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Analytic expressions in stochastic max-plus-linear algebra

Ton J.J. van den Boom[1] and Bart De Schutter[1]

***Abstract*— In stochastic max-plus-linear systems one often needs to compute the expectation of a max-plus-scaling function. The algorithms available in literature are either too computationally expensive or only give an approximation. In this paper we derive an analytic expression for this expectation in the case of a uniform distribution, resulting in a piecewise polynomial function in the components of the free control variable. This function can be evaluated in a quick and efficient way.**

I. INTRODUCTION

Discrete-event models such as queuing systems, (extended) state machines, formal language models, automata, temporal logic models, generalized semi-Markov processes, Petri nets, etc. are in general nonlinear in conventional algebra. However, there exists an important class of discrete-event systems, namely the max-plus-linear (MPL) systems, for which the model is linear in the max-plus algebra. The class of max-plus linear systems consists of discrete-event systems with synchronization but no choice. Synchronization requires the availability of several resources at the same time, whereas choice appears, e.g., when a user has to choose among several resources [1]. Typical examples of such systems are serial production lines, production systems with a fixed routing schedule, and railway networks. In stochastic discrete-event systems, processing times and/or transportation times are assumed to be stochastic quantities, since in practice stochastic fluctuations in their values can, e.g. be caused by machine failure or depreciation [8]. To model this stochasticity in discrete-event systems one often uses stochastic max-plus-linear expressions or stochastic max-plus-scaling functions [1], [2], [5], [7]–[9]

To control stochastic max-plus-linear systems, an efficient control approach is model predictive control (MPC) [6]. MPC is an online model-based approach, in which at each event step an optimal control sequence is computed. This optimization is done over a finite sequence of events, and for each event step, only the first sample of the optimal control sequence will be applied to the system. For the next step, the horizon will be shifted forward and a new optimal control sequence will be computed.

Note that in the algorithm to solve the stochastic MPC problem, an optimization problem has to be solved at each event step. In stochastic systems, the objective function defined in the MPC optimization problem usually consists of an expected value of stochastic max-plus-scaling functions [10]. In general, the expected value is computed using either numerical integration or some available analytic approaches, which are all very time-consuming. Hence, solving this optimization problem creates a considerable computational complexity due to the presence of the expected value [4], [11].

In literature two approaches have been proposed to reduce the computational burden. The first approach [11] considers a method based on variability expansion. In particular, it has been shown that the computational load is reduced if one decreases the level of 'randomness' in the system. Another method [4] uses an approximation approach that is based on the pth order raw moments of a random variable. This method results in a much lower computational complexity and a much lower computation time. However because of the approximation, the performance will often degrade.

In this paper we aim for an *exact* computation of the expectation within a reasonable time. In particular, in this paper we derive an analytic solution to compute the expected value of max-plus-scaling functions in the presence of a uniform distribution. We show that the expectation is piecewise polynomial in general.

II. THE EXPECTATION OF A MAX-PLUS-SCALING FUNCTION

In this section we will give the problem definition.

Assume $\mathcal{E}$ and $\mathcal{W}$ are bounded polyhedral sets:

$$\mathcal{E} = \{e \in \mathbb{R}^{n\times 1} | C_e e \leq d_e\}$$

$$\mathcal{W} = \{w \in \mathbb{R}^{p\times 1} | C_w w \leq d_w\}$$

where $C_e \in \mathbb{R}^{n_e \times n}$, $d_e \in \mathbb{R}^{n_e \times 1}$, $C_w \in \mathbb{R}^{n_w \times p}$, and $d_w \in \mathbb{R}^{n_w \times 1}$. Let e be a stochastic variable with the probability density function $p(\cdot)$ with domain $\mathcal{E}$. In this paper we consider a uniform distribution, so the probability density function is given by:

$$p(e) = \begin{cases} 2^{-n} & \text{for } |e_i| \leq 1, \text{ for all } i = 1, \ldots, n \\ 0 & \text{elsewhere} \end{cases}$$

