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Robust H_∞ Control of a Class of Switched Nonlinear Systems with Application to Macroscopic Urban Traffic Control

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***Abstract*— This paper presents stability analysis and robust H_∞ control for nonlinear switched systems bounded in sectors with arbitrary boundaries. By proposing new and more general multiple Lyapunov functions that incorporate nonlinearities in the system, we formulate the stability conditions under arbitrary switching in the form of linear matrix inequalities. Moreover, an optimization problem subject to bilinear matrix inequalities is established in order to determine the minimum L_2-gain along with the optimal matrices for the Lyapunov functions and for the robust state feedback gains. Finally, the optimization problem is recast as a bi-level convex optimization problem using loop transformation and other linear matrix inequalities techniques. Furthermore, in order to illustrate the performance of the proposed switching control scheme, results for control of an urban network partitioned into sub-regions and modeled using a high-level hybrid model are presented.**

I. INTRODUCTION

Switched systems are a class of hybrid systems characterized by a set of linear and/or nonlinear subsystems and a switching signal selecting the active subsystems [1]. Stability analysis, stabilization, and control synthesis for such systems have been studied in recent years [1]–[3].

In this work, we consider a class of switched systems composed of several nonlinear subsystems. Further, a state and/or time dependent switching signal determines the active subsystem. The nonlinear functions are assumed to belong to sector sets with arbitrary (and possibly asymmetric) slopes for the sector boundaries. Thus, we cover more general cases of nonlinear functions compared e.g. to the Lure' type systems studied by [4], [5] and to the systems that admit diagonal-type Lyapunov functions investigated by [6], [7]).

To motivate the research and to provide a practical application, we draw the attention to the hybrid high-level model developed by [8] for large-scale urban traffic networks. In this model, the evolution of the traffic states is represented by several nonlinear dynamics, each corresponding to a particular traffic signal timing plan and a switching controller that determines the operating mode of the system. The switching controller needs to be designed in a way that guarantees stability and high overall performance of the whole system which is exposed to uncertain trip demands and other sources of disturbance. The required design scheme should be able to control the system in real-time. However, achieving the aforementioned performance goals is challenging, mainly due to the existence of multiple nonlinear functions bounded in sector sets with arbitrary slopes and the switching between nonlinear subsystems.

For this purpose, we use a set of Lyapunov functions that have both quadratic functions of states and also the integrals of nonlinearities in the subsystems. Since the proposed Lyapunov functions are general and include the nonlinear dynamics, we expect that this choice leads to less conservative stability conditions compared to e.g. the choice of quadratic functions (see [4], [9] for a specific non-switched case). Furthermore, we will utilize some matrix inequality techniques to eventually establish sufficient conditions for the design of robust stabilizing switching control laws in the form of an optimization problem constrained by bilinear matrix inequalities. The optimization problem gives the minimum possible L_2-gain for the switched system under control along with the optimal state feedback gains.

In order to further improve the efficiency, we reformulate the optimization problem using a transformation technique to normalize the sector boundaries and additional congruence transformations. At the end, we obtain a bi-level optimization problem with high level problem that is non-convex only in a single scalar variable, while the low level optimization problem is convex. Hence, we are able to solve it efficiently using a line search method along with a convex optimization method and subject to LMI constraints.

The rest of the paper is organized as follows. In Section II the particular class of switched nonlinear systems under study is presented. Section III presents the stability conditions under arbitrary switching patterns. A set of new Lyapunov functions that contain the nonlinearities in the model is used in Section IV, as basis for the design of robust stabilizing controllers. Finally, the performance of the proposed scheme is evaluated using an urban network control case study in Section V.

II. PROBLEM STATEMENT AND BACKGROUND

Consider the switched nonlinear system:

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t) + E_\sigma f\big(x(t)\big) + H_\sigma \omega(t), \quad (1)$$

$$u(t) = K_\sigma x(t) + F_\sigma f(x), \quad (2)$$

$$y(t) = C_\sigma g\big(x(t)\big), \quad (3)$$

with $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ the state, $u \in \mathbb{R}^{n_u}$ the control input, $\omega \in \mathbb{R}^{n_\omega}$ the disturbance input, $y \in \mathbb{R}^{n_y}$ the output, and $f : \mathbb{R}^n \to \mathbb{R}^n : x_i \to f_i(x_i)$, $g : \mathbb{R}^n \to \mathbb{R}^n : x_i \to g_i(x_i)$ nonlinear vector functions. Moreover, the switching signal σ is defined as a piecewise constant function, $\sigma(\cdot) : [0, +\infty) \to \{1, \ldots, N\}$.

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Assumption 1: The scalar functions f_i are continuous and belong to the class $\mathcal{S}_{c1}$ defined as:

$$\mathcal{S}_{c1} = \{\phi : \mathbb{R} \to \mathbb{R} | \ \big(\phi(\zeta) - \alpha\zeta\big)\big(\phi(\zeta) - \beta\zeta\big) \leq 0\}, \quad (4)$$

for all $\zeta \in \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$.

