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Backward Reachability of Autonomous Max-Plus-Linear Systems ⋆

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Abstract: This work discusses the backward reachability of autonomous Max-Plus-Linear (MPL) systems, a class of continuous-space discrete-event models that are relevant for applications dealing with synchronization and scheduling. Given an MPL system and a continuous set of final states, we characterize and compute its "backward reach tube" and "backward reach sets," namely the set of states that can reach the final set within a given event interval or at a fixed event step, respectively. We show that, in both cases, the computation can be done exactly via manipulations of difference-bound matrices. Furthermore, we illustrate the application of the backward reachability computations over safety and transient analysis of MPL systems.

Keywords: Backward reachability analysis, backward reach tube, max-plus-linear systems, piecewise affine systems, difference-bound matrices, safety and transient analysis

1. INTRODUCTION

Max-Plus-Linear (MPL) systems are discrete-event models (Baccelli et al., 1992; Hillion and Proth, 1989; Cuninghame-Green, 1979) with a continuous state space characterizing the timing of the underlying discrete events. MPL systems are predisposed to describe the timing synchronization between interleaved processes, under the assumption that timing events are linearly dependent (within the max-plus algebra) on previous event occurrences (cf. Section 2). These models are widely employed in the analysis and scheduling of infrastructure networks, such as communication and railway systems (Heidergott et al., 2006), production and manufacturing lines (Roset et al., 2005; van Eekelen et al., 2006), as well as in biological systems (Brackley et al., 2012). They cannot model concurrency and are related to a subclass of timed Petri nets, namely timed-event graphs (Baccelli et al., 1992).

Reachability analysis of MPL systems from a *single* initial condition has been investigated in (Gazarik and Kamen, 1999; Gaubert and Katz, 2003), by computing the reachability matrix as in the case of discrete-time linear dynamical systems. It has been shown in (Gaubert and Katz, 2006, Sec. 4.13) that the reachability problem for autonomous MPL systems with a single initial condition is decidable – this result however does not hold for a general, uncountable set of initial conditions. Furthermore, the existing literature does not deal with backward reachability analysis, which would require expressing the set of final conditions as a max-plus convex cone (Gaubert and Katz, 2007). Furthermore, the computation would need the system matrix to be max-plus invertible. A matrix is max-plus invertible iff there is a single finite element (not equal to $-\infty$) in each row and in each column. In conclusion, these assumptions limit the applicability of the approach.

In this work, we extend the state-of-the-art results in backward reachability analysis of MPL systems by presenting a computational approach that can handle state matrices that are not max-plus invertible and further manage problems over an arbitrary (possibly uncountable) set of final conditions. We start by characterizing MPL systems alternatively by Piece-wise Affine (PWA) systems, and show that the dynamics can be fully represented by Difference-Bound Matrices (DBM) (Dill, 1990, Sec. 4.1), which are structures that are quite simple to manipulate computationally. Furthermore, one can show that DBM are closed under PWA dynamics, which leads to being able to compute a set of states that is mapped to given DBM-sets through an MPL system. Given a set of final states, we then compute its "backward reach tube" and the collection of "backward reach sets," namely the set of states that can arrive at the final states in any number of steps and in a fixed number of steps, respectively. We further describe two alternative approaches to compute the latter quantities.

Closely related to backward reachability is the problem of safety analysis (Mitchell, 2007): given an unsafe set over the state space, it is of interest to determine whether trajectories of the model enter the unsafe set – either at a given event step, or over an events interval. If the model

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is not safe, we can seek the subset of initial conditions leading to the unsafe set by using backward reachability analysis.

In addition to general safety analysis, we show that backward reachability is specifically helpful in the transient analysis of MPL systems. According to the max-plus algebra analogue of the Perron-Frobenius theorem (Baccelli et al., 1992, Sec. 3.7), if the system matrix is irreducible, there exists a periodic behavior ensuing after some event index. The smallest of such indices is called the length of the transient part, which is used in the literature to characterize model performance. For example in transportation networks, whenever there is a delay, the transient determines the worst-case recovery time. Moreover in the case of link reversal routing (Gafni and Bertsekas, 1987), it is equal to the time complexity of the routing algorithm. Hartmann and Arguelles (1999) established an upper bound on the length of the transient part of general MPL systems via graph-theoretical techniques. Under the assumption of integer delays Charron-Bost et al. (2013) employed algebraical approaches to obtain an upper bound. In this work, we show how backward reachability analysis can be used to determine the length of the transient part of a model (given via its system matrix), for any desired initial state: this generalizes related results in the literature. The set of final conditions for this backward reachability problem is defined as the set of states with zero length of the transient part, namely the states for which the periodic behavior occurs immediately.