Let $\alpha \in \mathbb{R}^{m\times 1}$, $\beta \in \mathbb{R}^{m\times p}$, $\gamma \in \mathbb{R}^{m\times n}$, $w \in \mathcal{W}$, $e \in \mathcal{E} \subset \mathbb{R}^{n\times 1}$, and define the Max-Plus-Scaling (MPS) function $f : (\mathbb{R}\times\mathbb{R}) \to \mathbb{R}$ as follows:

$$f(w,e) = \max_j(\alpha_j + \beta_j\, w + \gamma_j\, e)$$

where β_j and γ_j stand for the jth row of β and γ, respectively. In this paper we aim to compute an analytic expression for the expectation of an MPS function of the form

$$h(w) = \mathbb{E}[f(w,e)] = \mathbb{E}[\max_j(\alpha_j + \beta_j\, w + \gamma_j\, e)] \quad (1)$$

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where $\mathbb{E}[f]$ denotes the expectation of the function f.

In [10] we derived an algorithm to compute this expectation as follows:

$$h(w) = \mathbb{E}[f(w,e)] = \underset{\mathcal{E}}{\int\cdots\int} \max_j \Big(\alpha_j + \beta_j w + \gamma_j e\Big)\, p(e)\, de_1 \cdots de_n$$

In [10] we have shown that for a given $j \in \{1, 2, \ldots, m\}$ and $w \in \mathcal{W}$ one can compute the non-degenerate polyhedral set $\Phi_j(w)$ such that for all $e \in \Phi_j(w)$ there holds:

$$f(w,e) = \alpha_j + \beta_j w + \gamma_j e$$

and

$$\bigcup_{j=1}^{m} \Phi_j(w) = \mathcal{W}$$

III. AN ANALYTIC EXPRESSION FOR $h(w)$

In this section we derive an analytic expression for $h(w)$. We do this in three steps. In the first step we show that the vertices of the polytopic sets $\Phi_j(w)$, $j \in \{1, 2, \ldots, m\}$ are piecewise affine in the parameter w (see Lemma 1). In the second step we divide each polytope $\Phi_j(w)$ into simplices[1]. In the third step we derive $h(w)$ by integration over the derived simplices (see Lemma 2).

Define for each j the set $\mathcal{I}_j = \{i_{j,1}, \ldots, i_{j,m-1}\} = \{1, 2, 3, \ldots, m\} \backslash \{j\}$, and define matrices $A_{w,j} \in \mathbb{R}^{(m-1)\times p}$, $A_{e,j} \in \mathbb{R}^{(m-1)\times n}$, $b_j \in \mathbb{R}^{(m-1)\times 1}$,

$$\begin{aligned} [b_j]_s &= \alpha_j - \alpha_{i_{j,s}} \ , \quad \text{for } i_{j,s} \in \mathcal{I}_j, \\ [A_{w,j}]_s &= -\beta_j + \beta_{i_{j,s}} \ , \quad \text{for } i_{j,s} \in \mathcal{I}_j, \\ [A_{e,j}]_s &= -\gamma_j + \gamma_{i_{j,s}} \ , \quad \text{for } i_{j,s} \in \mathcal{I}_j, \end{aligned}$$

for all $s = 1, \ldots, m-1$, where $[A]_s$ denotes the sth row of the matrix A. Define $\bar{A}_{e,j} \in \mathbb{R}^{q\times n}$, $\bar{A}_{w,j} \in \mathbb{R}^{q\times p}$, and $\bar{b}_j \in \mathbb{R}^{q\times 1}$ as follows:

$$\bar{A}_{e,j} = \begin{bmatrix} A_{e,j} \\ C_e \\ 0 \end{bmatrix}, \ \bar{A}_{w,j} = \begin{bmatrix} A_{w,j} \\ 0 \\ C_w \end{bmatrix}, \ \bar{b}_j = \begin{bmatrix} b_j \\ d_e \\ d_w \end{bmatrix}$$

Note that $q > n$. Let $\mathcal{S}_j = \{S_{j,1}, \ldots, S_{j,L_j}\}$ be the set of all $n \times q$ submatrices of the $q \times q$ identity matrix such that the matrix $S_{j,\ell}\, \bar{A}_{e,j}$ is invertible. For any matrix $S_{j,\ell} \in \mathcal{S}_j$ the remaining part of the $q \times q$ identity matrix will be denoted by $T_{j,\ell} \in \mathbb{R}^{(q-n)\times q}$.