Assumption 2: The scalar functions g_i are continuous and belong to the class $\mathcal{S}_{c2}$ defined as follows:

$$\mathcal{S}_{c2} = \{\psi : \mathbb{R} \to \mathbb{R} | \ \exists \delta : \ |\psi(\zeta)| \leq \delta|\zeta|, \ \forall \zeta \in \mathbb{R}\}, \quad (5)$$

For the non-switched form of system (1) with $A_\ell = 0, \forall \ell$, it is proved [6] that the equilibrium $x_e = 0$ is globally asymptotically stable if and only if: 1) $x_i f_i(x_i) > 0$, $\forall i$, 2) $\int_0^{x_i} f_i(\xi)\mathrm{d}\xi \to +\infty$ as $|x_i| \to \infty$, $\forall i$, 3) E is a Hurwitz diagonally stable matrix. Moreover, they proved that

$$V(x) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(\xi)\mathrm{d}\xi, \quad (6)$$

is a diagonal-type Lyapunov function. However, stability of a switched system cannot be concluded from the stability of subsystems. According to [1], it is sufficient to find a common Lyapunov function (CLF) for a switched system in order to prove stability. In a special case of (1)–(3) with $A_\ell = 0, \forall \ell \in \{1, \ldots, N\}$, and with the external inputs u and ω equal to zero, a CLF can be constructed by combining the integrals of the nonlinearities of the model [7]. However, extension of the results for arbitrary switching proposed by [7] to the general system (1)–(3) and to the stabilization and robust control problem is not possible. Therefore, in the next sections, we use a different Lyapunov function that still contains the nonlinearities in the model and at the same time, it is suitable for the design of robust disturbance rejection switching laws. Another major advantage of our proposed methodology will be establishing efficient convex optimization problems subject to LMI constraints.

III. STABILITY ANALYSIS UNDER ARBITRARY SWITCHING

For the system (1) with $u(t), \omega(t) = 0 \ \forall t$, the following common Lyapunov function is proposed:

$$V(x) = x^{\mathrm{T}} P x + 2 \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(\xi)\mathrm{d}\xi. \quad (7)$$

The time derivative of (7) along the trajectories of the switched system is obtained as follows (the time index t is dropped for simplicity):

$$\dot{V}(x) = \begin{bmatrix} x \\ f(x) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} PA_\ell + A_\ell^{\mathrm{T}} P & PE_\ell + A_\ell^{\mathrm{T}} \Lambda \\ E_\ell^{\mathrm{T}} P + \Lambda A_\ell & \Lambda E_\ell + E_\ell^{\mathrm{T}} \Lambda \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix} \quad (8)$$

with $\Lambda = \mathrm{diag}\{\lambda_i\}$ and ℓ index of the active subsystem. Now noticing that the nonlinear functions f_i belong to the class $\mathcal{S}_{c1}$, the following theorem provides sufficient conditions for the asymptotic stability of (1) with $u, \omega \equiv 0$.

Theorem 1: Assume there exists a symmetric matrix P, a diagonal matrix Λ, and a positive definite and diagonal matrix $\mathcal{T} = \mathrm{diag}\{\tau_1, \ldots, \tau_n\}$, such that the LMIs:

$$\begin{bmatrix} PA_\ell + A_\ell^{\mathrm{T}} P - \mathcal{T}\mathcal{D}_\alpha\mathcal{D}_\beta & \star \\ E_\ell^{\mathrm{T}} P + \Lambda A_\ell + \frac{1}{2}\mathcal{T}(\mathcal{D}_\alpha + \mathcal{D}_\beta) & \Lambda E_\ell + E_\ell^{\mathrm{T}}\Lambda - \mathcal{T} \end{bmatrix} < 0$$

$$\forall \ell \in \{1, \ldots, N\}, \quad (9)$$

$$P + \mathcal{D}_\alpha \Lambda > 0, \quad (10)$$

$$P + \mathcal{D}_\beta \Lambda > 0, \quad (11)$$

with $\mathcal{D}_\alpha = \mathrm{diag}\{\alpha_1, \ldots, \alpha_n\}$, $\mathcal{D}_\beta = \mathrm{diag}\{\beta_1, \ldots, \beta_n\}$, are feasible, then the switched system (1) with $u, \omega \equiv 0$ will be asymptotically stable under arbitrary switching.

