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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: `https://www.dcsc.tudelft.nl`

Model predictive control for max-plus-linear systems via optimistic optimization

J. Xu, B. De Schutter, T. J. J. van den Boom

Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands
(e-mail: {j.xu-3, b.deschutter, a.j.j.vandenboom}@tudelft.nl)

Abstract: The model predictive control problem for max-plus-linear discrete-event systems generally leads to a nonlinear optimization problem, which may be hard to solve efficiently. In this paper, we propose to apply optimistic optimization to resolve this problem. The algorithm builds a tree where each selected control sequence corresponds to a node of the tree. An optimistic exploration of the tree is implemented, where the most promising control sequences are explored first. We give an example to illustrate the effectiveness of the method.

Keywords: Discrete-event systems; Max-plus-linear systems; Model predictive control; Optimistic optimization

1. INTRODUCTION

Many complex man-made systems such as flexible manufacturing systems, telecommunication networks, railway networks, traffic control systems, and logistic systems, can be modeled by discrete-event systems. Usually in conventional algebra discrete-event systems lead to nonlinear descriptions, but there is a subclass of discrete-event systems for which we can get a "linear" model in the max-plus algebra (Baccelli et al., 1992; Cuninghame-Green, 1979; Heidergott et al., 2006), i.e. max-plus-linear (MPL) systems. In these systems, only synchronization and no choice is considered. Many results have been achieved for control or analysis of MPL systems (Cassandras et al., 1995; Boimond and Ferrier, 1996; Gazarik and Kamen, 1999; Cottenceau et al., 2001; De Schutter and van den Boom, 2001; Katz, 2007; Hardouin et al., 2010; Maia et al., 2011; Houssin et al., 2013).

Model predictive control (MPC) has been widely used in the process industry since its introduction in the 1980s (Richalet et al., 1978; Garcia et al., 1989). A key advantage of MPC is that it can handle constraints on inputs and outputs. In essence, MPC uses a prediction model in combination with on-line optimization to determine a sequence of control inputs. An objective function is optimized subject to various operational constraints over a given prediction horizon. MPC has been extended to MPL discrete-event systems (De Schutter and van den Boom, 2001; van den Boom and De Schutter, 2004; Necoara et al., 2008).

The MPC problem for MPL systems can be formulated as an optimization problem subject to constraints on inputs and outputs. For some special cases, namely, if the objective function is a monotonically non-decreasing piecewise affine function of the output and an affine function of the input and if the constraints are linear and monotonically non-decreasing as a function of the output, we get a linear programming problem, which can be solved very efficiently. However, in general MPL-MPC will result in a nonlinear optimization problem. Mixed integer linear programming (MILP) can be used to solve the problem, but this method will be less efficient if the prediction horizon increases. In this paper we focus on applying optimistic optimization to solve the MPL-MPC problem and we develop a corresponding dedicated semi-metric. Optimistic optimization uses a tree architecture to represent the possible sequences of control inputs and implements an optimistic exploration of the tree, where the most promising nodes of the tree are explored first (Munos, 2011; Valko et al., 2013; Munos, 2013). The main advantage of optimistic optimization is that one can specify the computation budget (e.g. the number of function evaluations) in advance. The computation complexity of optimistic optimization depends on the control horizon instead of the prediction horizon. In particular, the algorithm with the proposed semi-metric will be more efficient than MILP for small control horizons and large prediction horizons.

This paper is organized as follows. In Section 2 we introduce MPL discrete-event systems. Section 3 discusses the MPC problem for MPL systems. Section 4 presents the background of optimistic optimization. In Section 5 we apply optimistic optimization to the MPL-MPC problem. Next, we consider an example to illustrate the effectiveness of this algorithm. Finally, conclusions are given.

2. MAX-PLUS-LINEAR SYSTEMS

Consider a single-input single-output [1] MPL system of the following form

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \tag{1}$$
$$y(k) = C \otimes x(k) \tag{2}$$

[1] The approach presented in this paper can be extended to multi-input multi-output systems too.

where k is the event counter, $x(k) \in \mathbb{R}_\varepsilon^n$ is the state, $u(k) \in \mathbb{R}_\varepsilon$ is the input, $y(k) \in \mathbb{R}_\varepsilon$ is the output, with $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, and where $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times 1}$, and $C \in \mathbb{R}_\varepsilon^{1\times n}$ are the coefficient matrices.