The following lemma shows that the vertices of the set $\Phi_j(w)$ are piecewise affine in w.

Lemma 1: Define for $\ell = 1, \ldots, L_j$ the vectors $\sigma_{j,\ell} \in \mathbb{R}^{n\times 1}$, $g_{j,\ell} \in \mathbb{R}^{q-n\times 1}$ and matrices $\tau_{j,\ell} \in \mathbb{R}^{n\times p}$, $F_{j,\ell} \in \mathbb{R}^{(q-n)\times p}$ as follows:

$$\sigma_{j,\ell} = (S_{j,\ell}\, \bar{A}_{e,j})^{-1} S_{j,\ell} \bar{b}_j \tag{2}$$

$$\tau_{j,\ell} = (S_{j,\ell}\, \bar{A}_{e,j})^{-1} S_{j,\ell} \bar{A}_{w,j} \tag{3}$$

$$F_{j,\ell} = T_{j,\ell}\, (I - \bar{A}_{e,j}(S_{j,\ell}\, \bar{A}_{e,j})^{-1} S_{j,\ell}) \bar{A}_{w,j} \tag{4}$$

$$g_{j,\ell} = T_{j,\ell}\, (I - \bar{A}_{e,j}(S_{j,\ell}\, \bar{A}_{e,j})^{-1} S_{j,\ell}) \bar{b}_j \tag{5}$$

Now let $w \in \mathcal{W}_{j,\ell} = \{w | F_{j,\ell} w \le g_{j,\ell}\}$. Then

$$v_{j,\ell}(w) = \sigma_{j,\ell} + \tau_{j,\ell}\, w \tag{6}$$

is a vertex of $\Phi_j(w)$.

Proof: Let us consider all $e \in \mathcal{E}$ and $w \in \mathcal{W}$ such that

$$\alpha_j + \beta_j\, w + \gamma_j\, e \ge \alpha_i + \beta_i\, w + \gamma_i\, e \ , \ \ \forall i \ne j \tag{7}$$

Now finding all $e \in \mathcal{E}$ and $w \in \mathcal{W}$ for which condition (7) holds, can be replaced by finding all $e \in \mathbb{R}^{n\times 1}$ and $w \in \mathbb{R}^{p\times 1}$ such that

$$\begin{bmatrix} A_{e,j} \\ C_e \\ 0 \end{bmatrix} e + \begin{bmatrix} A_{w,j} \\ 0 \\ C_w \end{bmatrix} w \le \begin{bmatrix} b_j \\ d_e \\ d_w \end{bmatrix}$$

or

$$\bar{A}_{e,j}\, e + \bar{A}_{w,j}\, w \le \bar{b}_j$$

or

$$\bar{A}_{e,j}\, e \le \bar{b}_j - \bar{A}_{w,j}\, w \tag{8}$$

The matrix $S_{j,\ell}$ will select constraints from (8) that are active and $T_{j,\ell}$ will select constraints from (8) that are inactive. For $v_{j,\ell}$ to be a vertex of the polyhedral set $\Phi_j(w)$, we need the following properties:

$$S_{j,\ell}\, \bar{A}_{e,j}\, v_\ell = S_{j,\ell}(\bar{b}_j - \bar{A}_{w,j}\, w) \tag{9}$$

$$T_{j,\ell}\, \bar{A}_{e,j}\, v_\ell \le T_{j,\ell}(\bar{b}_j - \bar{A}_{w,j}\, w) \tag{10}$$

$$\det(S_{j,\ell}\, \bar{A}_{e,j}) \ne 0 \tag{11}$$

From (9) and (11) we derive

$$v_{j,\ell} = (S_{j,\ell}\, \bar{A}_{e,j})^{-1}\, S_{j,\ell}(\bar{b}_j - \bar{A}_{w,j}\, w) \tag{12}$$

