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Design of Stabilizing Switching Laws for Mixed Switched Affine Systems

Mohammad Hajiahmadi, Bart De Schutter, and Hans Hellendoorn

***Abstract*—This paper presents stability analysis and stabilization for a general class of switched systems characterized by nonlinear functions. The proposed approach is composed of approximating the switched nonlinear system with a switched affine system that has a mixture of controlled and autonomous switching behavior. Utilizing a joint polyhedral partitioning approach, a stabilizing switching law based on quadratic Lyapunov functions and with considering the autonomous switching between polyhedral regions is proposed. To ensure the decrease of the overall Lyapunov function, two approaches are proposed, 1) guarantee continuity of the Lyapunov function over boundaries of polyhedral regions, 2) relax the continuity requirement by using additional matrix inequalities. The second approach is less conservative but with more variables and matrix inequalities than in the first method. With fixing one scalar variable, the stabilization conditions will have the form of linear matrix inequalities (LMIs). Further, the sufficient conditions for stabilizing the original switched nonlinear system using the proposed switching schemes are presented. Finally, through two examples, the performance of the proposed stabilization methods is demonstrated.**

***Index Terms*—Switched nonlinear system, piecewise affine functions, switched affine system, stability**

I. Introduction

Switched systems comprise several linear and/or nonlinear subsystems along with some switching rules that orchestrate the switching between subsystems [1]. Stability analysis and control synthesis for such systems have been extensively studied in recent years [1]–[9]. However, stability analysis of switched *nonlinear* systems has been investigated only for particular cases [10]–[12] and there is concrete procedure for the stabilization of general nonlinear systems.

Therefore, in this work we aim at tackling the stability problem for switched nonlinear systems with smooth nonlinear functions using a different and novel framework. The main idea is to approximate each nonlinear subsystem with piecewise affine (PWA) functions. In this way, we obtain a switched system composed of PWA subsystems and a controllable switching signal that orchestrates the switching between PWA subsystems. Note that there also exists an autonomous type of switching between affine functions of each PWA subsystem. This autonomous switching makes the stability analysis and control of such system hard. As a starting point, we use a joint polyhedral partitioning of the entire state which helps to merge and unify the partitions of all subsystems and to have affine subsystems inside each polyhedral region.

This work has three main contributions. The first contribution is the design of a stabilizing switching law for the mixed switched affine system. The design conditions are in the form of linear matrix inequalities (LMIs) with fixing one scalar variable. Compared to the existing approaches for stability analysis of PWA systems and switched affine systems [13], [14], our method is less conservative as it allows to have affine subsystems in the regions containing the origin and as it also uses more general multiple Lyapunov functions compared to [6], [13]. Also, instead of limiting the matrices of the Lyapunov functions to take a particular structure [15], we use additional equality constraints to impose continuity over boundaries. This choice will also help to extend our proposed stabilization approach for robust state feedback control design in the form of LMIs. The second contribution is to even relax the continuity of the Lyapunov functions over the boundaries of partitions (which is essential for the methods proposed in [13], [14]) using additional LMIs. Finally, the last main contribution is to present the sufficient conditions under which the proposed switching control schemes would be able to stabilize the original switched nonlinear system.

The rest of the paper is organized as follows. In Section II, the switched nonlinear system and the way it is approximated by a switched affine system is presented. Section III presents two procedures for the design of stabilizing switching rules. Stabilization of the original switched nonlinear system using the proposed methods is discussed in Section IV. Next, through two examples in Section V, the performance of the proposed stabilization methods are illustrated.

II. Problem statement

Consider the following switched nonlinear system:

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t)\big) \tag{1}$$

with $x \in \mathbb{R}^n$ the state and $\sigma(\cdot)$ the switching signal which is assumed to be piecewise constant over time. The variable $\sigma(t)$ takes values from a pre-defined index set. In other words, for each value that $\sigma(t)$ assumes, the state space model (1) is governed by different vector functions $f_i(x)$ from the set $f_{\sigma(t)} \in \{f_1, \dots, f_N\}$. A function $\phi : \Omega \to \mathbb{R}^m$ is PWA if there exists a polyhedral partition $\{\Omega_i\}_{i\in\mathcal{I}}$ ($\cup_{i\in\mathcal{I}}\Omega_i = \Omega,\ \Omega_i \neq \varnothing,\ \mathrm{int}(\Omega_i) \cap\ \mathrm{int}(\Omega_j) = \varnothing,\ \forall i \neq j$) of $\Omega \subseteq \mathbb{R}^n$ such that ϕ is affine on each polyhedron Ω_i. By considering a sufficiently large number of regions, one can smoothly proximate nonlinear functions f_i by PWA functions with

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arbitrary accuracy. The piecewise affine approximation of f_i will have the following form:

$$f_i(x) \cong \big(A_{i,\ell} \cdot x + b_{i,\ell}\big), \qquad \text{if } x \in \mathcal{X}_{i,\ell}, \tag{2}$$

with $A_{i,\ell}(n \times n)$ and $b_{i,\ell}(n \times 1)$ the PWA matrices, $\mathcal{X}_{i,\ell}$ the corresponding polyhedron, and $\ell \in \mathcal{M}_i = \{1, \ldots, M_i\}$, with M_i the number of polyhedral partitions for function f_i.