Proof: It is easy to verify that condition (4), for the function f_i, can be written in the following quadratic form:

$$\begin{bmatrix} x_i \\ f_i(x_i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \alpha_i\beta_i & -\frac{\alpha_i+\beta_i}{2} \\ -\frac{\alpha_i+\beta_i}{2} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ f_i(x_i) \end{bmatrix} \leq 0, \quad (12)$$

The main idea is that the derivative (8) should be negative whenever (12) holds for all $i \in \{1, \ldots, n\}$. Using the so-called S-procedure [10], the inequalities (8) and (12) can be combined, resulting in the LMI (9). Furthermore, we have for any $x_i \in \mathbb{R}$:

$$\sum_{i=1}^{n} \lambda_i \int_0^{x_i} \alpha_i \xi \mathrm{d}\xi \leq \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(\xi)\mathrm{d}\xi$$

$$\leq \sum_{i=1}^{n} \lambda_i \int_0^{x_i} \beta_i \xi \mathrm{d}\xi \quad (13)$$

Therefore, in order to guarantee that $V(x) > 0$, we need:

$$x^{\mathrm{T}} P x + x^{\mathrm{T}} \mathcal{D}_\alpha \Lambda x > 0, \quad (14)$$

$$x^{\mathrm{T}} P x + x^{\mathrm{T}} \mathcal{D}_\beta \Lambda x > 0. \quad (15)$$

Hence, it is sufficient to have matrices $P + \mathcal{D}_\alpha\Lambda$ and $P + \mathcal{D}_\beta\Lambda$ positive definite as in (10)–(11). ■

IV. DESIGN OF ROBUST STABILIZING SWITCHING LAWS

Before, we discussed the stability analysis for switched systems (1)–(3) under given switching signals. In this section, we synthesize switching laws together with the control input u in order to stabilize the switched system and moreover, to minimize the effects of disturbances on the output of the system. We assume that the disturbance vector ω belongs to the space of square integrable functions. The system has L_2-gain $\gamma > 0$ under some switching law σ if $\|y\|_{L_2[0,T]} \leq \gamma\|\omega\|_{L_2[0,T]}$ for all nonzero $\omega \in L_2[0, T]$ $(0 \leq T < \infty)$ and for initial condition $x(0) = 0$.

A Lyapunov-like function is proposed as follows:

$$\mathcal{V}(x) = \min_{\ell=1,\ldots,N} V_\ell(x), \quad (16)$$

with V_ℓ selected as:

$$V_\ell(x) = x^{\mathrm{T}} P_\ell x + 2 \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(\xi)\mathrm{d}\xi. \quad (17)$$

Further, we define the class of Metzler matrices $\mathcal{M}$, with elements $\mu_{ij} \geq 0 \ \forall i \neq j$, $\sum_{i=1}^{N} \mu_{ij} = 0 \ \forall j$ [11]. The main results are summarized in the following theorem.

Theorem 2: Suppose there exist a Metzler matrix $M \in \mathcal{M}$ with elements μ_{ij}, positive definite matrices P_ℓ, a diagonal matrix Λ, matrices K_ℓ and F_ℓ, and positive diagonal matrices $\mathcal{T}_\ell = \mathrm{diag}\{\tau_{1,\ell}, \dots, \tau_{n,\ell}\}$ that give an optimal solution for the problem (18)–(20), then the control input:

$$u(t) = K_\ell^* x + F_\ell^* f(x), \tag{21}$$

along with the min-switching law:

$$r(x(t)) = \arg \min_{\ell=1,\dots,N} V_\ell(x(t)), \tag{22}$$

make the closed-loop switched system (1)–(3) asymptotically stable in the absence of disturbance and moreover, ensure the L_2-gain $\gamma^* = \sqrt{\rho^*}$ from ω to the output y.

Proof: The Lyapunov function (16) is piecewise differentiable. Therefore, we define the so-called Dini derivative [11], [12]:

$$\mathbf{D}^+\big(\mathcal{V}(x(t))\big) = \lim_{\delta t \to 0^+} \sup \frac{\mathcal{V}(x(t+\delta t)) - \mathcal{V}(x(t))}{\delta t} \tag{23}$$

Assume that at a time instant $t \geq 0$, the switching law is given by $\sigma(t) = r(x(t)) = \ell$ for some $\ell \in \mathcal{I}(x(t)) = \{\ell : \mathcal{V}(x) = V_\ell(x)\}$. Hence, from (23) and (1), we have ([13]):

$$\begin{aligned}\mathbf{D}^+\big(\mathcal{V}(x(t))\big) &= \min_{i \in \mathcal{I}(x(t))} \Big[\frac{\partial V_i}{\partial x}\Big(A_\ell x + E_\ell f(x)\Big)\Big] \\ &\leq \frac{\partial V_\ell}{\partial x}\Big(A_\ell x + E_\ell f(x)\Big)\end{aligned} \tag{24}$$

where ℓ denotes the index of the active subsystem calculated from (22). Applying the Schur complement to (19) with respect to the fourth row and column, rearranging terms and pre- and post-multiplying it by $[x^{\mathrm{T}}, f^{\mathrm{T}}(x), \omega^{\mathrm{T}}]$ and its transpose yields:

$$\begin{aligned}\frac{\partial V_\ell}{\partial x}\big(A_\ell x + B_\ell u + E_\ell f(x) + H_\ell \omega\big) &< -\sum_{j=1}^{N} \mu_{j\ell} x^{\mathrm{T}} P_j x \\ &+ \begin{bmatrix} x \\ f(x) \end{bmatrix}\begin{bmatrix} \mathcal{T}_\ell \mathcal{D}_\alpha \mathcal{D}_\beta & \star \\ -\frac{1}{2}\mathcal{T}_\ell(\mathcal{D}_\alpha + \mathcal{D}_\beta) & \mathcal{T}_\ell \end{bmatrix}\begin{bmatrix} x \\ f(x) \end{bmatrix} \\ &- \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}} \Delta^2 x + \rho \omega^{\mathrm{T}} \omega\end{aligned} \tag{25}$$