The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y$$

for numbers $x, y \in \mathbb{R}_\varepsilon$ and

$$\left[A \oplus B\right]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$\left[A \otimes C\right]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\dots,n}(a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$.

3. THE MPC PROBLEM FOR MPL SYSTEMS

MPC uses a prediction model in combination with on-line optimization to determine a sequence of control inputs by optimizing a performance criterion subject to various constraints over a given prediction horizon. De Schutter and van den Boom (2001) have extended the MPC framework to MPL systems as follows. Assume N_p is the prediction horizon and denote the estimate of the output at event step $k+j$ based on the information available at event step k by $\hat{y}(k+j|k)$. The evolution of the output can be estimated by successive substitution of (1) into (2). In matrix notation we obtain

$$\tilde{y}(k) = H \otimes \tilde{u}(k) \oplus g(k) \tag{3}$$

where

$$\tilde{y}(k) = \begin{bmatrix} \hat{y}(k|k) \\ \vdots \\ \hat{y}(k+N_\mathrm{p}-1|k) \end{bmatrix}, \quad \tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_\mathrm{p}-1) \end{bmatrix},$$

$$H = \begin{bmatrix} C \otimes B & \varepsilon & \cdots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{N_\mathrm{p}-1}} \otimes B & C \otimes A^{\otimes^{N_\mathrm{p}-2}} \otimes B & \cdots & C \otimes B \end{bmatrix},$$

$$g(k) = \begin{bmatrix} C \otimes A \\ C \otimes A^{\otimes^2} \\ \vdots \\ C \otimes A^{\otimes^{N_\mathrm{p}}} \end{bmatrix} \otimes x(k-1).$$

De Schutter and van den Boom (2001) consider a performance criterion J that reflects the difference between output times and due dates (J_out) and just-in-time production (J_in):

$$J(\tilde{y}(k), \tilde{u}(k)) = J_\mathrm{out}(\tilde{y}(k)) + \beta J_\mathrm{in}(\tilde{u}(k))$$

where β is a non-negative scalar.

The MPC problem at event step k for MPL system involves finding the input sequence $u(k), \dots, u(k+N_\mathrm{p}-1)$ that minimizes the performance criterion $J(\tilde{y}(k), \tilde{u}(k))$ subject to (3) and other constraints on the inputs and outputs.

MPL-MPC uses a receding horizon approach. This means that once the optimal input sequence is determined, only the first element of the input sequence is applied to the system, after which the new state of the system is measured. At the next event step the prediction horizon is shifted and the whole process is repeated.

In this paper we consider the following objective functions

$$\begin{aligned} J_\mathrm{out}(\tilde{y}(k)) &= \sum_{j=0}^{N_\mathrm{p}-1} \left|\hat{y}(k+j|k) - r(k+j)\right| \\ &= \sum_{j=1}^{N_\mathrm{p}} \left|\tilde{y}_j(k) - \tilde{r}_j(k)\right| \end{aligned} \tag{4}$$

$$J_\mathrm{in}(\tilde{u}(k)) = -\sum_{j=0}^{N_\mathrm{p}-1} u(k+j) = -\sum_{j=1}^{N_\mathrm{p}} \tilde{u}_j(k) \tag{5}$$

where $\tilde{r}(k) = \left[r(k) \ \cdots \ r(k+N_\mathrm{p}-1)\right]^T$ is the due date signal.

Remark 1. The output cost function (4) reflects the differences between the due dates r and the actual output time instants. The input function (5) leads to a maximization of the input instants.

Note that

$$u(k+j) = u(k-1) + \sum_{s=0}^{j} \Delta u(k+s), \text{ for } j = 0, \dots, N_\mathrm{p}-1,$$

with $\Delta u(k+s) = u(k+s) - u(k+s-1)$.