Now the switched system (1) can be approximated by the following switched affine system:

$$\dot{x}(t) = A_{\sigma(t),\ell}x(t) + b_{\sigma(t),\ell}, \qquad \text{if } x \in \mathcal{X}_{\sigma(t),\ell}, \tag{3}$$

where the controlled switching signal σ takes values from the set $\mathcal{N} = \{1, \ldots, N\}$, with N the total number of subsystems.

Note that two types of switching are integrated in (3), one associated with switching between affine functions describing the dynamics of each subsystem i; this type of switching is therefore uncontrolled, and the other one is the controlled switching between subsystems driven by σ. In the following sections, we focus on the stabilization of (3) and next, on connecting the obtained results to the stability of the original switched nonlinear system (1). Before proceeding, a useful lemma from the literature is presented.

Lemma 1 (Finsler Lemma [16]): Let $x \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(B) < n$. The following statements are equivalent:

- $x^{\mathrm{T}}Qx < 0, \quad \forall x \neq 0$ such that $Bx = 0$.
- ${B^{\perp}}^{\mathrm{T}}QB^{\perp} < 0$.
- $\exists\lambda \in \mathbb{R} : Q - \lambda B^{\mathrm{T}}B < 0$.
- $\exists\zeta \in \mathbb{R}^{n\times m} : Q + \zeta B + B^{\mathrm{T}}\zeta^{\mathrm{T}} < 0$.

where $B^{\perp}$ is a basis for the null space of B. This lemma is used to combine conditions on the values of the Lyapunov functions at the boundaries of polyhedral regions in the proofs of Theorems 1 and 2.

III. STABILIZATION USING STATE-BASED SWITCHING

The main aim is to drive the state of system (3), with $u, \omega \equiv 0$, to a desired state x_{r}. Given the desired state x_{r} and the switched system (3), the error system is formulated as follows:

$$\begin{aligned} &\dot{e}(t) = A_{\sigma(t),\ell}e(t) + q_{\sigma(t),\ell}, \qquad \text{if } e \in \mathcal{E}_{\sigma(t),\ell} \\ &e(t) = x(t) - x_{\mathrm{r}}, \qquad q_{\sigma(t),\ell} = b_{\sigma(t),\ell} + A_{\sigma(t),\ell}x_{\mathrm{r}} \end{aligned} \tag{4}$$

Now the aim is redefined as to design a switching rule that asymptotically steers the state of the error system to the origin.

Before proceeding with the main results, note that in the process of approximating nonlinear subsystems by piecewise affine subsystems, the number of affine functions for each subsystem and also the polyhedral regions and their boundaries may differ for different subsystems. Therefore, in order to help with the design approach and obtain stability conditions that can be easily evaluated, we redefine the polyhedral partitions and their boundaries in such a way that in each region we only have affine subsystems. Since we provide a common partitioning for all subsystems, from now we use $\mathcal{E}_\ell$ instead of $\mathcal{E}_{\sigma(t),\ell}$. Each polyhedral region $\mathcal{E}_\ell$ is characterized by:

$$\underbrace{\begin{bmatrix} F_\ell & f_\ell \end{bmatrix}}_{\bar{F}_\ell} \begin{bmatrix} e \\ 1 \end{bmatrix} \succeq 0, \qquad \text{iff } e \in \mathcal{E}_\ell, \tag{5}$$

where the inequality is element-wise. Furthermore, the boundary hyperplane for each pair of neighboring regions $\mathcal{E}_\ell$ and $\mathcal{E}_{\ell'}$ is represented by:

$$h_{\ell\ell'}^{\mathrm{T}}e + g_{\ell\ell'} = 0 \Leftrightarrow \underbrace{\begin{bmatrix} h_{\ell\ell'}^{\mathrm{T}} & g_{\ell\ell'} \end{bmatrix}}_{\bar{h}_{\ell\ell'}^{\mathrm{T}}} \begin{bmatrix} e \\ 1 \end{bmatrix} = 0 \tag{6}$$

Moreover, for each polyhedral region $\mathcal{E}_\ell, \ell \in \mathcal{M} = \{1, \ldots, M\}$, with M the total number of polyhedral regions (number of affine functions associated to each subsystem), the following auxiliary functions are defined:

$$V_{i,\ell}(e) = \begin{bmatrix} e \\ 1 \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix}}_{\bar{P}_{i,\ell}} \underbrace{\begin{bmatrix} e \\ 1 \end{bmatrix}}_{\bar{e}}, \quad \forall i \in \mathcal{N}, \forall \ell \in \mathcal{M}. \tag{7}$$

with $P_{i,\ell} \in \mathbb{R}^{n\times n}$ symmetric, $s_{i,\ell} \in \mathbb{R}^n$, and $r_{i,\ell} \in \mathbb{R}$. For each $\mathcal{E}_\ell$, a Lyapunov function is proposed as follows:

$$\mathcal{V}_\ell(e) = \min_{i\in\mathcal{N}} V_{i,\ell}(e), \tag{8}$$

The following theorem presents the design procedure for a stabilizing switching rule that brings the state of the error system (4) to the origin, provided that at least for one subsystem $q_{i,\ell} = 0$ in the polyhedral regions containing the origin. This means that the desired state x_{r} is an (unstable or stable) equilibrium of at least one of the subsystems of (3).