$$= \sigma_{j,\ell} + \tau_{j,\ell}\, w \tag{13}$$

Substitution of (12) in (10) gives

$$T_{j,\ell}\, \bar{A}_{e,j} (S_{j,\ell}\, \bar{A}_{e,j})^{-1}\, S_{j,\ell}(\bar{b}_j - \bar{A}_{w,j}\, w) \le T_{j,\ell}(\bar{b}_j - \bar{A}_{w,j}\, w) \tag{14}$$

or

$$F_{j,\ell} w \le g_{j,\ell} \tag{15}$$

This means that (13) is a vertex if (15) is satisfied. ■

We now use the following recursive procedure to divide the non-degenerate polytope $\Phi_j(w)$ into $K_j(w)$ non-degenerate simplices $\Omega_{j,k}(w),\ k = 1, \ldots, K_j(w)$. We start by considering each 2-dimensional face of the polytope. We select the geometric center of the face and connect that to

[1] A simplex in $\mathbb{R}^n$ is a n-dimensional polytope which is the convex hull of its $n+1$ vertices.

each of the vertices of the given face. In this way each 2-dimensional face can be partitioned into simplices with 3 vertices. We consider all 3-dimensional faces and construct 3-dimensional simplices by connecting the geometric center of each of the 3-dimensional faces with all the vertices of the simplices of the 2-dimensional subfaces of the given 3-dimensional face. We continue in this way until the full n-dimensional polytope $\Phi_j(w)$ has been divided into n-dimensional simplices. Note that the geometric center of a polytope is a convex combination of the vertices of that polytope. This means that the vertices of the n-dimensional simplices are convex combinations of the vertices of the polytope $\Phi_j(w)$.

Consider one of the simplices $\Omega_{j,k}(w)$ and denote the vertices of this simplex by $\bar{v}_{j,k,0}, \bar{v}_{j,k,1}, \bar{v}_{j,k,2}, \cdots, \bar{v}_{j,k,n}$. The simplex $\Omega_{j,k}(w)$ is now given by[2]:

$$\Omega_{j,k}(w) = \mathbf{Co}(\bar{v}_{j,k,0}, \bar{v}_{j,k,1}, \bar{v}_{j,k,2}, \cdots, \bar{v}_{j,k,n})$$

Define

$$h_{j,k}(w) = \int\cdots\int_{\Omega_{j,k}(w)} \Big(\alpha_j + \beta_j w + \gamma_j e\Big) p(e)\, de_1 \cdots de_n$$

Then $h(w)$ can be computed by

$$h(w) = \sum_{j=1}^{m} \sum_{k=1}^{K_j(w)} h_{j,k}(w) \tag{16}$$

For a fixed j and $w \in \mathcal{W}$, let $v_{j,\ell}(w)$, $\ell = 1, \ldots, L_j$ be the vertices of the polytope $\Phi_j(w)$. The vertices $\bar{v}_{j,k,i}(w)$, $i = 0, \ldots, n$ of the simplex $\Omega_{j,k}(w)$, $k \in \{1, \ldots, K_j\}$ will be convex combinations of the vertices $v_{j,\ell}(w)$, $\ell = 1, \ldots, L_j$. In other words, there exist parameters $\lambda_{j,i,k,\ell}$ such that

$$\bar{v}_{j,k,i}(w) = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell} v_{j,\ell}(w)$$

where $\lambda_{j,i,k,\ell}$ does not depend on w. (Because we used geometric centers to construct the simplices. If the vertices of a polytope are affine in w, also the geometric center will be affine in w.) Now define $\bar{\sigma}_{j,k,i} = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell} \sigma_{j,\ell}$ and $\bar{\tau}_{j,k,i} = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell} \tau_{j,\ell}$ then using (6) we find

$$\bar{v}_{j,k,i}(w) = \bar{\sigma}_{j,k,i} + \bar{\tau}_{j,k,i}\, w\,. \tag{17}$$

The following lemma gives an analytic expression for the value $h_{j,k}(w)$.