The article is structured as follows. Section 2 introduces models and notions needed to tackle the problem at hand. Section 3 discusses the procedure for backward reachability analysis. Section 4 describes applications of backward reachability in safety and transient analysis. Finally, Section 5 presents conclusions and discusses future work.

2. MODELS AND PRELIMINARIES

2.1 Max-Plus-Linear Systems

Define $\mathbb{R}_\varepsilon$, ε, and e respectively as $\mathbb{R} \cup \{\varepsilon\}$, $-\infty$, and 0. For $\alpha, \beta \in \mathbb{R}_\varepsilon$, introduce the two operations:

$$\alpha \oplus \beta = \max\{\alpha, \beta\} \qquad \text{and} \qquad \alpha \otimes \beta = \alpha + \beta,$$

where the element ε is considered to be absorbing w.r.t. $\otimes$ (Baccelli et al., 1992, Definition 3.4). Given $\beta \in \mathbb{R}$, the max-algebraic power of $\alpha \in \mathbb{R}$ is denoted by $\alpha^{\otimes\beta}$ and corresponds to $\alpha\beta$ in the conventional algebra. The rules for the order of evaluation of the max-algebraic operators correspond to those of conventional algebra: max-algebraic power has the highest priority, and max-algebraic multiplication has a higher precedence than max-algebraic addition (Baccelli et al., 1992, Sec. 3.1).

The basic max-algebraic operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m\times n}; C \in \mathbb{R}_\varepsilon^{n\times p}$; and $\alpha \in \mathbb{R}_\varepsilon$

$$\begin{aligned}
[\alpha \otimes A](i,j) &= \alpha \otimes A(i,j),\\
[A \oplus B](i,j) &= A(i,j) \oplus B(i,j),\\
[A \otimes C](i,j) &= \bigoplus_{k=1}^{n} A(i,k) \otimes C(k,j),
\end{aligned}$$

for all i, j. Notice the analogy between $\oplus$, $\otimes$ and $+$, $\times$ for matrix and vector operations in conventional algebra. Given $m \in \mathbb{N}$, the m-th max-algebraic power of $A \in \mathbb{R}_\varepsilon^{n\times n}$ is denoted by $A^{\otimes m}$ and corresponds to $A \otimes \cdots \otimes A$ (m times). Notice that $A^{\otimes 0}$ is an n-dimensional max-plus identity matrix, i.e. the diagonal and nondiagonal elements are e and ε, respectively. In this paper, the following notation is adopted for reasons of convenience. A vector with each component that is equal to 0 (resp., $-\infty$) is also denoted by e (resp., ε). Furthermore, for practical reasons, the state space is taken to be $\mathbb{R}^n$, which also implies that the system matrix A has to be row-finite (cf. Definition 1).

Definition 1. (Cuninghame-Green, 1979). A max-plus matrix is called regular (or row-finite) if it contains at least one element different from ε in each row. □

An autonomous MPL system (Baccelli et al., 1992, Remark 2.75) is defined as:

$$x(k) = A \otimes x(k-1),$$

where $A \in \mathbb{R}_\varepsilon^{n\times n}$, $x(k-1) = [x_1(k-1) \dots x_n(k-1)]^T \in \mathbb{R}^n$ for $k \in \mathbb{N}$. The independent variable k denotes an increasing discrete-event counter, whereas the state variable x defines the (continuous) timing of the discrete events. Autonomous MPL systems are characterized by deterministic dynamics, namely they are not affected by exogenous inputs.

Related to matrix A is the notion of precedence (or communication) graph and of regular (or row-finite) matrix.

