip toolbox and the SeDuMi solver), the best value obtained for γ is 0.1318 for all ω that belong to the L_2 space. Moreover, the obtained matrices for the Lyapunov functions and the feedback gain matrices are:

$$P_1=\begin{bmatrix}10.3307 & -10.3314\\ -10.3314 & 10.3323\end{bmatrix},\ P_2=\begin{bmatrix}9.4144 & -9.4169\\ -9.4146 & 9.4197\end{bmatrix},\ P_3=\begin{bmatrix}11.4676 & -11.4666\\ -11.4666 & 11.4658\end{bmatrix},\ \Lambda=\begin{bmatrix}0.5078 & 0\\ 0 & 0.2979\end{bmatrix}$$

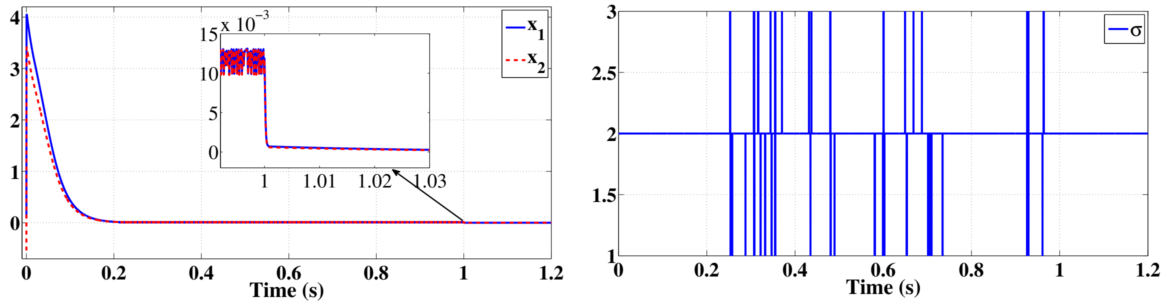
$$K_1=10^4\cdot\Big[4.8258\ -5.9892\Big],\ K_2=10^4\cdot\Big[4.2731\ -5.0686\Big],\ K_3=10^5\cdot\Big[-1.2504\ 1.0872\Big],$$

$$F_1=\Big[4.6502\ -5.2399\Big],\ F_2=\Big[1.6438\ -3.3400\Big],\ F_3=\Big[2.3197\ -4.6176\Big]$$

As depicted in Fig. 1, the designed switching control strategy is able to reduce the effects of severe disturbance signals and further, makes the closed-loop system stable. The steady state response of the system contains oscillations with a very small amplitude mainly because of function f_1. For the simulated example and the given disturbance inputs, the actual L_2-gain is calculated 0.0529, which is indeed smaller than $\gamma^*=0.1318$.

6 Conclusions

Stability analysis and H_∞ control for a class of switched nonlinear systems with general point symmetric sector conditions were presented. Combining multiple Lyapunov functions that contain both quadratic functions of the states and integrals of the nonlinearities, we have formulated stability conditions under arbitrary switching and the design of robust stabilizing controllers in the form of matrix inequalities. This is a great advantage in contrast to the general case for switched nonlinear systems, which is based on searching for Lyapunov functions without a pre-defined structure and which involves solving multi-parametric optimization problems. Possible extensions to the current work are 1) introducing different sector bounds for different quadrants to better characterize the sector-bounded nonlinear functions and to reduce the conservatism even more, 2) investigating the possibility of using polynomial Lyapunov functions (Chesi et al., 2012) for stability analysis, 3) calculating the largest sector bounds that ensure stability, 4) designing consistent switching control schemes (Geromel et al., 2013).



(a) (b)

Fig. 1. (a) State evolution of the closed-loop system, (b) Switching signal selecting the active subsystem

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