Denote

$$\Delta\tilde{u}(k) = \begin{bmatrix} \Delta u(k) \\ \vdots \\ \Delta u(k+N_\mathrm{p}-1) \end{bmatrix} = \begin{bmatrix} u(k) - u(k-1) \\ \vdots \\ u(k+N_\mathrm{p}-1) - u(k+N_\mathrm{p}-2) \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{N_\mathrm{p}\times N_\mathrm{p}}, \quad u_\mathrm{prev}(k) = u(k-1).$$

Then

$$\tilde{u}(k) = L\ \Delta\tilde{u}(k) + \begin{bmatrix} u_\mathrm{prev}(k) \\ \vdots \\ u_\mathrm{prev}(k) \end{bmatrix}. \tag{6}$$

Using (3) and (6) we can rewrite the objective function as follows:

$$J(\tilde{y}(k), \tilde{u}(k)) = \sum_{j=1}^{N_\mathrm{p}} \left|\tilde{y}_j(k) - \tilde{r}_j(k)\right| - \beta \sum_{j=1}^{N_\mathrm{p}} \tilde{u}_j(k)$$

$$= \sum_{j=1}^{N_{\mathrm{p}}} \left| \tilde{y}_j(k) - \tilde{r}_j(k) \right| - \beta \sum_{j=1}^{N_{\mathrm{p}}} \left(\left(L\, \Delta\tilde{u}(k) \right)_j + u_{\mathrm{prev}}(k) \right)$$
$$= \sum_{j=1}^{N_{\mathrm{p}}} \left| \tilde{y}_j(k) - \tilde{r}_j(k) \right| - \beta \sum_{j=1}^{N_{\mathrm{p}}} \left(\sum_{s=1}^{j} \Delta\tilde{u}_s(k) + u_{\mathrm{prev}}(k) \right)$$
$$=: J_\Delta(\Delta\tilde{u}(k)).$$

This defines a new objective function J_Δ.

In this paper the input rate is assumed to be bounded, which can be guaranteed by the following assumption:

Assumption 1.

$$0 \le a \le \Delta u(k+j) \le b, \text{ for } j = 0, \ldots, N_{\mathrm{p}} - 1, \quad (7)$$

where $a < b$ and b is finite.

Now we can write the MPL-MPC problem at event step k as follows:

$$\min_{\Delta\tilde{u}(k)} J_\Delta(\Delta\tilde{u}(k)) \quad (8)$$

subject to

$$\Delta\tilde{u}(k) \in [a, b]^{N_{\mathrm{p}}}. \quad (9)$$

4. OPTIMISTIC OPTIMIZATION

Denote $\mathcal{X} = [a, b]^{N_{\mathrm{p}}}$, then[2] $\Delta\tilde{u} \in \mathcal{X}$. Optimistic optimization is a tree search algorithm that can perform an efficient exploration of the search space $\mathcal{X}$. This algorithm is called optimistic because it explores the most promising areas first (Munos, 2011). The implementation of optimistic optimization is based on a hierarchical partitioning of the search space $\mathcal{X}$. Consider a set of partitions of the space $\mathcal{X}$ at all scales $h = 0, 1, \ldots$, i.e. for any integer h, $\mathcal{X}$ is partitioned into K^h sets $X^{h,i}$ (called cells), where $i = 0, \ldots, K^h - 1$. This partitioning may be represented by a K-ary tree (i.e. a rooted tree in which each node has no more than K children) where each cell $X^{h,i}$ corresponds to a node (h, i) of the tree such that each node (h, i) possesses K child nodes $\{(h+1, i_k)\}_{k=1,\ldots,K}$. In addition, the set of cells of the children $\{X^{h+1,i_k}\}_{k=1}^{K}$ forms a partition of the parent cell $X^{h,i}$. The root of the tree (i.e. the cell $X_{0,0}$) corresponds to the whole domain $\mathcal{X}$. To each cell $X^{h,i}$ we assign a representative point $\Delta\tilde{u}^{h,i} \in X^{h,i}$ where f may be evaluated. In this paper, $\Delta\tilde{u}^{h,i}$ is the center of $X^{h,i}$.

We also need the following definition:

Definition 1 (Semi-metric). A semi-metric on a set $\mathcal{S}$ is a function $\ell : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ satisfying the following conditions for any $x, y \in \mathcal{S}$:
1) $\ell(x, y) = \ell(y, x) \ge 0$;
2) $\ell(x, y) = 0$ if and only if $x = y$.

To use optimistic optimization, the following requirements should be satisfied:

Requirement 1. There exists a semi-metric ℓ in the search space $\mathcal{X}$.

Requirement 2. There exists at least one global optimizer $\Delta\tilde{u}^* \in \mathcal{X}$ of J_Δ (i.e. $J_\Delta(\Delta\tilde{u}^*) = \min_{\Delta\tilde{u} \in \mathcal{X}} J_\Delta(\Delta\tilde{u})$) such that for all $\Delta\tilde{u} \in \mathcal{X}$,

$$J_\Delta(\Delta\tilde{u}) - J_\Delta(\Delta\tilde{u}^*) \le \ell(\Delta\tilde{u}, \Delta\tilde{u}^*).$$

[2] For simplicity, the index k is dropped from now on.