Definition 2. (Baccelli et al., 1992, p. 39). The precedence graph of $A \in \mathbb{R}_\varepsilon^{n\times n}$, denoted by $\mathcal{G}(A)$, is a weighted directed graph with vertices $1, \dots, n$ and arc (j, i) with weight $A(i, j)$ for each $A(i, j) \neq \varepsilon$. □

Example 1. Consider the following autonomous MPL system from (Heidergott et al., 2006, Sec. 0.1), representing the scheduling of train departures from two connected stations $i = 1, 2$ ($x_i(k)$ denotes the time of the k-th departure from station i):

$$\begin{aligned}
x(k) &= \begin{bmatrix} 2 & 5 \\ 3 & 3 \end{bmatrix} \otimes x(k-1), \text{ or equivalently,}\\
\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} &= \begin{bmatrix} \max\{2 + x_1(k-1), 5 + x_2(k-1)\} \\ \max\{3 + x_1(k-1), 3 + x_2(k-1)\} \end{bmatrix}.
\end{aligned} \tag{1}$$

Notice that A is a row-finite matrix. Its precedence graph is shown in Fig. 1. □

The notion of irreducible matrix, to be used shortly, can be given via that of precedence graph.

Definition 3. (Baccelli et al., 1992, Th. 2.14). A max-plus matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is called irreducible if its precedence graph $\mathcal{G}(A)$ is strongly connected. □

Recall that a directed graph is strongly connected if for any pair of distinct vertices i, j of the graph, there exists a path from i to j (Baccelli et al., 1992, p. 37). From a max-algebraic perspective, a matrix A is irreducible if the non-diagonal elements of $\bigoplus_{k=1}^{n-1} A^{\otimes k}$ are finite (not equal to ε).

Example 2. For the preceding example (1), since $A(1,2) \neq \varepsilon \neq A(2,1)$, the matrix A is irreducible. Equivalently, notice that the precedence graph in Fig. 1 is strongly connected. $\square$

If a max-plus matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible, there exists a unique max-plus eigenvalue $\lambda \in \mathbb{R}$ (Baccelli et al., 1992, Th. 3.23) and a corresponding eigenspace $E(A) = \{x \in \mathbb{R}^n : A \otimes x = \lambda \otimes x\}$ (Baccelli et al., 1992, Sec. 3.7.2). Recall that an eigenspace is a max-plus linear combination of finitely many vectors, which can be formulated as a union of finitely many DBM (Adzkiya et al., 2013, p. 3047). It has been studied under the names of max-plus convex cone in (Gaubert and Katz, 2007) or semimodules in (Cohen et al., 2004) or of idempotent space in (Litvinov et al., 2001).

Proposition 4. (Baccelli et al., 1992, Sec. 3.7). Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ be an irreducible matrix with max-plus eigenvalue $\lambda \in \mathbb{R}$. There exist $k_0, c \in \mathbb{N}$ such that $A^{\otimes(k+c)} = \lambda^{\otimes c} \otimes A^{\otimes k}$, for all $k \geq k_0$. The smallest k_0 and c verifying the property are defined as the length of the transient part and the cyclicity, respectively. $\square$

Proposition 4 allows to establish the existence of a periodic behavior. Given an initial condition $x(0) \in \mathbb{R}^n$, there exists a finite $k_0(x(0))$, such that $x(k+c) = \lambda^{\otimes c} \otimes x(k)$, for all $k \geq k_0(x(0))$. Notice that we can seek a specific length of the transient part $k_0(x(0))$, in general less conservative than the global $k_0 = k_0(A)$, as in Proposition 4. Upper bounds for the length of transient part k_0, and for its computation, have been discussed in (Charron-Bost et al., 2013; Hartmann and Arguelles, 1999).

The complete set of periodic behaviors are encompassed by the eigenspace of $A^{\otimes c}$, where c is the cyclicity of A. It is formulated as $E(A^{\otimes c}) = \{x \in \mathbb{R}^n : A^{\otimes c} \otimes x = \lambda^{\otimes c} \otimes x\}$ and contains the eigenspace of A, i.e. $E(A) \subseteq E(A^{\otimes c})$.