Since (19) is valid for some $M \in \mathcal{M}$ and $V_j \geq V_\ell$ for all $j \in \{1, \dots, N\} \setminus \{\ell\}$, and based on (12), we obtain:

$$\begin{aligned}\mathbf{D}^+\big(\mathcal{V}(x(t))\big) &\leq \frac{\partial V_\ell}{\partial x}\Big(A_\ell x + B_\ell u + E_\ell f(x) + H_\ell \omega\Big) \\ &< -\sum_{j=1}^{N} \mu_{j\ell} x^{\mathrm{T}} P_j x - \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}} \Delta^2 x + \rho \omega^{\mathrm{T}} \omega \\ &< -x^{\mathrm{T}} P_\ell x \underbrace{\sum_{j=1}^{N} \mu_{j\ell}}_{=0} - \|C_\ell\|_{\mathrm{F}}^2 x^{\mathrm{T}} \Delta^2 x + \rho \omega^{\mathrm{T}} \omega \\ &< -y^{\mathrm{T}} y + \rho \omega^{\mathrm{T}} \omega.\end{aligned} \tag{26}$$

The last inequality is justified using:

$$\begin{aligned}y^{\mathrm{T}} y = \|C_{\sigma(t)} g(x)\|_2^2 &\leq \|C_{\sigma(t)}\|_{\mathrm{F}}^2 \cdot \|g(x)\|_2^2 \\ &\leq \|C_{\sigma(t)}\|_{\mathrm{F}}^2 x^{\mathrm{T}} \Delta^2 x,\end{aligned} \tag{27}$$

where the first inequality is obtained using relation (2.3.7) of [14] and $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm. ■

The optimization problem (18)–(20) involves solving Bilinear Matrix Inequalities (BMIs), which is in general a computationally hard problem. Therefore, using some transformation techniques, we reformulate the problem as a convex optimization problem.

First, we use a transformation to bring the functions f_i into the sector $[0, 1]$. The transformed system, with functions $\bar{f}_i$ bounded in the sector $[0, 1]$, has the following form:

$$\dot{x}(t) = \bar{A}_\sigma x(t) + B_\sigma u(t) + \bar{E}_\sigma \bar{f}\big(x(t)\big) + H_\sigma \omega(t), \tag{28}$$

$$u(t) = \bar{K}_\sigma x(t) + \bar{F}_\sigma f(x), \tag{29}$$

$$y(t) = C_\sigma g\big(x(t)\big), \tag{30}$$

with system matrices:

$$\begin{aligned}\bar{A}_{\sigma(t)} &= A_{\sigma(t)} + E_{\sigma(t)} \mathcal{D}_\alpha, \quad \bar{E}_{\sigma(t)} = E_{\sigma(t)} \Gamma, \\ \bar{K}_{\sigma(t)} &= K_{\sigma(t)} + F_{\sigma(t)} \mathcal{D}_\alpha, \quad \bar{F}_{\sigma(t)} = F_{\sigma(t)} \Gamma,\end{aligned} \tag{31}$$

where $\mathcal{D}_\alpha = \mathrm{diag}\{\alpha_i\}$, and $\Gamma = \mathrm{diag}\{\beta_i - \alpha_i\}$, $i = 1, \dots, n$. Moreover, the Lyapunov function (17) has to be adapted to the transformed system. Therefore, we have:

$$\bar{V}_\ell(x) = x^{\mathrm{T}} \bar{P}_\ell x + 2\sum_{i=1}^{n} \bar{\lambda}_i \int_0^{x_i} \bar{f}_i(\xi) \mathrm{d}\xi, \tag{32}$$

$$\bar{P}_\ell = P_\ell + \mathrm{diag}\Big\{\alpha_1 \lambda_1, \dots, \alpha_n \lambda_n\Big\}, \tag{33}$$

$$\bar{\lambda}_i = \lambda_i(\beta_i - \alpha_i). \tag{34}$$

The following theorem provides the design tools for robust control of the transformed system (28)–(30).