Requirement 3. There exists a decreasing sequence $\delta(h) > 0$, such that for any level $h \ge 0$, for any cell $X^{h,i}$ of level h, we have $\sup_{\Delta\tilde{u} \in X^{h,i}} \ell(\Delta\tilde{u}^{h,i}, \Delta\tilde{u}) \le \delta(h)$.

Requirement 4. There exists a $\nu > 0$ such that for any level $h \ge 0$, any cell $X^{h,i}$ contains an ℓ-ball of radius $\nu\delta(h)$ centered in $\Delta\tilde{u}^{h,i}$.

5. OPTIMISTIC OPTIMIZATION FOR THE MPL-MPC PROBLEM

In this section, we will look at the implementation of optimistic optimization for the MPL-MPC problem (8)-(9). To show that Requirements 1-4 hold for the problem, the following lemma will be needed.

Lemma 1. Suppose that $\tilde{y}^*$ and $\tilde{u}^*$ are the outputs and inputs corresponding to the global optimizer $\Delta\tilde{u}^*$ of the problem (8)-(9). Then for any $\Delta\tilde{u}$ and corresponding input sequence $\tilde{u}$ and output sequence $\tilde{y}$, it holds that

$$\left| \tilde{y}_j - \tilde{r}_j \right| - \left| \tilde{y}_j^* - \tilde{r}_j \right| \le \max_{i=1,\ldots,j} \left| \sum_{s=1}^{i} \Delta\tilde{u}_s - \sum_{s=1}^{i} \Delta\tilde{u}_s^* \right|$$

for all $j = 1, \ldots, N_{\mathrm{p}}$.

Proof. Due to the triangle inequality, it is easy to verify that

$$\left| \tilde{y}_j - \tilde{r}_j \right| - \left| \tilde{y}_j^* - \tilde{r}_j \right| \le \left| \tilde{y}_j - \tilde{y}_j^* \right|$$

for all $j = 1, \ldots, N_{\mathrm{p}}$.

From (3) we have

$$\tilde{y}_j = \max\left(\max_{p=1,\ldots,j} h_{jp} + \tilde{u}_p, g_j \right).$$

Given $\tilde{u}$ and $\tilde{u}^*$ corresponding to a given $\Delta\tilde{u} \in \mathcal{X}$ and the optimizer $\Delta\tilde{u}^*$, we can define p_0 and q_0 as follows:

$$p_0 = \arg \max_{p=1,\ldots,j} h_{jp} + \tilde{u}_p,$$
$$q_0 = \arg \max_{q=1,\ldots,j} h_{jq} + \tilde{u}_q^*.$$

Now

$$\begin{aligned}
\tilde{y}_j - \tilde{y}_j^* &= \max\left(\max_{p=1,\ldots,j} h_{jp} + \tilde{u}_p, g_j \right) \\
&\quad - \max\left(\max_{q=1,\ldots,j} h_{jq} + \tilde{u}_q^*, g_j \right) \\
&= \max(h_{jp_0} + \tilde{u}_{p_0}, g_j) \\
&\quad - \max(h_{jq_0} + \tilde{u}_{q_0}^*, g_j) \\
&\le \max(h_{jp_0} + \tilde{u}_{p_0}, g_j) \\
&\quad - \max(h_{jp_0} + \tilde{u}_{p_0}^*, g_j) \\
&\le \max(\tilde{u}_{p_0} - \tilde{u}_{p_0}^*, 0) \\
&\le \left| \tilde{u}_{p_0} - \tilde{u}_{p_0}^* \right| \le \max_{i=1,\ldots,j} \left| \tilde{u}_i - \tilde{u}_i^* \right|.
\end{aligned}$$

In a similar way, we have

$$\tilde{y}_j^* - \tilde{y}_j \le \left| \tilde{u}_{q_0}^* - \tilde{u}_{q_0} \right| \le \max_{i=1,\ldots,j} \left| \tilde{u}_i - \tilde{u}_i^* \right|.$$

Therefore

$$\left| \tilde{y}_j - \tilde{r}_j \right| - \left| \tilde{y}_j^* - \tilde{r}_j \right| \le \left| \tilde{y}_j - \tilde{y}_j^* \right| \le \max_{i=1,\ldots,j} \left| \tilde{u}_i - \tilde{u}_i^* \right|.$$