Theorem 1: Assume there exists at least one subsystem $\hat{i}$ with $q_{\hat{i},\ell} = 0$ in regions containing the origin. Moreover, suppose there exist symmetric matrices $P_{i,\ell}$ and $\mathcal{T}_{i,j,\ell}$, vectors $s_{i,\ell}$, $\zeta_{\ell\ell'}$, scalars $r_{i,\ell}$, and symmetric matrices U_ℓ, Z_ℓ with nonnegative elements that satisfy (10)–(16) for a given positive scalar $\mu_{\min} > 0$. Then the switching rule[1]:

$$\sigma(t) = \arg\ \min_{i\in\mathcal{N}} V_{i,\ell}(e(t)), \qquad \text{if} \quad e(t) \in \mathcal{E}_\ell, \tag{9}$$

with $V_{i,\ell}$ defined as in (7), will asymptotically bring the state of the error system (4), with $u, \omega \equiv 0$, to the origin.

Proof: Suppose that at an arbitrary time instant $t \geq 0$ and based on the polyhedral region ℓ in which the state of the error system resides, the switching law is given by $\sigma(t) = r(e(t)) = i$ for some $i \in \mathcal{I}_\ell(e) = \{i : \ \mathcal{V}_\ell(e) = V_{i,\ell}(e)\}$. Hence, following the definition of the Dini derivative [3], [17], for our error system (4), we have

$$\begin{aligned} \mathbf{D}^{+}\big(\mathcal{V}_\ell(e)\big) &= \min_{j\in\mathcal{I}_\ell(e(t))} \left[\frac{\partial V_{j,\ell}}{\partial e}\Big(A_{j,\ell}e + q_{j,\ell}\Big)\right] \\ &\leq \frac{\partial V_{i,\ell}}{\partial e}\Big(A_{i,\ell}e + q_{i,\ell}\Big) \end{aligned} \tag{17}$$

where i denotes the index of the active subsystem in region ℓ determined from (9). Pre-multiplying (10) by $[e^{\mathrm{T}}, 1]$ and post-multiplying by its transpose, we obtain (18). Using the fact that for the polyhedral region ℓ, (5) holds, and U_ℓ has nonnegative entries, and since for the active subsystem i, $V_{i,\ell} \leq V_{j,\ell}, \forall j \neq i, j \in \{1, \ldots, N\}$, the last inequality in (18) is less than zero, which means that the derivative of the Lyapunov function $V_{i,\ell}$ along the trajectory of the subsystem i in the polyhedral region ℓ is negative.

[1]Note that in (9), we take the minimum argument, in case of having multiple minima $V_{i,\ell}$.

$$\begin{bmatrix} P_{i,\ell}A_{i,\ell} + A_{i,\ell}^{\mathrm{T}}P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}}A_{i,\ell} + q_{i,\ell}^{\mathrm{T}}P_{i,\ell} & q_{i,\ell}^{\mathrm{T}}s_{i,\ell} + s_{i,\ell}^{\mathrm{T}}q_{i,\ell} \end{bmatrix} + \mu_{\min}\sum_{j\in\mathcal{N}}\left(\begin{bmatrix} P_{j,\ell} & \star \\ s_{j,\ell}^{\mathrm{T}} & r_{j,\ell}\end{bmatrix} - \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell}\end{bmatrix}\right) + \bar{F}_\ell^{\mathrm{T}}U_\ell\bar{F}_\ell < 0,$$
$$\forall (i,\ell) \in \ \{(i,\ell)\in\mathcal{N}\times\mathcal{M} | i \neq \hat{\imath}\} \cup \{(\hat{\imath},\ell)\in\mathcal{N}\times\mathcal{M} | 0 \notin \mathcal{E}_\ell\}, \quad (10)$$

$$P_{\hat{\imath},\ell}A_{\hat{\imath},\ell} + A_{\hat{\imath},\ell}^{\mathrm{T}}P_{\hat{\imath},\ell} - \sum_{j\in\mathcal{N}, j\neq\hat{\imath}}\mathcal{T}_{\hat{\imath},j,\ell} + F_\ell^{\mathrm{T}}U_\ell F_\ell < 0, \qquad \forall \ell\in\mathcal{M} : 0 \in \mathcal{E}_\ell, \quad (11)$$

$$\sum_{j\in\mathcal{N}, j\neq\hat{\imath}}\begin{bmatrix}\mathcal{T}_{\hat{\imath},j,\ell} & 0 \\ 0 & 0\end{bmatrix} < \mu_{\min}\sum_{j\in\mathcal{N}}\left(\begin{bmatrix} P_{\hat{\imath},\ell} & 0 \\ 0 & r_{\hat{\imath},\ell}\end{bmatrix} - \begin{bmatrix} P_{j,\ell} & \star \\ s_{j,\ell}^{\mathrm{T}} & r_{j,\ell}\end{bmatrix}\right), \qquad \forall \ell\in\mathcal{M} : 0 \in \mathcal{E}_\ell, \quad (12)$$

$$\begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell}\end{bmatrix} - \bar{F}_\ell^{\mathrm{T}}Z_\ell\bar{F}_\ell > 0, \qquad \forall i\in\mathcal{N},\ \ell\in\mathcal{M}, \quad (13)$$

$$s_{\hat{\imath},\ell} = 0, \qquad \forall \ell\in\mathcal{M} : 0 \in \mathcal{E}_\ell, \quad (14)$$

$$r_{\hat{\imath},\ell} < r_{j,\ell}, \qquad \forall j\in\mathcal{N},\ j\neq\hat{\imath}, \quad \forall \ell\in\mathcal{M} : 0 \in \mathcal{E}_\ell, \quad (15)$$