Example 3. For the numerical example in (1), from Proposition 4 we obtain a max-plus eigenvalue $\lambda = 4$, cyclicity $c = 2$, and a (global) length of the transient part $k_0 = 2$. The specific length of the transient part for $x(0) = [0, 1]^T$ can be computed observing the following trajectory

$$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}, \begin{bmatrix}9\\9\end{bmatrix}, \begin{bmatrix}14\\12\end{bmatrix}, \begin{bmatrix}17\\17\end{bmatrix}, \begin{bmatrix}22\\20\end{bmatrix}, \begin{bmatrix}25\\25\end{bmatrix}, \begin{bmatrix}30\\28\end{bmatrix}, \ldots$$

Notice that the periodic behavior occurs (only) after 1 event step, i.e. $k_0([0, 1]^T) = 1$, and shows a period equal to 2, namely $x(3) = 4^{\otimes 2} \otimes x(1) = 8 + x(1)$. Furthermore, we have that $x(k + 2) = 8 \otimes x(k)$, for $k \geq 1$.

The eigenspace of A is $E(A) = \{x \in \mathbb{R}^2 : x_1 - x_2 = 1\}$ and the complete periodic behaviors are $E(A^{\otimes 2}) = \{x \in \mathbb{R}^2 : 0 \leq x_1 - x_2 \leq 2\}$. $\square$

2.2 Piecewise Affine Systems

This section discusses Piece-wise Affine (PWA) systems (Sontag, 1981) generated by an autonomous MPL system. In the following section, PWA systems will play an important role in backward reachability analysis. PWA systems are characterized by a cover of the state space, and by affine (linear plus constant) dynamics that are active within each set of the cover.

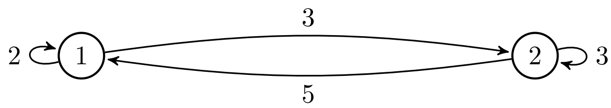

Fig. 1. Precedence graph of A for the MPL system in (1).

Every autonomous MPL system characterized by a row-finite matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ can be expressed as a special PWA system in the event domain. More precisely, the affine dynamics can be obtained, along with the corresponding region in the state space, from an n-tuple $g = (g_1, \ldots, g_n) \in \{1, \ldots, n\}^n$. For each i, the coefficient g_i characterizes the maximal term in the i-th state equation $x_i(k) = \max\{A(i,1) + x_1, \ldots, A(i,n) + x_n\}$, namely it holds that $A(i,j) + x_j \leq A(i,g_i) + x_{g_i}$, $\forall j \in \{1, \ldots, n\}$. It follows that the set of states corresponding to a particular g, denoted by R_g, is

$$R_g = \bigcap_{i,j=1}^{n} \{x \in \mathbb{R}^n : A(i, g_i) + x_{g_i} \geq A(i,j) + x_j\}. \quad (2)$$

The affine dynamics that are active in R_g follow directly from the definition of g (see previous paragraph), namely

$$x_i(k) = x_{g_i}(k-1) + A(i, g_i), \qquad i = 1, \ldots, n. \quad (3)$$

Notice that the discussed affine dynamics are a special case of general PWA dynamics as in (Sontag, 1981, Sec. 1).

Example 4. With reference to the autonomous MPL example in (1), the obtained PWA system is

$$x(k) = \begin{cases} \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} x(k-1) + \begin{bmatrix}2\\3\end{bmatrix}, \text{ if } x(k-1) \in R_{(1,1)}, \\ \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} x(k-1) + \begin{bmatrix}5\\3\end{bmatrix}, \text{ if } x(k-1) \in R_{(2,1)}, \\ \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} x(k-1) + \begin{bmatrix}5\\3\end{bmatrix}, \text{ if } x(k-1) \in R_{(2,2)}, \end{cases}$$

where $R_{(1,1)} = \{x \in \mathbb{R}^2 : x_1 - x_2 \geq 3\}$, $R_{(2,1)} = \{x \in \mathbb{R}^2 : e \leq x_1 - x_2 \leq 3\}$, $R_{(2,2)} = \{x \in \mathbb{R}^2 : x_1 - x_2 \leq e\}$, as depicted in Fig. 2 (left). $\square$

2.3 Difference-Bound Matrices

This section introduces the definition of a DBM (Dill, 1990, Sec. 4.1) and of its canonical-form representation. DBM provide a simple and computationally advantageous representation of the MPL dynamics, and will be further used in the next section to represent the set of final states, the backward reach tube, and the backward reach sets.