Because

$$\tilde{u}_i = u_{\mathrm{prev}} + \sum_{s=1}^{i} \Delta\tilde{u}_s, \qquad \tilde{u}_i^* = u_{\mathrm{prev}} + \sum_{s=1}^{i} \Delta\tilde{u}_s^*,$$

we have

$$\left|\tilde{y}_j - \tilde{r}_j\right| - \left|\tilde{y}^*_j - \tilde{r}_j\right| \leq \max_{i=1,\ldots,j} \left|\sum_{s=1}^{i} \Delta\tilde{u}_s - \sum_{s=1}^{i} \Delta\tilde{u}^*_s\right|$$

for all $j = 1, \ldots, N_\mathrm{p}$. □

Define a mapping $\ell : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, such that for all $\Delta\tilde{u}, \Delta\tilde{v} \in \mathcal{X}$,

$$\ell(\Delta\tilde{u}, \Delta\tilde{v}) = (1+\beta) \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left|\Delta\tilde{u}_s - \Delta\tilde{v}_s\right|. \quad (10)$$

Now we show that Requirements 1-4 are satisfied for the problem (8)-(9).

Theorem 2. The function ℓ defined by (10) is a semi-metric.

Proof. From the definition of the mapping ℓ and since $\beta \geq 0$, it is obvious that

$$\ell(\Delta\tilde{u}, \Delta\tilde{v}) = \ell(\Delta\tilde{v}, \Delta\tilde{u}) \geq 0,$$
$$\ell(\Delta\tilde{u}, \Delta\tilde{v}) = 0, \text{ if } \Delta\tilde{u} = \Delta\tilde{v}.$$

Now we only need to prove that

$$\ell(\Delta\tilde{u}, \Delta\tilde{v}) = 0 \quad \text{implies} \quad \Delta\tilde{u} = \Delta\tilde{v}.$$

If $\ell(\Delta\tilde{u}, \Delta\tilde{v}) = 0$, then we have

$$\sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left|\Delta\tilde{u}_s - \Delta\tilde{v}_s\right| = 0$$

i.e.

$$\left|\Delta\tilde{u}_s - \Delta\tilde{v}_s\right| = 0$$

for $s = 1, \ldots, N_\mathrm{p}$. Therefore, $\Delta\tilde{u} = \Delta\tilde{v}$.

This completes the proof. □

Theorem 3. Suppose $\Delta\tilde{u}^* \in \mathcal{X}$ is a global optimizer of $J_\Delta(\Delta\tilde{u})$, i.e. $J_\Delta(\Delta\tilde{u}^*) = \min_{\Delta\tilde{u} \in \mathcal{X}} J_\Delta(\Delta\tilde{u})$. Then for all $\Delta\tilde{u} \in \mathcal{X}$, it holds that

$$J_\Delta(\Delta\tilde{u}) - J_\Delta(\Delta\tilde{u}^*) \leq \ell(\Delta\tilde{u}, \Delta\tilde{u}^*). \quad (11)$$

Proof. Note that for a sequence of real numbers $\alpha_1, \ldots, \alpha_j$ we have

$$\max_{i=1,\ldots,j} \left|\sum_{s=1}^{i} \alpha_s\right| \leq \max_{i=1,\ldots,j} \sum_{s=1}^{i} \left|\alpha_s\right| \leq \sum_{s=1}^{j} \left|\alpha_s\right|. \quad (12)$$

Based on Lemma 1 and (12), we have

$$\begin{aligned}
&J_\Delta(\Delta\tilde{u}) - J_\Delta(\Delta\tilde{u}^*) \\
&= \sum_{j=1}^{N_\mathrm{p}} \left[\left|\tilde{y}_j - \tilde{r}_j\right| - \left|\tilde{y}^*_j - \tilde{r}_j\right|\right] - \beta \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left[\Delta\tilde{u}_s - \Delta\tilde{u}^*_s\right] \\
&\leq \sum_{j=1}^{N_\mathrm{p}} \max_{i=1,\ldots,j} \left|\sum_{s=1}^{i} \Delta\tilde{u}_s - \sum_{s=1}^{i} \Delta\tilde{u}^*_s\right| \\
&\quad - \beta \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left[\Delta\tilde{u}_s - \Delta\tilde{u}^*_s\right] \\
&\leq \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left|\Delta\tilde{u}_s - \Delta\tilde{u}^*_s\right| - \beta \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left[\Delta\tilde{u}_s - \Delta\tilde{u}^*_s\right] \\
&\leq (1+\beta) \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left|\Delta\tilde{u}_s - \Delta\tilde{u}^*_s\right| \\
&= \ell(\Delta\tilde{u}, \Delta\tilde{u}^*).
\end{aligned}$$

This completes the proof. □

Remark 2. The global optimizer of the MPL-MPC problem exists because the objective function is convex and the constraints are feasible and closed.