Lemma 2: Consider the simplex

$$\Omega_{j,k}(w) = \mathbf{Co}(\bar{v}_{j,k,0}(w), \bar{v}_{j,k,1}(w), \cdots, \bar{v}_{j,k,n}(w)) \tag{18}$$

with vertices affine in w according to (17). Define

$$\begin{aligned} &V_{j,k}(w) \\ &= \big[\, \bar{v}_{j,k,1}(w) - \bar{v}_{j,k,0}(w) \;\; \bar{v}_{j,k,2}(w) - \bar{v}_{j,k,0}(w) \\ &\qquad \cdots \;\; \bar{v}_{j,k,n}(w) - \bar{v}_{j,k,0}(w) \,\big] \\ &= V_{j,k,0} + \sum_{\ell=1}^{p} V_{j,k,\ell}\, w_\ell \end{aligned}$$

then

$$\begin{aligned} h_{j,k}(w) &= \int\cdots\int_{\Omega_{j,k}(w)} \Big(\alpha_j + \beta_j w + \gamma_j e\Big) p(e)\, de_1 \cdots de_n \\ &= \frac{\det V_{j,k}(w)}{(n+1)!} \Big((n+1)(\alpha_j + \beta_j w) + \gamma_j \bar{\sigma}_{j,k,0} + \\ &\quad + \gamma_j \bar{\tau}_{j,k,0}\, w + \gamma_j \bar{\sigma}_{j,k,1} + \gamma_j \bar{\tau}_{j,k,1}\, w + \\ &\quad + \cdots + \gamma_j \bar{\sigma}_{j,k,n} + \gamma_j \bar{\tau}_{j,k,n}\, w \Big) \end{aligned}$$

Hence, $h_{j,k}(w)$ is an $(n+1)$st order polynomial function in w.

Proof: First consider the simplex

$$\Omega' = \{\lambda \in \mathbb{R}^n \mid \lambda_i \geq 0, \sum \lambda_i \leq 1\}$$

and the integral over this simplex of an affine function $(\kappa + \mu\lambda)$, where $\kappa \in \mathbb{R}$, and $\mu \in \mathbb{R}^{1\times n}$. Then we can write

$$\begin{aligned} &\int_0^1 \int_0^{1-\lambda_1} \int_0^{1-\lambda_1-\lambda_2} \cdots \int_0^{1-\lambda_1-\lambda_2\cdots-\lambda_n} \\ &\qquad (\kappa + \mu\lambda) d\lambda_1 d\lambda_2 d\lambda_3 \cdots d\lambda_n \\ &\qquad\qquad = \frac{(n+1)\kappa + \mu_1 + \mu_2 + \cdots + \mu_n}{(n+1)!} \end{aligned}$$

Simplex (18) can now be written as

$$\Omega_{j,k} = \{e = \bar{v}_{j,k,0} + V_{j,k}(w)\, \lambda | \lambda \in \Omega'\}$$

Consider the coordinate transformation:

$$e = \bar{v}_{j,k,0}(w) + V_{j,k}(w)\, \lambda$$

Then

$$de_1 de_2 \cdots de_n = \det V_{j,k}(w) d\lambda_1 d\lambda_2 \cdots d\lambda_n$$

$$\begin{aligned} &\alpha_j + \beta_j w + \gamma_j e \\ &\quad = \alpha_j + \beta_j w + \gamma_j\, \bar{v}_{j,k,0}(w) + \gamma_j\, V_{j,k}(w)\, \lambda \\ &\quad = \kappa(w) + \mu(w)\, \lambda \end{aligned}$$

[2]The convex hull of the vectors $(x_1, \ldots, x_n)$ is defined by: $\mathbf{Co}(x_1, \ldots, x_n) = \{y | y = \lambda_1 x_1 + \ldots + \lambda_n x_n,\ 0 \leq \lambda_i \leq 1,\ \lambda_1 + \ldots + \lambda_n = 1\ \}$