Theorem 3: Suppose there exist positive definite matrices Q_ℓ and S_ℓ, positive diagonal matrices Z and U_ℓ, matrices W_ℓ, Y_ℓ, and scalar $\bar{\mu} < 0$, such that the problem (35)–(37) has an optimal solution ρ^*, then the switching rule:

$$\bar{\sigma}(t) = \bar{r}(x(t)) = \arg \min_{\ell=1,\dots,N} \bar{V}_\ell(x(t)), \tag{37}$$

with $\bar{P}_\ell = Q_\ell^{-1}$ and $\bar{\Lambda} = Z^{-1}$, along with the controller:

$$u(t) = \bar{K}_\ell x(t) + \bar{F}_\ell \bar{f}(x), \tag{38}$$

with $\bar{K}_\ell = W_\ell Q_\ell^{-1}$, $\bar{F}_\ell = Y_\ell Z^{-1}$, make the closed-loop switched system (28)–(30) globally asymptotically stable in the absence of disturbances, and further, guarantee an upper bound $\gamma^* = \sqrt{\rho^*}$ for L_2-gain.

Proof: We use a backward reasoning approach. First, we consider a Metzler matrix with equal diagonal elements, i.e. $\mu_{ii} = \bar{\mu}$, $\bar{\mu} < 0$. Based on the definition of Metzler matrices, this implies that $\bar{\mu}^{-1} \sum_{j=1, j\neq\ell}^{N} \mu_{j\ell} = 1$. Taking this into account, the Schur complement is performed to (36) with respect to the last row and column. We multiply the result by $\mu_{j\ell}$, sum up for all $j \neq \ell$, and then multiply by $\bar{\mu}^{-1}$. Now, the resulting matrix is pre- and post-multiplied by the matrix $\mathrm{diag}\{Q_\ell^{-1}, Z^{-1}, I, I\}$ with $Q_\ell^{-1} = \bar{P}_\ell$, $Z^{-1} = \bar{\Lambda}$, and the variables $W_\ell Q_\ell^{-1} = \bar{K}_\ell$, $Y_\ell Z^{-1} = \bar{F}_\ell$, $S_\ell Q_\ell^{-1} = \bar{\mathcal{T}}_\ell$, $U_\ell Z^{-1} = \bar{\mathcal{T}}_\ell$ are assigned. Furthermore, a new variable $\mathcal{T}_\ell$ for the positive diagonal matrix $\bar{\Lambda}\bar{\mathcal{T}}_\ell$ is defined. Now the final

$$\min_{P_\ell,\Lambda,K_\ell,F_\ell,\mathcal{T}_\ell,\rho,\mu_{j\ell}} \rho \tag{18}$$

subject to:

$$\begin{bmatrix} P_\ell(A_\ell+B_\ell K_\ell)+(A_\ell+B_\ell K_\ell)^{\mathrm{T}}P_\ell-\mathcal{T}_\ell\mathcal{D}_\alpha\mathcal{D}_\beta+\sum_{j=1}^{N}\mu_{j\ell}P_j & \star & \star & \star \\ (E_\ell+B_\ell F_\ell)^{\mathrm{T}}P_\ell+\Lambda(A_\ell+B_\ell K_\ell)+\frac{1}{2}\mathcal{T}_\ell(\mathcal{D}_\alpha+\mathcal{D}_\beta) & \Lambda(E_\ell+B_\ell F_\ell)+(E_\ell+B_\ell F_\ell)^{\mathrm{T}}\Lambda-\mathcal{T}_\ell & \star & \star \\ H_\ell^{\mathrm{T}}P_\ell & H_\ell^{\mathrm{T}}\Lambda & -\rho I & \star \\ \|C_\ell\|_{\mathrm{F}}\Delta & O & O & -I \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} P_\ell+\mathcal{D}_\alpha\Lambda & \star \\ O & P_\ell+\mathcal{D}_\beta\Lambda \end{bmatrix} > 0, \qquad P_\ell > 0,\ \rho > 0, \qquad \forall \ell \in \{1,\ldots,N\}. \tag{20}$$

$$\min_{Q_\ell,W_\ell,Y_\ell,S_\ell,U_\ell,Z,\rho,\bar{\mu}} \rho \tag{35}$$

subject to:

$$\begin{bmatrix} \bar{A}_\ell Q_\ell+Q_\ell\bar{A}_\ell^{\mathrm{T}}+B_\ell W_\ell+W_\ell^{\mathrm{T}}B_\ell^{\mathrm{T}}+\bar{\mu}Q_\ell & \star & \star & \star & \star \\ \bar{A}_\ell Q_\ell+B_\ell W_\ell+Y_\ell^{\mathrm{T}}B_\ell^{\mathrm{T}}+Z\bar{E}_\ell^{\mathrm{T}}+S_\ell & B_\ell Y_\ell+\bar{E}_\ell Z+Z\bar{E}_\ell^{\mathrm{T}}+Y_\ell^{\mathrm{T}}B_\ell^{\mathrm{T}}-U_\ell & \star & \star & \star \\ H_\ell^{\mathrm{T}} & H_\ell^{\mathrm{T}} & -\rho I & \star & \star \\ \|C_\ell\|_{\mathrm{F}}\Delta Q_\ell & O & O & -I & \star \\ -\bar{\mu}Q_\ell & O & O & O & \bar{\mu}Q_j \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} Q_\ell & Q_\ell \\ Q_\ell & \mathcal{D}_{\alpha,(1)}^{-1}(\mathcal{D}_\beta-\mathcal{D}_\alpha)Z \end{bmatrix} > 0, \qquad \forall \ell, j \in \{1,\ldots,N\},\ \ell \neq j.$$

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matrix inequality resembles (36), with $\alpha_i = 0, \beta_i = 1$ for the transformed system.