$$\begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell}\end{bmatrix} = \begin{bmatrix} P_{i,\ell'} & \star \\ s_{i,\ell'}^{\mathrm{T}} & r_{i,\ell'}\end{bmatrix} + \bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}} + \zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}}, \qquad \forall \ell,\ell'\in\mathcal{M} : \mathcal{E}_\ell\cap\mathcal{E}_{\ell'}\neq\emptyset, \quad \forall i\in\mathcal{N}, \quad (16)$$

$$\underbrace{\begin{bmatrix} e \\ 1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} P_{i,\ell}A_{i,\ell} + A_{i,\ell}^{\mathrm{T}}P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}}A_{i,\ell} + q_{i,\ell}^{\mathrm{T}}P_{i,\ell} & q_{i,\ell}^{\mathrm{T}}s_{i,\ell} + s_{i,\ell}^{\mathrm{T}}q_{i,\ell}\end{bmatrix}\begin{bmatrix} e \\ 1\end{bmatrix}}_{\frac{\partial V_{i,\ell}}{\partial e}\left(A_{i,\ell}e + q_{i,\ell}\right)} < -\begin{bmatrix} e \\ 1\end{bmatrix}^{\mathrm{T}}\bar{F}_\ell^{\mathrm{T}}U_\ell\bar{F}_\ell\begin{bmatrix} e \\ 1\end{bmatrix} + \mu_{\min}\sum_{j\in\mathcal{N}}\begin{bmatrix} e \\ 1\end{bmatrix}^{\mathrm{T}}(\bar{P}_{i,\ell} - \bar{P}_{j,\ell})\begin{bmatrix} e \\ 1\end{bmatrix} \quad (18)$$

The same procedure is applied to (11). Note that (11) provides the same condition as in (10) but with illuminating the rows and columns with zero elements. This condition along with (12) guarantee that the derivative of the Lyapunov function $V_{\hat{\imath},\ell}$ would be zero only when the state e is zero and when the value of the Lyapunov function is less than the other Lyapunov functions. Note that using the conditions (14) and (15) we ensure that the minimum of the overall Lyapunov function occurs at the origin and the derivative of the active Lyapunov function at the origin is zero.

Moreover, the Lyapunov functions (7) are not required to be positive definite in the entire space but only in the active polyhedral region. This is ensured using constraint (13) and it can be easily proved using (5).

In order to have global asymptotic stability, the decrease of the Lyapunov function inside the polyhedral regions is not enough. One way to tackle this problem is to equalize the values of the Lyapunov functions $\mathcal{V}_{i,\ell}$ and $\mathcal{V}_{i,\ell'}$ for the boundary hyperplane of neighboring regions $\mathcal{E}_\ell$ and $\mathcal{E}'_\ell$. Note that at the boundary between polyhedral regions an uncontrolled switching between affine functions of the same subsystem i occurs. Therefore, we only need to connect the Lyapunov functions associated with each subsystem i at the boundary between neighboring regions ℓ and ℓ'. Hence, we need

$$\bar{e}^{\mathrm{T}}\bar{P}_{i,\ell}\bar{e} = \bar{e}^{\mathrm{T}}\bar{P}_{i,\ell'}\bar{e},\ \forall e:\ \bar{h}_{\ell\ell'}^{\mathrm{T}}\bar{e} = 0 \quad (19)$$

In order to recast (19) as an LMI, we define auxiliary vectors $\zeta_{\ell\ell'}$ and combine the two equalities in (19) as follows (using the Finsler Lemma):

$$\bar{e}^{\mathrm{T}}\bar{P}_{i,\ell}\bar{e} = \bar{e}^{\mathrm{T}}\bar{P}_{i,\ell'}\bar{e} + \bar{e}^{\mathrm{T}}\bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}}\bar{e} + \bar{e}^{\mathrm{T}}\zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}}\bar{e}, \quad (20)$$

Since (20) should hold for all e, we can instead check the feasibility of the equality (16). ■

Remark 1: Note that in case there are multiple subsystems of (4) that have an equilibrium at the origin, we assign the index $\hat{\imath}$ to one of them arbitrarily and check the feasibility of the conditions in Theorem 1. In case the conditions are found to be infeasible, we can assign the index $\hat{\imath}$ to another subsystem and repeat the procedure.

Remark 2: By fixing $\mu_{\min}$, the conditions (10)–(16) become linear. Therefore, the overall feasibility problem can be solved using combined line search on $\mu_{\min}$ and an LMI feasibility checking algorithm.

Constraint (16) could be conservative in the sense that subsystem i in region ℓ might not become active right before the uncontrolled switching at the boundary between regions ℓ and ℓ'. Therefore, the following theorem is proposed in which constraint (16) is removed and instead, we impose constraints on the Lyapunov functions of the *active* subsystems at the boundary between polyhedral regions.

Theorem 2: Assume there exist symmetric matrices $P_{i,\ell}$, $\mathcal{T}_{i,j,\ell}$, $\mathcal{R}_{i,j,\ell}$, vectors $s_{i,\ell}$, $\zeta_{\ell\ell'}$, scalars $r_{i,\ell}$, $\beta_{\min} > 0, \mu_{\min} > 0$, and symmetric matrices U_ℓ, Z_ℓ with nonnegative elements that satisfy (10)–(15) and the following matrix inequalities:

$$\bar{P}_{i,\ell'} - \bar{P}_{j,\ell} - \mathcal{R}_{i,j,\ell} + \bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}} + \zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}} \leq 0, \quad (21)$$
$$\mathcal{R}_{i,j,\ell} < \beta_{\min}(\bar{P}_{i,\ell} - \bar{P}_{j,\ell}), \quad (22)$$
$$\forall i,j\in\mathcal{N},\ i\neq j, \quad \forall \ell,\ell'\in\mathcal{M} : \mathcal{E}_\ell\cap\mathcal{E}_{\ell'}\neq\emptyset,$$

then the switching rule (9) with $V_{i,\ell}$ defined as in (7), will asymptotically bring the state of system (4) to the origin.