Definition 5. (Difference-Bound Matrix). A DBM in $\mathbb{R}^n$ is the intersection of finitely many sets defined by $x_i - x_j \bowtie_{i,j} \alpha_{i,j}$, where $\bowtie_{i,j} \in \{<, \leq\}$, $\alpha_{i,j} \in \mathbb{R} \cup \{+\infty\}$, for $0 \leq i \neq j \leq n$ and where the value of x_0 is taken to be equal to 0. $\square$

The special variable x_0 is used to represent bounds on a single variable: $x_i \leq \alpha$ can be written as $x_i - x_0 \leq \alpha$.

Each DBM admits an equivalent and unique canonical-form representation, which is a DBM with the tightest possible bounds (Dill, 1990, Sec. 4.1) that can be obtained by application of the Floyd-Warshall algorithm. One advantage of the canonical-form representation is that it is easy to compute orthogonal projections w.r.t. a subset of its variables, which is simply performed by deleting rows and columns corresponding to the complementary variables (Dill, 1990, Sec. 4.1).

Example 5. Let us project the following DBM $\{x \in \mathbb{R}^4 : x_3 - x_2 = 5, x_4 - x_2 = 3, e \leq x_3 - x_4 \leq 2\}$ over the variables $\{x_1, x_2\}$. First we compute the canonical-form representation $\{x \in \mathbb{R}^4 : x_3 - x_2 = 5, x_4 - x_2 = 3, x_3 - x_4 = 2\}$ – notice that the bounds for $x_3 - x_4$ are tighter. Then we eliminate inequalities involving $\{x_3, x_4\}$. The projection is $\mathbb{R}^2$, since all inequalities contain $\{x_3, x_4\}$. $\square$

Proposition 6. (Adzkiya et al., 2013, Th. 1). The inverse image of a DBM with respect to affine dynamics (in particular the PWA expression (3) generated by an MPL system) is a DBM. $\square$

In other words, DBM structures are closed under MPL (backward) dynamics – this fact is clearly fundamental for (backward) reachability computations. The procedure to compute the inverse image of a DBM in $\mathbb{R}^n$ w.r.t. the affine dynamics in (3) involves: 1) computing the cross product of $\mathbb{R}^n$ and the DBM; then 2) determining the DBM generated by the expression of the affine dynamics (each equation can be expressed as the difference between variables at event step k and $k-1$); 3) intersecting the DBM obtained in steps 1 and 2; 4) generating the canonical-form representation of the intersection; finally 5) projecting this DBM over the variables at event step $k-1$, i.e. $\{x_1(k-1), \ldots, x_n(k-1)\}$.

Example 6. Considering the autonomous MPL system in (1), let us compute the inverse image of the complete periodic behavior $E(A^{\otimes c})$ w.r.t. the affine dynamics that are active in $R_{(2,2)}$. First we compute the cross product $\{[x(k-1)^T, x(k)^T]^T \in \mathbb{R}^4 : 0 \leq x_1(k) - x_2(k) \leq 2\}$. Then we intersect it with the DBM generated by the affine dynamics active in $R_{(2,2)}$, obtaining $\{[x(k-1)^T, x(k)^T]^T \in \mathbb{R}^4 : 0 \leq x_1(k) - x_2(k) \leq 2, x_1(k) - x_2(k-1) = 5, x_2(k) - x_2(k-1) = 3\}$. The obtained DBM coincides with that in Example 5, which leads to obtaining the inverse image as the whole space $\mathbb{R}^2$. $\square$

Generalizing the result in Proposition 6, we are in a position to compute the inverse image of a DBM in $\mathbb{R}^n$ w.r.t. the MPL system characterized by a row-finite matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$. In order to do so, for each affine dynamics of the corresponding PWA system, first we compute the inverse image of the DBM, then intersect the result with the region where the affine dynamics are active.

Example 7. With reference to the example in (1) characterized by the system matrix A, let us compute the set of states that can arrive at the complete periodic behavior $E(A^{\otimes c})$ in one step. Skipping the details of the computation, the inverse images of $E(A^{\otimes c})$ w.r.t. the affine dynamics that are active in regions $R_{(1,1)}$, $R_{(2,1)}$, and $R_{(2,2)}$, are the sets $\emptyset$, $E(A^{\otimes c})$, and $\mathbb{R}^2$, respectively. After intersecting each inverse image with the corresponding region, we obtain the set $\{x \in \mathbb{R}^2 : x_1 - x_2 \leq 2\}$. $\square$

3. BACKWARD REACHABILITY ANALYSIS

The objective of backward reachability analysis is to determine the set of states that enter a given set of final conditions over a given events horizon. We distinguish two notions related to this problem.