The positive-definiteness of the Lyapunov functions (17) defined for the original system should be preserved under the proposed transformation. Pre- and post-multiplying (37) by the diagonal matrix $\mathrm{diag}\{Q_\ell^{-1}, I\}$, and then performing the Schur complement to the result, we obtain:

$$\bar{P}_\ell - \mathcal{D}_{\alpha,(1)}(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} > 0, \tag{39}$$

since $Q_\ell^{-1} = \bar{P}_\ell$ and $Z^{-1} = \bar{\Lambda}$. We use the fact that $\mathcal{D}_\alpha = \mathrm{diag}\{\alpha_i\}$ can be written as subtraction of two positive definite diagonal matrices, i.e. $\mathcal{D}_\alpha = \mathcal{D}_{\alpha,(1)} - \mathcal{D}_{\alpha,(2)}$. Therefore, (39) guarantees $\bar{P}_\ell - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} > 0$. On the other hand, using (33)–(34), we get:

$$\begin{aligned} 0 &< \bar{P}_\ell - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}\bar{\Lambda} = \\ &= P_\ell + \mathcal{D}_\alpha\Lambda - \mathcal{D}_\alpha(\mathcal{D}_\beta - \mathcal{D}_\alpha)^{-1}(\mathcal{D}_\beta - \mathcal{D}_\alpha)\Lambda = P_\ell \end{aligned} \tag{40}$$

Finally, adding and subtracting the term $\mathcal{D}_\alpha\Lambda$ to the condition $Q_\ell^{-1} = \bar{P}_\ell > 0$, yields:

$$0 < \bar{P}_\ell - \mathcal{D}_\alpha\Lambda + \mathcal{D}_\alpha\Lambda = P_\ell + \mathcal{D}_\alpha\Lambda. \tag{41}$$

Condition (41) is in fact similar to the original condition (20). Hence, (17) is ensured to be positive definite under the proposed transformation. ■

Remark 1: Note that if the variable $\bar{\mu}$ is fixed, the optimization problem (35)–(37) can be efficiently solved by any LMI solver. Therefore, the optimal value of $\bar{\mu}$ corresponding to the minimum gain $\gamma = \sqrt{\rho}$ can be obtained by a line search method together with LMIs feasibility checking.

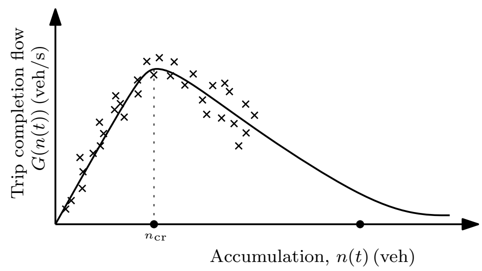

Fig. 1. Schematic macroscopic fundamental diagram.

Proposition 1: The switching law (22) with

$$\lambda_i = (Z^{-1})_{ii} \cdot (\beta_i - \alpha_i)^{-1}, \tag{42}$$

$$P_\ell = Q_\ell^{-1} - Z^{-1} \cdot \mathrm{diag}\Big\{\frac{\alpha_1}{\beta_1 - \alpha_1}, \ldots, \frac{\alpha_n}{\beta_n - \alpha_n}\Big\}, \tag{43}$$

together with the state feedback control (2) with:

$$F_\ell = \bar{F}_\ell\Gamma^{-1}, \quad K_\ell = \bar{K}_\ell - F_\ell\mathcal{D}_\alpha, \tag{44}$$

make the closed-loop switched system (1)–(3) globally asymptotically stable for $\omega \equiv 0$, and guarantees an upper bound for the L_2-gain γ^* (obtained from (35)-(37)).

Proof: The proof follows directly from the relation between the transformed system and the original system. ■

V. CASE STUDY: URBAN NETWORK CONTROL

For urban networks, a low-scatter macroscopic fundamental diagram (MFD) (as depicted in Fig. 1) can be captured in case the congestion is evenly distributed in the network.

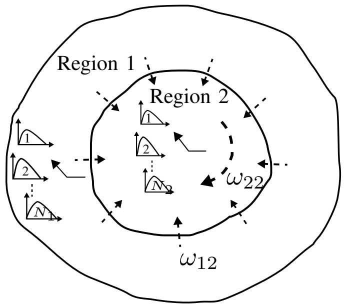

Fig. 2. Schematic two-region urban network.