Proof: We consider a transition from region $\mathcal{E}_\ell$ to region $\mathcal{E}_{\ell'}$. Pre- and post-multiplying (21) and (22) by $\bar{e}^{\mathrm{T}}$ and by its transpose respectively, will result in:

$$V_{i,\ell'} - V_{j,\ell} + \underbrace{\bar{e}^{\mathrm{T}}\bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}}\bar{e} + \bar{e}^{\mathrm{T}}\zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}}\bar{e}}_{=0,\ \text{if}\ e\in\mathcal{E}_\ell\cap\mathcal{E}_{\ell'}} \leq \bar{e}^{\mathrm{T}}\mathcal{R}_{i,j,\ell}\bar{e}$$
$$< \beta_{\min}(V_{i,\ell} - V_{j,\ell}),\ \forall \ell,\ell' \in \mathcal{M} : \mathcal{E}_\ell \cap \mathcal{E}_{\ell'} \neq \emptyset \quad (23)$$

Now if at the boundary between regions $\mathcal{E}_\ell$ and $\mathcal{E}_{\ell'}$, subsystem i is active, which means:

$$\beta_{\min}(V_{i,\ell} - V_{j,\ell}) \leq 0, \quad \forall j \in \mathcal{N}, \quad (24)$$

then, due to (23), $V_{i,\ell'} \leq V_{j,\ell},\ \forall j \in \mathcal{N}$. Hence, the value of the Lyapunov function $V_{i,\ell'}$ for the subsequent polyhedral region $\mathcal{E}_{\ell'}$ would be:

$$V_{i,\ell'} \leq \min_{j\in\mathcal{N}, j\neq i} V_{j,\ell} \quad (25)$$

The same reasoning holds for moving from region $\mathcal{E}_{\ell'}$ to $\mathcal{E}_\ell$. In contrast to condition (16), conditions (21)–(22) impose constraints only on the values of the Lyapunov functions of the *active* subsystems at the boundaries and moreover, these values no longer need to coincide with the ones of the respective subsystems in the previous regions.

In the end, based on (25) we can conclude that the overall Lyapunov function for the error system (4) will be decreasing and therefore, the error state would asymptotically approaches zero using the switching strategy (9). ■

Remark 3: With fixed scalar variables $\mu_{\min}$ and $\beta_{\min}$, conditions (10)–(15) and (21)–(22) will become LMIs. Therefore, the overall feasibility problem can be solved using LMI solvers along with line search on $\mu_{\min}$ and $\beta_{\min}$.

Remark 4 (sliding mode): The previous results are developed without taking into account the possible sliding modes inside polyhedral regions and/or on the boundaries. For inside the polyhedral regions we prove that even if a sliding mode occurs (as a result of switching between subsystems) it will be always stable. It can be shown that the time-derivative of the minimum Lyapunov function is strictly negative along the Filippov solution of the system (similar to the approach in Remark 2 of [4]), as follows ($0 \leq \theta_{i,\ell},\ \sum_{i\in\mathcal{N}}\theta_{i,\ell} = 1$):

$$\frac{\partial V_{j,\ell}}{\partial e}\sum_{i\in\mathcal{N}}\theta_{i,\ell}\Big(A_{i,\ell}e + q_{i,\ell}\Big)$$
$$\leq \sum_{i\in\mathcal{N}}\theta_{i,\ell}\frac{\partial V_{i,\ell}}{\partial e}\Big(A_{i,\ell}e + q_{i,\ell}\Big) < 0, \quad \text{for } e \in \mathcal{E}_\ell \quad (26)$$

where the last inequality is justified using the same reasoning as in (17) and (18), and the first inequality holds from the fact that under sliding mode a switching from subsystem j to subsystem i is allowed only if:

$$\frac{\partial V_{j,\ell}}{\partial e}(A_{i,\ell}e + q_{i,\ell}) \leq \frac{\partial V_{i,\ell}}{\partial e}(A_{i,\ell}e + q_{i,\ell}). \quad (27)$$

However, if there exist attractive sliding modes on the boundaries of polyhedral regions, they should be taken into account on the stability analysis as it is also studied by [15] for PWA systems. Similar to the approach in [15], if there exists a sliding set $\mathcal{S}$ of the following general form:

$$\mathcal{S} = \{e|\ \Phi\bar{e} \geq 0 \wedge \Psi\bar{e} = 0\}, \quad (28)$$

with Φ and Ψ the matrices characterizing the sliding set, then for neighboring polyhedral regions $\mathcal{E}_\ell$ and $\mathcal{E}_{\ell'}$ with $\mathcal{E}_\ell\cap\mathcal{S} \neq 0$, $\mathcal{E}_{\ell'}\cap\mathcal{S} \neq 0$, we need to have:

$$\frac{\partial V_{i,\ell}}{\partial e}\big(A_{i,\ell'}e + q_{i,\ell'}\big) < 0, \quad \forall e \in \mathcal{S}, \quad \forall i \in \mathcal{N}, \quad (29)$$