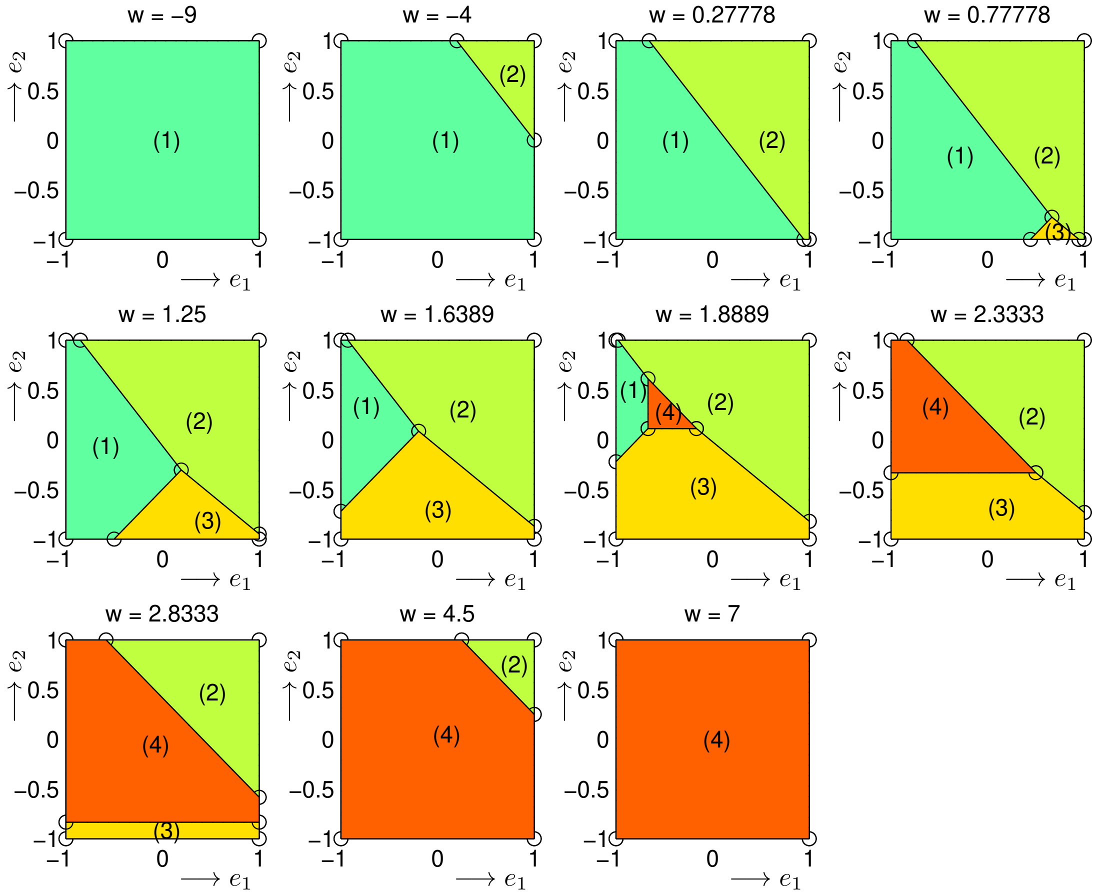

Fig. 1. The regions $\Phi_j(w)$, $j = 1, \ldots, 4$ for different values of w. The vertices are denoted by small circles (○) .

and (dropping (w) for easier notation)

$$\begin{aligned}
h_{j,k}(w) &= \int_{\Omega}\!\!\cdots\!\int \Big(\alpha_j+\beta_j w+\gamma_j e\Big)\, de_1 \cdots de_n \\
&= \int_0^1 \int_0^{1-\lambda_1} \int_0^{1-\lambda_1-\lambda_2} \cdots \int_0^{1-\lambda_1-\lambda_2\cdots-\lambda_n} \\
&\quad \det V_{j,k}(w)(\kappa+\mu\lambda) d\lambda_1 d\lambda_2 d\lambda_3 \cdots d\lambda_n \\
&= \frac{1}{(n+1)!}\Big((n+1)\kappa + \mu_1 + \mu_2 + \cdots + \mu_n\Big) \\
&= \frac{1}{(n+1)!}\Big((n+1)(\alpha_j+\beta_j w + \gamma_j\, \bar{v}_{j,k,0}) \\
&\quad + \gamma_j(\bar{v}_{j,k,1}-\bar{v}_{j,k,0}) + \gamma_j(\bar{v}_{j,k,2}-\bar{v}_{j,k,0}) + \\
&\quad + \cdots + \gamma_j(\bar{v}_{j,k,n}-\bar{v}_{j,k,0})\Big) \\
&= \frac{\det V_{j,k}(w)}{(n+1)!}\Big((n+1)(\alpha_j+\beta_j w) + \gamma_j \bar{v}_{j,k,0} \\
&\quad + \gamma_j \bar{v}_{j,k,1} + \gamma_j \bar{v}_{j,k,2} + \cdots + \gamma_j \bar{v}_{j,k,n}\Big) \\
&= \frac{\det V_{j,k}(w)}{(n+1)!}\Big((n+1)(\alpha_j+\beta_j w) + \gamma_j \bar{\sigma}_{j,k,0} + \\
&\quad + \gamma_j \bar{\tau}_{j,k,0}\, w + \gamma_j \bar{\sigma}_{j,k,1} + \gamma_j \bar{\tau}_{j,k,1}\, w + \\
&\quad + \cdots + \gamma_j \bar{\sigma}_{j,k,n} + \gamma_j \bar{\tau}_{j,k,n}\, w\Big)
\end{aligned}$$