Theorem 4. For any cell $X^{h,i}$ of level $h \in \{0, 1, \ldots\}$, there is a decreasing sequence

$$\delta(h) = \frac{N_\mathrm{p}(N_\mathrm{p}+1)(1+\beta)(b-a)}{2^{h+2}} \quad (13)$$

such that

$$\sup_{\Delta\tilde{u} \in X^{h,i}} \ell(\Delta\tilde{u}^{h,i}, \Delta\tilde{u}) \leq \delta(h)$$

where $\Delta\tilde{u}^{h,i}$ is the representative point of $X^{h,i}$.

Proof. From the constraint (9), we have

$$a \leq \Delta\tilde{u}_j \leq b, \quad j = 1, \ldots, N_\mathrm{p}$$

for any $\Delta\tilde{u} \in X^{h,i}$.

Denote d_h as the maximum distance along every axis between any two points in a cell at level h, i.e.

$$d_h = \frac{b-a}{2^h}. \quad (14)$$

Because $a < b$, we have $d_h > 0$.

Since $\Delta\tilde{u}^{h,i}$ is the center of $X^{h,i}$, then we have for any $\Delta\tilde{u} \in X^{h,i}$

$$\begin{aligned}
\ell(\Delta\tilde{u}^{h,i}, \Delta\tilde{u}) &= (1+\beta) \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \left|\Delta\tilde{u}^{h,i}_s - \Delta\tilde{u}_s\right| \\
&\leq (1+\beta) \sum_{j=1}^{N_\mathrm{p}} \sum_{s=1}^{j} \frac{d_h}{2} \\
&\leq (1+\beta) \frac{N_\mathrm{p}(N_\mathrm{p}+1)}{2} \frac{b-a}{2^{h+1}}.
\end{aligned}$$

So if we define $\delta(h)$ as in (13), then

$$\sup_{\Delta\tilde{u} \in X^{h,i}} \ell(\Delta\tilde{u}^{h,i}, \Delta\tilde{u}) \leq \delta(h).$$

□

Theorem 5. For any level $h \in \{0, 1, \ldots\}$, define

$$\nu \leq \frac{\rho}{N_\mathrm{p}(N_\mathrm{p}+1)(1+\beta)}.$$

Then any cell $X^{h,i}$ contains an ℓ-ball $\mathcal{B}^{h,i}$ of radius $\nu\delta(h)$ centered in $\Delta\tilde{u}^{h,i}$, where $\mathcal{B}^{h,i} = \{\Delta\tilde{u} \in \mathcal{X} | \ell(\Delta\tilde{u}^{h,i}, \Delta\tilde{u}) \leq \nu\delta(h)\}$.

Proof. According to Theorem 4, we can define a decreasing sequence $\delta(h)$ as in (13). Select $0 < \rho < 1$. The ℓ-ball $\mathcal{B}^{h,i}$ centered in $\Delta\tilde{u}^{h,i}$ is inside the cell $X^{h,i}$ if we select ν such that

$$\nu\delta(h) \leq \frac{\rho d_h}{2}$$

where d_h is defined as in (14).

Now we can choose ν such that

$$\nu \leq \frac{\rho d_h}{2\delta(h)} = \frac{2\rho}{N_\mathrm{p}(N_\mathrm{p}+1)(1+\beta)}.$$

This completes the proof. □

In MPC often a control horizon N_c is introduced with $N_\mathrm{c} < N_\mathrm{p}$ and the control input is taken to be constant