in order to ensure the stability of the Filippov solutions. Since the uncontrolled switching at the boundaries occurs only for the affine functions of the same subsystem, therefore in (29) we require the negativeness of $V_{i,\ell}$ only on the trajectories of the same subsystem i in the neighboring region ℓ'. Using the S-procedure and the Finsler Lemma, the following LMIs can be established:

$$\bar{P}_{i,\ell}\bar{A}_{i,\ell'} + \bar{A}_{i,\ell'}^{\mathrm{T}}\bar{P}_{i,\ell} + \Phi^{\mathrm{T}}\Lambda_{i,\ell,\ell'}\Phi + \eta_{i,\ell,\ell'}\Psi^{\mathrm{T}}\Psi < 0,$$
$$\forall i \in \mathcal{N},\ \forall \ell,\ell' \in \mathcal{M}, \quad (30)$$

with $\Lambda_{i,\ell,\ell'}$ symmetric matrices with nonnegative elements, and $\eta_{i,\ell,\ell'}$ scalar multipliers.

IV. Stabilization of the original switched nonlinear system

In this section, we discuss the stability of the switched nonlinear system (1) using the switching law designed based on the approximated switched affine system (3). For simplicity and without loss of generality we assume that $x_{\mathrm{r}} = 0$. The approximation error can be defined as follows:

$$\epsilon_i(x) = f_i(x) - \big(A_{i,\ell}x + b_{i,\ell}\big) \quad \forall i \in \mathcal{N},\ \text{for } x \in \mathcal{X}_\ell. \quad (31)$$

Suppose that the original switched nonlinear system (1) is controlled by the switching law (9). Therefore when $\sigma(t) = i$, the dynamics of (1) is governed by f_i. Hence, the derivative of the Lyapunov function (8) along the trajectories of (1) is:

$$\dot{V}_\ell = \begin{bmatrix} f_i(x) \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} f_i(x) \\ 0 \end{bmatrix} \quad (32)$$

for $x \in \mathcal{X}_\ell$ (note that since the continuity of V_ℓ on the boundaries of the polyhedral regions is preserved under conditions of Theorem 1, we therefore only consider the behavior of V_ℓ and $\dot{V}_\ell$ inside the polyhedral regions). Replacing $f_i(x)$ by $\epsilon_i(x) + A_{i,\ell}x + b_{i,\ell}$ yields:

$$\dot{V}_\ell = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{i,\ell}^{\mathrm{T}}P_{i,\ell} + P_{i,\ell}A_{i,\ell} & \star \\ b_{i,\ell}P_{i,\ell} + s_{i,\ell}^{\mathrm{T}}A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}}s_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$
$$+ 2\begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix}. \quad (33)$$

Now since the inequalities in (10) of Theorem 1 are strict, it implies that if (10) holds, there should exist a positive scalar variable denoted by α such that:

$$\begin{bmatrix} P_{i,\ell}A_{i,\ell} + A_{i,\ell}^{\mathrm{T}}P_{i,\ell} & \star \\ b_{i,\ell}^{\mathrm{T}}P_{i,\ell} + s_{i,\ell}^{\mathrm{T}}A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}}s_{i,\ell} \end{bmatrix} - \mu_{\min}\sum_{j\in\mathcal{N}}(\bar{P}_{j,\ell} - \bar{P}_{i,\ell})$$
$$+ \bar{F}_\ell^{\mathrm{T}}U_\ell\bar{F}_\ell < -\alpha I, \qquad \forall i \in \mathcal{N},\ \forall \ell \in \mathcal{M} \quad (34)$$

Now if (34) holds, we obtain:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell}A_{i,\ell} + A_{i,\ell}^{\mathrm{T}}P_{i,\ell} & \star \\ b_{i,\ell}^{\mathrm{T}}P_{i,\ell} + s_{i,\ell}^{\mathrm{T}}A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}}s_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < -\alpha\|\bar{x}\|_2^2, \quad (35)$$

for the active subsystem i in (3). Therefore, for (33) we have:

$$\dot{V}_\ell < -\alpha\|\bar{x}\|_2^2 + 2\begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix}, \qquad (36)$$

for $x \in \mathcal{X}_\ell$. Therefore, in order to have $\dot{V}_\ell < 0$ for the switched nonlinear system, we need to have:

$$2\begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix} < \alpha\|\bar{x}\|_2^2 \qquad (37)$$

The following proposition provides the sufficient condition for stabilization of the switched nonlinear system (1) using switching law (9).

Proposition 1: Assume there exist matrices $P_{i,\ell}$ and $\mathcal{T}_{i,j,\ell}$, vectors $s_{i,\ell}$, $\zeta_{\ell\ell'}$, scalars $r_{i,\ell}$, $\alpha > 0$ and symmetric matrices U_ℓ, Z_ℓ with nonnegative elements that satisfy (12)–(16) and (34) for a given positive scalar $\mu_{\min} > 0$. Then the switching rule (9) asymptotically stabilizes (1) provided that the norm of the PWA approximation error is bounded by:

$$\|\epsilon_i(x)\|_2 < \frac{\alpha\|\bar{x}\|_2}{2a_{\max}(\bar{P}_{i,\ell})}, \qquad \forall i \in \mathcal{N}, \text{ for } x \in \mathcal{X}_\ell, \qquad (38)$$

where $a_{\max}(\bar{P}_{i,\ell})$ denotes the largest singular value of $\bar{P}_{i,\ell}$.