Definition 7. (Backward Reach Set). Given an MPL system and a nonempty set of final positions $X_0 \subseteq \mathbb{R}^n$, the N-steps backward reach set X_{-N} is the set of all states $\{x(-N) : x(0) \in X_0\}$ obtained via MPL dynamics. $\square$

Definition 8. (Backward Reach Tube). Given an MPL system and a nonempty set of final positions $X_0 \subseteq \mathbb{R}^n$, the backward reach tube is defined by the set-valued function $k \mapsto X_{-k}$, for any given $k > 0$ where X_{-k} is defined. $\square$

Unless otherwise stated, in this work we focus on *finite-horizon* backward reachability: in other words, we compute the backward reach set for a finite index N (cf. Definition 7) and the backward reach tube for $k = 1, \ldots, N$, where $N < \infty$ (cf. Definition 8). While the backward reach set can be obtained as a by-product of the (sequential) computations used to obtain the backward reach tube, we will argue that it can be as well calculated by a tailored procedure (one-shot).

In the computation of the quantities defined above, the set of final conditions $X_0 \subseteq \mathbb{R}^n$ will be assumed to be a union of finitely many DBM. In the more general case of arbitrary sets, these can be over- or under-approximated by a (single or a union of) DBM. As it will become clear later, this will in general shape the backward reach set X_{-k} at event step $k > 0$ as a union of finitely many DBM.

3.1 Backward Reach Tube

Given a set of final conditions X_0, for each $k = 1, \ldots, N$ we determine the states that enter X_0 in k event steps by the following recursion:

$$X_{-k} = \mathcal{I}^{-1}(X_{-k+1}) = \{x \in \mathbb{R}^n : A \otimes x \in X_{-k+1}\}.$$

The mapping $\mathcal{I}^{-1}$ is also known in the literature as the *Pre* operator (Baier and Katoen, 2008, Definition 2.3).

Given a system matrix A and a set of final conditions X_0, the general procedure for obtaining the backward reach tube works as follows: first, we construct the PWA system generated by A; then, for each $k = 1, \ldots, N$, the backward reach set X_{-k} is obtained by computing $\mathcal{I}^{-1}(X_{-k+1})$ as the inverse image over the PWA representation of the MPL dynamics (cf. Section 2.3). Since X_0 is assumed to be a union of finitely many DBM, it can be shown that the backward reach set X_{-k} for each $k > 0$ is a union of finitely many DBM.

Given a set of final conditions X_0, the backward reach tube can be always computed if the event horizon N is finite. Furthermore, if the MPL system is irreducible and X_0 is not intersected with the complete periodic behavior, the *infinite-horizon* backward reach tube can be explicitly computed, as elaborated in the following proposition:

Proposition 9. Let $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ be an irreducible matrix with cyclicity $c \in \mathbb{N}$. If $X_0 \cap E(A^{\otimes c})$ is empty, there exists a finite $k_{\emptyset}$, such that X_{-k} is empty for all $k \geq k_{\emptyset}$.

Proof. Since the set of final conditions X_0 is not intersected with the complete periodic behavior $E(A^{\otimes c})$, then the minimum length of the transient part of X_0 is positive, i.e. $\min_{x \in X_0} k_0(x) > 0$. Furthermore the minimum length of the transient part of the backward reach set X_{-k} is increasing as k gets bigger, i.e. $\min_{x \in X_{-k}} k_0(x) = k + \min_{x \in X_0} k_0(x)$, if X_{-k} is not empty. Notice that X_{-k} is empty if $k \geq \max_{x \in \mathbb{R}^n} k_0(x)$. Recall that $\max_{x \in \mathbb{R}^n} k_0(x) \leq k_0(A)$ and $k_0(A)$ is finite (cf. Proposition 4). □