Due to (17) we know that

$$V_{j,k}(w) = V_{j,k,0} + \sum_{\ell=1}^{p} V_{j,k,\ell}\, w_\ell$$

and so $\det V_{j,k}(w)$ will be an nth order polynomial function in the components of w. This means that $h_{j,k}(w)$ is an $(n+1)$st order polynomial function in w. ■

Theorem 3: For a fixed $w \in \mathcal{W}$, let $s_1, s_2, \ldots, s_n$ be such that

$$w \in \mathcal{W}_{j,s_j} \text{ for } j = 1, \ldots, n$$

where $\mathcal{W}_{j,s_j}$ has been defined in Lemma 1. Define

$$h_j(w) = h_{j,s_j}(w) \text{ for } w \in \mathcal{W}_{j,s_j},\; j = 1, \ldots, n \qquad (19)$$

Then for $w \in \mathcal{W}_{j,s_j}$ we find that

$$h(w) = \sum_{j=1}^{m} h_j(w)$$

is a piecewise $(n+1)$th order polynomial function in w.

Proof: This immediately follows from (16) combined with Lemmas 1 and 2. ■

From [10] we know that h is also a continuous and convex function, and so a subgradient can easily be determined by computing the local derivative of the polynomial function h.

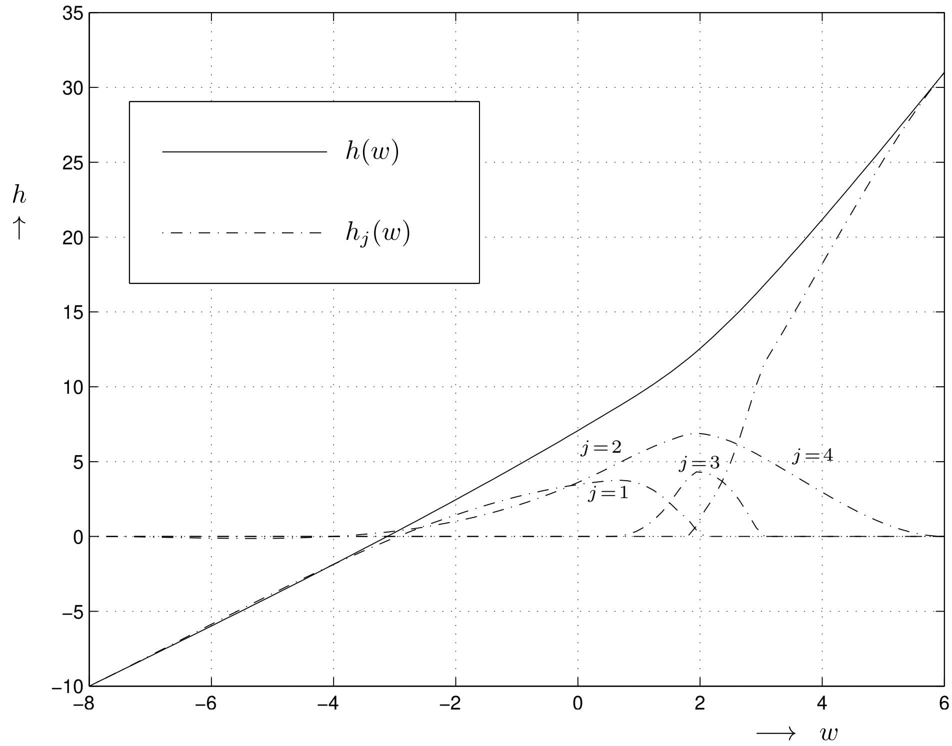

Fig. 2. The function $h(w)$ for $-8 \le w \le 6$ with the corresponding functions $h_j(w)$ as defined in (19).