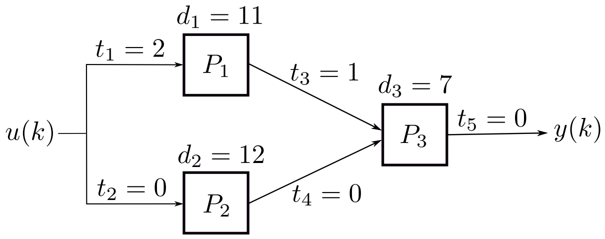

Fig. 1. A manufacturing system

from the event step $k + N_c$. The use of N_c leads to a reduction of the computational complexity. For the MPL-MPC problem in this paper, it is assumed that the input rate $\Delta u(k + j)$ stays constant from the event step $k + N_c$, i.e.

$$\Delta u(k + j) = \Delta u(k + N_c - 1)$$

or

$$\Delta^2 u(k + j) = \Delta u(k + j) - \Delta u(k + j - 1) = 0$$

for $j = N_c, \ldots, N_p - 1$.

Remark 3. Up to now we have in fact considered the case $N_c = N_p$. If we would use $N_c < N_p$ the previous results still remain valid; only the structure of $\Delta\tilde{u}$ changes in the sense that only the first N_c components are free, and the remaining ones will be set constant and equal to $\Delta\tilde{u}_{N_c}$. Moreover, this also implies that the actual search space is now $\mathcal{X}_c = [a, b]^{N_c}$ instead of $\mathcal{X} = [a, b]^{N_p}$.

Remark 4. In general, the MPL-MPC problem can be formulated as an MILP problem; the number of auxiliary binary variables that are used to convert the max operator into linear equations is proportional to the prediction horizon N_p. So the computational complexity of the MILP problem is in the worst case exponential in N_p. On the other hand, the computational complexity of optimistic optimization is exponential in the number of decision variables, i.e. the control horizon N_c. Thus optimistic optimization will be more efficient if $N_c \ll N_p$.

6. EXAMPLE

Consider the single-input single-output manufacturing system of Fig. 1, which was also used in (De Schutter and van den Boom, 2001). This system can be modeled as following MPL system

$$x(k+1) = \begin{bmatrix} 11 & \varepsilon & \varepsilon \\ \varepsilon & 12 & \varepsilon \\ 23 & 24 & 7 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \otimes u(k), \qquad (15)$$

$$y(k) = \begin{bmatrix} \varepsilon & \varepsilon & 7 \end{bmatrix} \otimes x(k) \qquad (16)$$

with $u(k)$ the release time of raw material for the kth time, $x(k)$ the starting time of the ith machine for the kth time, and $y(k)$ the finishing time of the kth product.

Assume that $k = 1$ and the prediction horizon $N_p = 8$. The initial conditions are given as $x(0) = \begin{bmatrix} 0 & 0 & 10 \end{bmatrix}^T$ and $u(0) = 0$. The reference signal sequence is

$$\begin{aligned} \tilde{r}(k) &= \begin{bmatrix} r(k) & \cdots & r(k + N_p - 1) \end{bmatrix}^T \\ &= \begin{bmatrix} 25 & 40 & 55 & 70 & 85 & 100 & 115 & 130 \end{bmatrix}^T. \end{aligned}$$