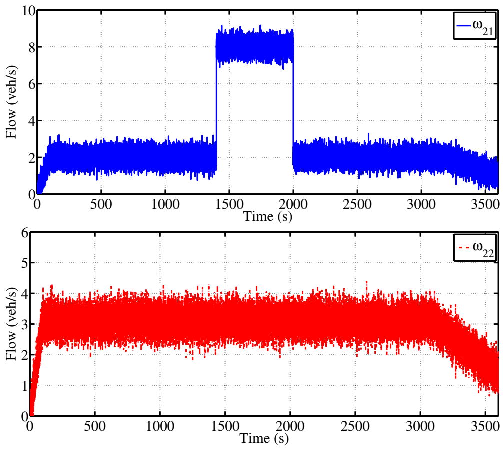

Fig. 3. Trip demands: region 1 to 2 (ω_{12}), and inside region 2 (ω_{22}).

The MFD relates the network's vehicle accumulation and the space-mean flow [15]. For an urban network divided into two regions (as in Fig. 2): the periphery (region 1) and the city center (region 2), a hybrid MFD-based model is formulated as follows (based on the two-state model in [16]):

$$\dot{n}_1(t) = -G_{1,j}(n_1(t)) \cdot u(t) + \omega_{12}(t), \tag{45}$$
$$\dot{n}_2(t) = -G_{2,j}(n_2(t)) + G_{1,j}(n_1(t)) \cdot u(t) + \omega_{22}(t), \tag{46}$$

with $n_i(t)$, the accumulation in region i at time t. The trip completion flow $G_{i,j}(n_i(t))$ (veh/s) is defined as the rate of vehicles reaching their destinations. The timing plans for intersections inside each region can be altered. Consequently, instead of one MFD, a set of MFDs (each corresponding to a different timing plan) is defined. Therefore, $G_{i,j}$, with $j = 1, \ldots, N_i$, constitute the MFDs for region i, with N_i the total number of MFDs for region i.

The perimeter control $u \in [0, 1]$ may restrict vehicles to transfer between regions (in our case, the flow of vehicles is restricted from region 1, the periphery, to region 2, the city center). The perimeter control can be realized by e.g. coordinating green and red durations of signalized intersections placed on the border between two regions.

We assume that each of the regions has three timing plans and therefore three MFDs ($N_1 = N_2 = 3$). Each MFD is modeled by an exponential function $G_{i,j}(n_2) = 1/3600 \cdot a_{i,j} \cdot n_i \cdot \exp(-1/2 \cdot (n_i/n_{i,\mathrm{crt},j})^2), i \in \{1, 2\}, j \in \{1, \ldots, 3\}$.

The parameters used in our simulation are as follows:

$$G_{1,j}: a_{1,1} = 17.8,\ a_{1,2} = 9.75,\ a_{1,3} = 13,\ n_{1,\mathrm{crt},j} = 3500$$
$$G_{2,j}: a_{2,j} = a_{1,j}/1.3,\ n_{2,\mathrm{crt},j} = n_{1,\mathrm{crt},j}/1.2,$$

Furthermore, the perimeter control input u can be assumed as a quantized input that can take values from a finite set. This is not a conservative assumption as in reality the perimeter control is realized by manipulating the green to red duration of traffic signals and investigations have shown that the evolution of flows is not very sensitive to small changes in the perimeter signal [8]. Therefore, we assume that the perimeter inputs can takes values from the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$.

By quantizing the perimeter input, the model (45)–(46) can be reformulated in the format of the switched system (1)–(3). The quantized perimeter input introduces 5 modes. Each region is assumed to have 3 MFDs. Therefore, the total number of modes (subsystems) will be $3 \times 3 \times 5 = 45$. The resulting system matrices are as follows[1]:

$$A_\ell = 0,\ B_\ell = 0,\ H_\ell = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_\ell = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{47}$$
$$E_1 = \begin{bmatrix} -0.1 \cdot a_{1,1}/3600 & 0 \\ 0.1 \cdot a_{1,1}/3600 & -a_{2,1}/3600 \end{bmatrix}, \cdots \tag{48}$$

The sector bounded nonlinear function $f = [f_1, f_2]^{\mathrm{T}}$ is:

$$f = \left[n_1 \cdot \exp^{\left(-0.5(\frac{n_1}{n_{1,\mathrm{crt}}})^2\right)}, n_2 \cdot \exp^{\left(-0.5(\frac{n_2}{n_{2,\mathrm{crt}}})^2\right)} \right]^{\mathrm{T}} \tag{49}$$

For the sector slopes we take $[\alpha_1, \beta_1] = [0.0168, 0.607]$ for f_1 and $[\alpha_2, \beta_2] = [0.0028, 0.655]$ for f_2. Moreover, the output function g in (3) is $[n_1, n_2]^{\mathrm{T}}$.