Example 8. Consider the unit square as the set of final conditions $X_0 = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$, and let us determine X_{-1}. The inverse images of X_0 w.r.t. the affine dynamics that are active in regions $R_{(1,1)}$, $R_{(2,1)}$, and $R_{(2,2)}$ are the sets $\{x \in \mathbb{R}^2 : x_1 = -2\}$, $\{x \in \mathbb{R}^2 : -3 \leq x_1 \leq -2, -5 \leq x_2 \leq -4\}$, and $\emptyset$, respectively. After intersecting each inverse image with the corresponding region, we obtain $\{x \in \mathbb{R}^2 : x_1 = -2, x_2 \leq -5\}$, $\{x \in \mathbb{R}^2 : -3 \leq x_1 \leq -2, -5 \leq x_2 \leq -4\}$, and $\emptyset$. Thus $X_{-1} = \{x \in \mathbb{R}^2 : x_2 \leq -5, x_1 = -2\} \cup \{x \in \mathbb{R}^2 : -3 \leq x_1 \leq -2, -5 \leq x_2 \leq -4\}$. One can show that $X_{-2} = \{x \in \mathbb{R}^2 : x_1 \leq -7, -8 \leq x_2 \leq -7\}$. The obtained regions are shown in Fig. 2 (middle). □

3.2 Backward Reach Set: One-Shot Computation

We discuss here a procedure for computing the backward reach set for a specific event step N using a tailored (one-shot) algorithm. Given a set of final conditions X_0, we compute the N-steps backward reach set as

$$X_{-N} = \mathcal{I}^{-N}(X_0) = \{x \in \mathbb{R}^n : A^{\otimes N} \otimes x \in X_0\},$$

where $\mathcal{I}^{-N} = \mathcal{I}^{-1} \circ \cdots \circ \mathcal{I}^{-1}$ (N times). Given a system matrix A and a set of final conditions X_0, the general procedure for obtaining the N-steps backward reach set works as follows: first, we construct the PWA system generated by $A^{\otimes N}$; then, the backward reach set X_{-N} is obtained by computing the inverse image of X_0 w.r.t. the obtained PWA system.

4. APPLICATIONS OF BACKWARD REACHABILITY

4.1 Safety Analysis

We consider the following safety problem: given an unsafe set, determine whether the states of an MPL system starting from a given initial set enter that set during the event interval $0, \ldots, N$. This problem can be solved by computing the N-steps backward reach tube, where the set of final conditions is tagged as the unsafe set, then checking whether the intersection of the backward reach tube and the set of initial conditions is empty. If the intersection is empty, the system is deemed to be safe. If instead the system is not safe (namely, if the intersection is not empty), then the obtained intersection denotes the subset of the set of initial conditions that are related to "unsafe dynamics."

Example 9. Considering the autonomous MPL system in (1), suppose that there is a requirement on the departure times at station 2 to be not later than those at station 1. The unsafe states correspond to departure times at station 2 that are later than those at station 1 and characterized by $X_0 = \{x \in \mathbb{R}^2 : x_1 - x_2 < 0\}$. The set X_0 does not intersect with the complete periodic behavior. By backward reachability computation, we obtain that $X_{-1} = \{x \in \mathbb{R}^2 : x_1 - x_2 > 2\}$ and that $X_{-k} = \emptyset$ for $k \geq 2$ (cf. Proposition 9). Thus, as long as the set of initial conditions is a subset of $\{x \in \mathbb{R}^2 : 0 \leq x_1 - x_2 \leq 2\}$, the system is deemed safe. □

4.2 Transient Analysis

Classical results in the literature on transient analysis of MPL systems can be enhanced by computing a partition of $\mathbb{R}^n$ based on the length of the transient part k_0 via backward reachability analysis, as described next. First the set of final conditions X'_0 is defined as the complete set of periodic behaviors $E(A^{\otimes c}) = \{x \in \mathbb{R}^n : k_0(x) = 0\}$. Recall that the eigenspace can be formulated as a union of finitely many DBM (cf. Section 2.1). Then for each $k \in \mathbb{N}$, the set of states associated with $k_0 = k$ is obtained by

$$X'_{-k} = \begin{cases} \mathcal{I}^{-1}(X'_0) \setminus X'_0, & \text{if } k = 1, \\ \mathcal{I}^{-1}(X'_{-k+1}), & \text{if } k > 1. \end{cases}$$

Notice that the complete periodic behavior is a subset of its inverse image, i.e. $E(A^{\otimes c}) \subseteq \mathcal{I}^{-1}(E(A^{\otimes c}))$. Further one can see that $X'_{-k} = \{x \in \mathbb{R}^n : k_0(x) = k\}$, for each $k \in \mathbb{N} \cup \{0\}$. The procedure is finite in time, since each point in $\mathbb{R}^n$ has a finite k_0, i.e. $\max_{x \in \mathbb{R}^n} k_0(x) < +\infty$.