Proof: First, it is can be easily proved that:

$$\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix} \le \|\bar{x}\|_2 a_{\max}(\bar{P}_{i,\ell})\|\epsilon_i(x)\|_2 \qquad (39)$$

Therefore, using (38) we obtain:

$$2\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix} \le 2\|\bar{x}\|_2 a_{\max}(\bar{P}_{i,\ell})\|\epsilon_i(x)\|_2 \le \alpha\|\bar{x}\|_2^2 \quad (40)$$

which yields $\dot{V}_\ell < 0$ as in (36) and hence, asymptotic stability of the switched nonlinear system (1) is ensured. ■

Remark 5: As can be inferred from (38), the upper bound on the approximation error $\epsilon_i(x)$ depends on the maximum singular values of the $\bar{P}_{i,\ell}$ matrices. Therefore, the upper bound on the approximation error can be further relaxed if we formulate an optimization problem to minimize the maximum singular values of $\bar{P}_{i,\ell}$ matrices that satisfy (13)–(16) and (34).

V. Illustrative examples

In this section, two examples are presented to evaluate and compare the performance of the stabilizing approaches proposed in Section III.

Example 1: In this example, we use the conditions presented in Theorem 1 to design a stabilizing switching law. We directly use the error model (4) with the following matrices:

$$F_1 = -F_3 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \; F_2 = -F_4 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$
$$\bar{h}_{12} = \bar{h}_{34} = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}, \; \bar{h}_{23} = \bar{h}_{41} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}},$$
$$A_{1,1} = \begin{bmatrix} 3 & 1 \\ -5 & -8 \end{bmatrix}, \; A_{2,1} = \begin{bmatrix} -2 & 6 \\ 2 & 9 \end{bmatrix}, \; A_{3,1} = \begin{bmatrix} 4 & 4 \\ -2 & 3 \end{bmatrix},$$

$$A_{1,2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}, \; A_{2,2} = \begin{bmatrix} -4 & 1 \\ -2 & 6 \end{bmatrix}, \; A_{3,2} = \begin{bmatrix} 2.5 & 7 \\ 2 & -9 \end{bmatrix},$$
$$A_{1,3} = \begin{bmatrix} 5 & 3 \\ -2 & -4 \end{bmatrix}, \; A_{2,3} = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}, \; A_{3,3} = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix},$$
$$A_{1,4} = \begin{bmatrix} 6 & -2 \\ -4 & 5 \end{bmatrix}, \; A_{2,4} = \begin{bmatrix} -5 & 1 \\ -2 & 3 \end{bmatrix}, \; A_{3,4} = \begin{bmatrix} -1 & -3 \\ 2 & 8 \end{bmatrix}.$$

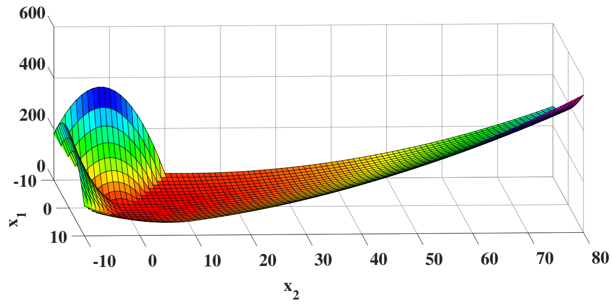


Fig. 1. Example 1: Plot of the overall Lyapunov function, its continuity is preserved over boundaries of the regions.

Note that none of the subsystems is stable. Using line search and the Yalmip toolbox (with SeDuMi solver), the feasibility problem (10)–(16) is solved. The obtained matrices for the Lyapunov functions are reported in [18]. Fig. 1 illustrates the overall Lyapunov function (obtained by taking the minimum of the Lyapunov functions in each region). As depicted in Fig. 2, the designed switching control strategy is able to steer the error state to the origin for different initial conditions.

Example 2: In this example, we aim at stabilizing a switched nonlinear system formulated as [19]:

$$\dot{x}_1(t) = -G_{1,\sigma}(x_1(t)) + \omega_1(t), \qquad (41)$$
$$\dot{x}_2(t) = -G_{2,\sigma}(x_2(t)) + G_{1,\sigma}(x_1(t)) + \omega_2(t), \qquad (42)$$

with $G_{j,i}(x_j) = C_{j,i} \cdot x_j \cdot \exp(-1/2 \cdot (x_j/x_{j,\mathrm{cr}})^2), j \in \{1, 2\}$. These exponential functions can be approximated by piecewise affine functions. The result is a switched affine system with mixed switching and with the following system matrices:

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; F_2 = \begin{bmatrix} 1 & 0 & -x_{1,\mathrm{cr}} \\ 0 & -1 & x_{2,\mathrm{cr}} \\ 0 & 1 & 0 \\ -1 & 0 & x_{1,\max} \end{bmatrix},$$
$$F_3 = \begin{bmatrix} -1 & 0 & x_{1,\mathrm{cr}} \\ 0 & 1 & -x_{2,\mathrm{cr}} \\ 1 & 0 & 0 \\ 0 & -1 & x_{2,\max} \end{bmatrix}, \; F_4 = \begin{bmatrix} 1 & 0 & -x_{1,\mathrm{cr}} \\ 0 & 1 & -x_{2,\mathrm{cr}} \\ -1 & 0 & x_{1,\max} \\ 0 & -1 & x_{2,\max} \end{bmatrix},$$
$$\bar{h}_{12} = \bar{h}_{34} = \begin{bmatrix} 1 & 0 & -x_{1,\mathrm{cr}} \end{bmatrix}^{\mathrm{T}}, \; \bar{h}_{13} = \bar{h}_{24} = \begin{bmatrix} 0 & 1 & -x_{2,\mathrm{cr}} \end{bmatrix}^{\mathrm{T}},$$
$$A_{i,1} = \frac{1}{3600} \cdot \begin{bmatrix} -u_i \cdot 10.28 & 0 \\ u_i \cdot 10.28 & -8.4 \end{bmatrix}, \; b_{i,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_{i,2} = \frac{1}{3600} \cdot \begin{bmatrix} u_i \cdot 6.4 & 0 \\ -u_i \cdot 6.4 & -8.4 \end{bmatrix}, \; b_{i,2} = \begin{bmatrix} -u_i \cdot 16.22 \\ u_i \cdot 16.22 \end{bmatrix},$$
$$A_{i,3} = \frac{1}{3600} \cdot \begin{bmatrix} -u_i \cdot 10.28 & 0 \\ u_i \cdot 10.28 & 4.5 \end{bmatrix}, \; b_{i,3} = \begin{bmatrix} 0 \\ -10.75 \end{bmatrix},$$
$$A_{i,4} = \frac{1}{3600} \cdot \begin{bmatrix} u_i \cdot 6.4 & 0 \\ -u_i \cdot 6.4 & 4.5 \end{bmatrix}, \; b_{i,4} = \begin{bmatrix} -u_i \cdot 16.22 \\ u_i \cdot 16.22 - 10.75 \end{bmatrix}$$

with $x_{1,\mathrm{cr}} = 3500,\ x_{2,\mathrm{cr}} = 3000,\ x_{1,\max} = 10000,\ x_{2,\max} = 9000$ and $u_i \in \{0.1, 0.35, 0.65, 0.9\}$.

Further, the system is exposed to the bounded disturbances illustrated in Fig. 3(a). In order to stabilize the switched nonlinear system and to minimize the effect of the disturbances, Theorem 2 (conditions of Theorem 1 were found infeasible for this example) is extended for L_2-gain minimization. Since there is no state feedback controller u, the extension is quite

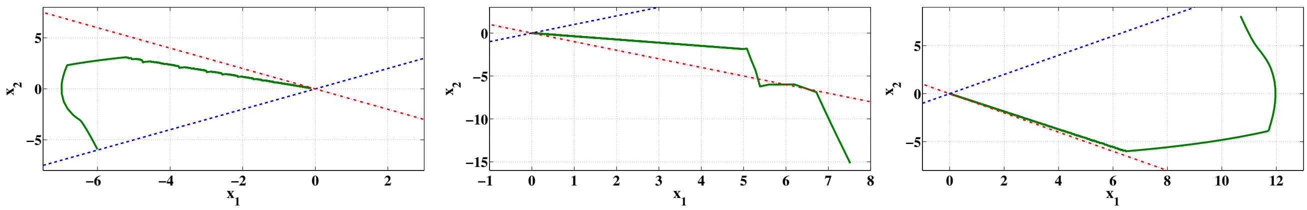
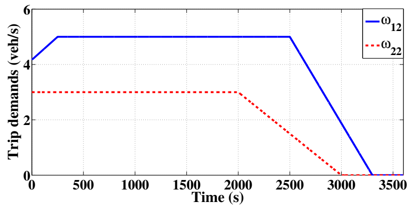
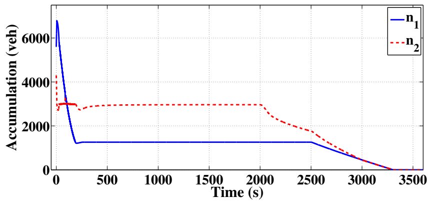


Fig. 2. Simulation of the closed-loop system for different initial states. The dashed blue and red lines represent the boundaries.



(a)



(b)

Fig. 3. Example 2: (a) disturbance signals, (b) result of using switching control, (c) uncontrolled case.

straightforward and the presentation is skipped in this paper. Note that we use the LMI solver SeDuMi and the Yalmip toolbox along with search on $\mu_{\min}$ and $\beta_{\min}$ (from 0 to 200 for each, with steps of 1). The obtained values for $\mu_{\min}$ and $\beta_{\min}$ corresponding to the minimum upper bound for the L_2-gain are 159 and 10, respectively. The Lyapunov matrices are presented in [18]. The switching control scheme is connected to the original switched nonlinear system. The results are depicted in Fig. 3. As inferred from Fig. 3(b), the switching control stabilizes the system and also significantly reduces the effects of the disturbances, while in the no control case, the states grow unbounded, as shown in Fig. 3(c). Furthermore, setting the initial accumulations to zero, the actual L_2-gain is $0.0881 \cdot 3600$ which is lower than the theoretical value $0.1332 \cdot 3600$ obtained by solving the optimization problem.

VI. Conclusion

Design of stabilizing controllers for switched affine systems with mixed switching types have been presented. The switched system has both autonomous and controlled switching patterns. To obtain less conservative control design approaches, first we have used a joint polyhedral partitioning of the entire state space and next, we have relaxed the continuity requirement for the Lyapunov functions over boundaries of polyhedral regions. Finally, we have presented sufficient conditions for stabilizing switched nonlinear systems using the proposed control schemes. Possible extensions of the current work are, 1) further reducing the conservatism using a joint time- and state-based switching strategy and the concept of average dwell-time [7], [20], 2) extending the approach for robust H_∞ control as also partially done in [18].

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