The assumed trip demands are depicted in Fig. 3. The uncertainty in the demands is modeled using zero mean white Gaussian noise with variance 0.1 (veh/s) added to the average profiles. Moreover, we have included a sudden jump in ω_{22} to evaluate the robustness of our control approach.

The matrices of the Lyapunov functions along with the minimum L_2-gain are determined *offline* by solving (35)–(37) using Yalmip toolbox. The measured accumulations are supplied to (22) to determine the active subsystem (and to obtain the specific MFD and the proper perimeter input). The results are depicted in Fig. 4. To demonstrate the effectiveness of the proposed method, the results are compared with two simple control strategies. In the first one, a greedy feedback controller is designed as follows: if both regions are uncongested, the perimeter input is maximized and if both regions are congested, the perimeter input is set to the minimum value, if region 2 is more congested than region 1, otherwise to maximum (further, we choose the MFDs with highest maximum flow for both regions). The second strategy (called protecting the center) is to protect the city center. If the center's accumulation is higher than the critical one, the perimeter control is set to minimum and

[1] Due to space limitation, we only mention E_1. For other subsystems the structure of the E matrix is the same, only the MFD coefficients $a_{i,j}$ and the value for the perimeter control input differ.

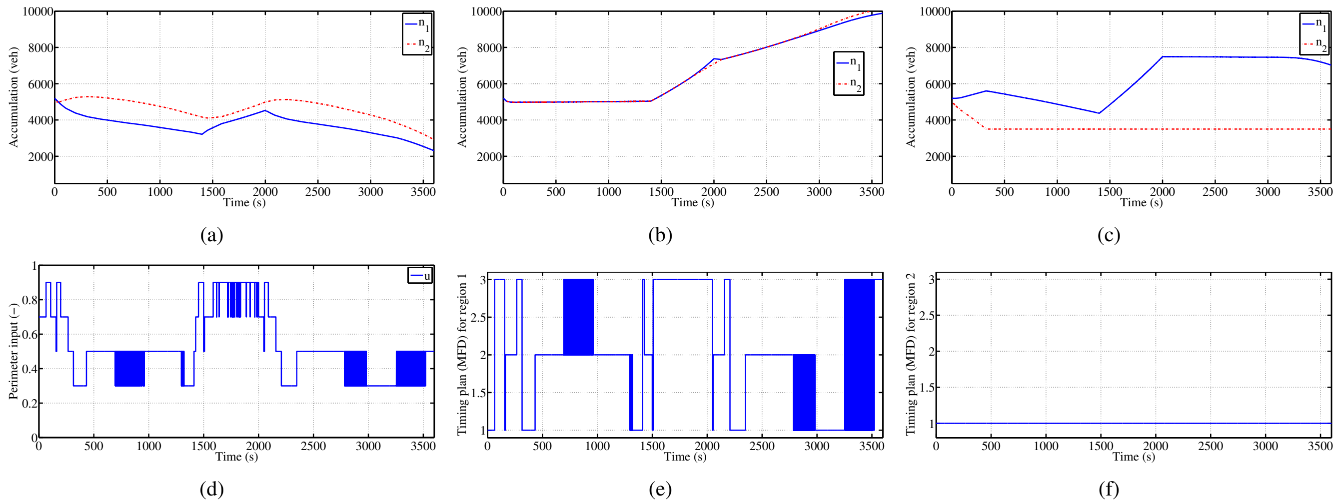

Fig. 4. Accumulations: (a) Robust switching control, (b) protecting the center, (c) greedy feedback control. Converted control inputs from the designed switching signal: (d) perimeter signal, (e) switching between MFDs of region 1, (f) switching between MFDs of region 2.

otherwise to maximum. As inferred from Fig. 4(a)-(c), the switching control stabilizes the system and also significantly reduces the effects of the trip demands, while in the other methods, one or both accumulations grow unboundedly.

Note that the proposed method is computationally efficient and can be applied in real-time (as it only requires computing 45 Lyapunov functions and determining the index of the active sub-system). This is a significant advantage over other existing approaches like MPC, which often needs online optimization. Furthermore, setting the initial accumulations to zero, the actual L_2-gain is $0.1237 \cdot 3600$ which is lower than the theoretical value $0.1418 \cdot 3600$ obtained by solving optimization problem (35)–(37).

VI. Conclusions and Future Work

We have presented stability analysis and H_∞ control for a class of switched nonlinear systems with arbitrary sector bounds. By combining multiple Lyapunov functions that contain both quadratic functions of states and integrals of nonlinearities, we formulated the robust state feedback controllers along with switching laws by solving an optimization problem subject to matrix inequalities. Further, to improve the computational efficiency, we proposed a transformation method together with LMI techniques to eventually obtain a convex optimization problem. Finally, we applied the proposed method to the problem of resolving congestion in urban networks modeled on a high-level. The obtained results showed significant performance in stabilizing the network and reducing the impacts of uncertain trip demands.

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