IV. EXTENSION TO OTHER PROBABILITY DENSITY FUNCTIONS

In the previous sections we have considered a uniform distribution. Also other probability density functions (pdfs) are possible. For piecewise affine or piecewise polynomial pdfs the methods derived in this paper can be extended and the expectation h can be computed analytically. Also pdfs that are piecewise multiply integrable (so consisting of polynomials, exponentials, sine and cosine functions) lead to an analytic solution. A disadvantage then is that the complexity of the analytic expression will increase with an increasing number of regions in the piecewise functions. In addition, note that any smooth distribution can be approximated satisfactory by the above piecewise pdfs.

V. EXAMPLE

In this example we compute a piecewise polynomial expression for the following expression:

$$h(w) = \mathbb{E}\Big\{\max(6+2w+e_2, 5+3w+5e_1+5e_2, 3+4w+e_1, 1+5w+e_1+e_2)\Big\}$$

so for

$$\alpha = \begin{bmatrix} 6 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & 1 \\ 5 & 5 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

if the MPS function is written in the form (1).

We compute the vertices of the regions $\Phi_j(w)$ for $j = 1, \ldots, 4$. Fig. 1 shows these regions $\Phi_j(w)$ for different values of w. The functions $F_{j,k}$, $g_{j,k}$, $\tau_{j,k}$ and $\sigma_{j,k}$ can be computed using Lemma 1, and with these we can compute $h_{j,k}$ and using Theorem 3 we then compute h_j and h. The resulting function h is given in Table I. The functions h and h_j, $j = 1, 2, 3, 4$ are plotted in Fig. 2 for $-8 \le w \le 6$. We clearly see that the function h is a piecewise polynomial function in the variable w.

VI. CONCLUSION

In this paper we have presented an analytic piecewise polynomial function expression for the expectation of a max-plus linear expression, in which the terms in the max-plus expression are affine in a control variable and affine in a stochastic variable with a uniform distribution.
In future research we will study the application of the analytic expression for the use in MPC of stochastic max-plus-linear systems and/or stochastic max-min-plus-scaling (MMPS) systems [3].

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$w \in [-\infty, -8)$:	$h(w) = 2w + 6$
$w \in [-8, 0)$:	$h(w) = 0.00208w^3 + 0.05w^2 + 2.4w + 7.067$
$w \in [0, 0.556)$:	$h(w) = 0.05w^2 + 2.4w + 7.067$
$w \in [0.556, 1)$:	$h(w) = 0.169w^3 - 0.231w^2 + 2.56w + 7.03$
$w \in [1, 1.5)$:	$h(w) = 0.167w^3 - 0.225w^2 + 2.55w + 7.4$
$w \in [1.5, 1.78)$:	$h(w) = -0.167w^3 + 1.275w^2 + 0.3w + 8.1648$
$w \in [1.78, 2)$:	$h(w) = 1.52w^3 - 7.72w^2 + 16.3w - 1.32$
$w \in [2, 2.67)$:	$h(w) = 0.0917w^3 - 0.275w^2 + 3.65w + 5.62$
$w \in [2.67, 3)$:	$h(w) = -0.0208w^3 + 0.625w^2 + 1.25w + 7.75$
$w \in [3, 6)$:	$h(w) = -0.0208w^3 + 0.375w^2 + 2.75w + 5.5$
$w \in [6, \infty)$:	$h(w) = 5w + 1$

TABLE I

THE FUNCTION $h(w)$ FOR DIFFERENT RANGES OF w (FOR EASE OF NOTATION WE LIST ROUNDED VALUES FOR THE PARAMETERS).

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