Example 10. Let us display the procedure on the autonomous MPL system (1). Recall that the states associated with $k_0 = 0$ encompass the complete periodic behavior $E(A^{\otimes c}) = \{x \in \mathbb{R}^2 : 0 \leq x_1 - x_2 \leq 2\}$. By using the result obtained in Example 7, the states corresponding to $k_0 = 1$ are given by $\{x \in \mathbb{R}^2 : x_1 - x_2 \leq 2\} \setminus \{x \in \mathbb{R}^2 : 0 \leq x_1 - x_2 \leq 2\} = \{x \in \mathbb{R}^2 : x_1 - x_2 < 0\}$. Finally the set of states such that $k_0 = 2$ can be computed by using the backward reachability analysis, which yields $\{x \in \mathbb{R}^2 : x_1 - x_2 > 2\}$. The graphical representation of the state-space partition is shown in Fig. 2 (right). □

5. CONCLUSIONS AND FUTURE WORK

This work has discussed the backward reachability of Max-Plus-Linear systems with applications in safety and transient analysis. A distinct procedure for forward reachability analysis, with the analysis of related applications, is discussed in (Adzkiya et al., 2014). Computationally, we plan to implement the method as a part of VeriSiMPL toolbox (Adzkiya and Abate, 2013). Theoretically, we are also interested in extending the method for the forward and backward reachability of *non-autonomous* models, which embed non-determinism in the form of a control input.

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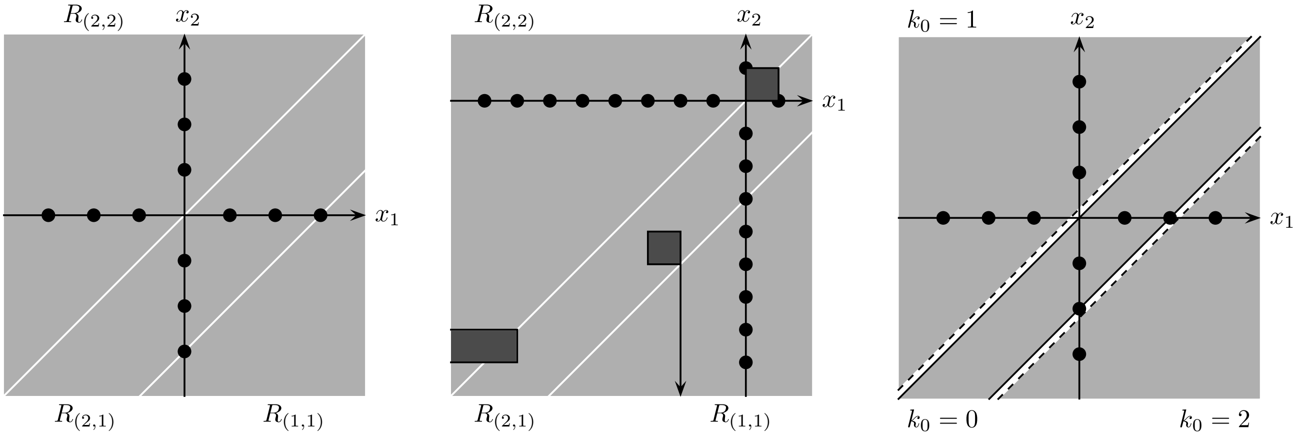

Fig. 2. (Left plot) Regions associated with the PWA system generated by the autonomous MPL system in (1). (Middle plot) Backward reach tube for autonomous MPL system over 2 event steps and the unit square. The down-pointing arrow in the middle square indicates a half-line: that set can be expressed as a union of two DBM. The left rectangle is unbounded in the left direction. (Right plot) Partition of $\mathbb{R}^2$ for the MPL system in (1) based on the length of transient part k_0. Black dots denote integer values.

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