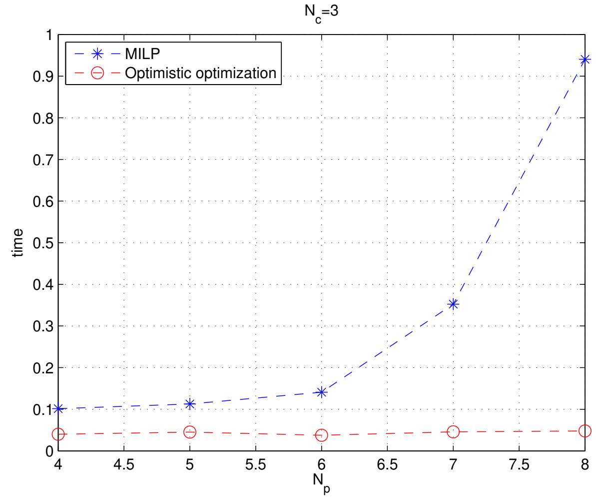

Fig. 2. The CPU time for $N_c = 3$ and $N_p = 4, \ldots, 8$

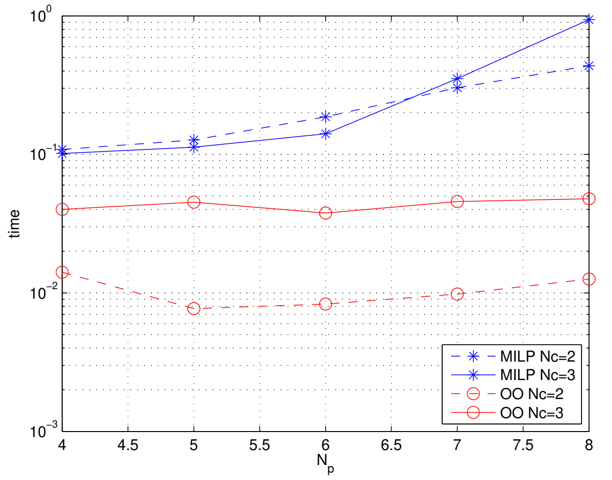

Fig. 3. The CPU time with logarithmic scale

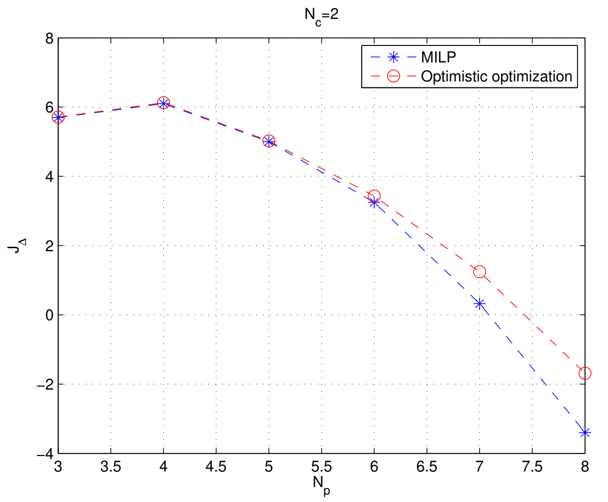

Fig. 4. Optimal values of J_Δ for $N_c = 2$ and $N_p = 3, 4, \ldots, 8$

Consider one step of the MPC problem for the MPL system (15)-(16) with the objective function J_Δ and the constraints $10 \leq \Delta u(k+j) \leq 20$ for $j = 0, \ldots, N_c - 1$. This

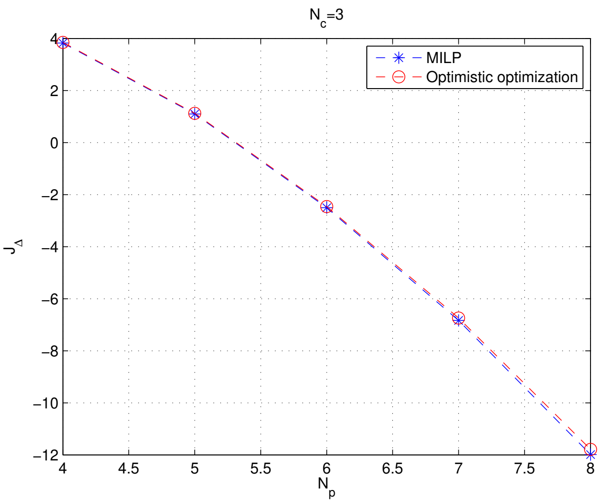

Fig. 5. Optimal values of J_Δ for $N_c = 3$ and $N_p = 4, 5, \ldots, 8$

MPL-MPC problem is solved by optimistic optimization and MILP method. It should be noted that optimistic optimization is implemented in Matlab and the MILP method uses the solver of 'mipSolve' in Tomlab. And CPU times in Fig. 2 and Fig. 3 are average values over 10 runs.

The CPU time needed to compute the optimal input rate sequence $\Delta\tilde{u}$ for $N_c = 3$ is shown in Fig. 2 as a function of N_p. The CPU time for the MILP method increases exponentially as the prediction horizon N_p increases. Fig. 3 illustrates that the computational complexity of optimistic optimization depends on the control horizon N_c instead of N_p. From Fig. 4 and Fig. 5, it can be seen that optimistic optimization with $N_c = 3$ yields a better approximation of the optimal value of the objective function J_Δ than the case with $N_c = 2$. And the performance of optimistic optimization is almost as good as MILP method when $N_c = 3$.

7. CONCLUSIONS

In this paper, we have considered the MPL-MPC problem of De Schutter and van den Boom (2001). Optimistic optimization has been proved to be able to solve the MPL-MPC problem. The method in this paper is more efficient than MILP when the control horizon is small and the prediction horizon is large. In the future, we will extend this method to the MPC problem for max-plus-linear systems with more general objective functions.

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