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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Minimal conjunctive normal expression of continuous piecewise affine functions

Jun Xu, Ton J. J. van den Boom, Bart De Schutter, and Xionglin Luo

***Abstract*—Continuous piecewise affine (PWA) functions arise in many aspects of control. For this kind of function, we propose the minimal conjunctive normal expression (CNE). The CNE can be expressed as the minimum of a collection of terms, each of which is the maximum of a set of affine functions. The minimal CNE is defined to contain the smallest number of parameters. Analogous to Boolean algebra, we propose implicants and prime implicants for continuous PWA functions. After obtaining all prime implicants, the problem of finding minimal CNEs can then be cast as a binary programming problem. A sharp bound on the number of boolean variables in the binary programming problem is given. In two worked examples, minimal CNEs are derived for given continuous PWA functions.**

***Index Terms*—Continuous piecewise affine, minimal expression, conjunctive normal expression.**

I. Introduction

For a continuous piecewise affine (PWA) function, the domain is partitioned into finite nonoverlapping polyhedra, in each of which the continuous PWA function is affine. Continuous PWA functions form the "simplest" extension of linear functions and they can approximate smooth nonlinear functions in a compact set with arbitrary precision [1]. Continuous PWA functions have been introduced for the modeling of nonlinear circuits [2], and they find extensive applications in modeling and control, such as dynamic system modeling, model predictive control (MPC), and constructing Lyapunov functions [3], [4].

In the context of modeling of dynamic system, continuous PWA systems are equivalent to several classes of hybrid systems [5]. Typical examples of hybrid systems are manufacturing systems, telecommunication and computer networks, traffic control systems, digital circuits, and logistic systems. Nonlinear smooth systems can also be approximated with continuous PWA systems [6], [7].

Continuous PWA functions also appear in MPC, in which an optimization problem is solved at each time step. For the optimization problems with affine constraints, if the cost function is convex quadratic or polyhedral, i.e., the epigraph of the cost is a polyhedron, the optimal control law of the optimization problem is continuous PWA with respect to the state; moreover, the optimal MPC objective function is continuous PWA if the cost is polyhedral. Continuous PWA functions can also express cost functions of min-max control problems for uncertain linear systems [8] or uncertain continuous PWA systems [9], [10].

There are several representation forms for continuous PWA functions. In [11], continuous PWA functions are expressed by the polyhedral regions and affine functions defined on them. Then in several papers [1], [12], [13], continuous PWA functions are expressed via basis functions expansion, i.e., linear combinations of a set of basis functions. There is also another representation form of continuous PWA functions, which is min-max (or max-min) function, i.e., the minimum (maximum) of a set of terms, each of which is the maximum (minimum) of several affine functions. In fact, these representations first appeared in [14], and nowadays find applications in generating nonlinear functions, discrete event system and model predictive control [15], [16]. In this paper, the min-max form is called the conjunctive normal expression (CNE) and the max-min form is called the disjunctive normal expression (DNE), which follows the terminology in Boolean algebra.

It is mentioned in [10] that the CNE will facilitate the research on structural properties of the system, such as controllability, reachability, and observability. Besides, if the MPC objective function is expressed as a CNE, only a set of linear programming problems have to be solved at each time step and the MPC optimization is simplified [15]. There are also approaches to remove redundant parameters in CNE or DNE [16]–[18]. However, none of these results has the minimal number of parameters. Other papers like [19], [20] reduce the complexity of the continuous PWA function through merging polyhedra in the function, and [20] finds the minimal number of polyhedra. In the current paper we consider minimal CNEs with the minimal number of parameters. It is important to note that the results can easily be extended to DNEs due to duality.

The paper is organized as follows. Section II introduces continuous PWA functions and CNEs. Then Section III defines implicants and prime implicants. In Section IV, minimal CNEs are derived through solving a binary programming problem. Besides, a sharp bound on the number of binary variables is given and the time complexity is discussed. Section V gives two examples for which minimal CNEs are obtained. The paper ends with conclusions and future work in Section VI.

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Jun Xu is with Department of Automation, China University of Petroleum (Beijing), 102249, Beijing, China and also with Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands (e-mail: xujunqgy@gmail.com).

Ton J. J. van den Boom and Bart De Schutter are with Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands (email: a.j.j.vandenBoom@tudelft.nl; b.deschutter@tudelft.nl).

Xionglin Luo is with Department of Automation, China University of Petroleum (Beijing), 102249, Beijing, China (luoxionglin@gmail.com).

II. Continuous PWA function and CNE

A scalar-valued continuous PWA function is defined as follows:

Definition 1. *A function $f : \mathbb{D} \to \mathbb{R}$ where $\mathbb{D} \subseteq \mathbb{R}^n$ is convex, is said to be continuous PWA if it is continuous on the domain $\mathbb{D}$ and the following conditions are satisfied:*

- *There are a finite number of nonempty, nonoverlapping full-dimensional polyhedra $\Omega_1, \ldots, \Omega_{\hat{N}}$ with $\Omega_i \subseteq \mathbb{D}$, the union of which covers $\mathbb{D}$, i.e.,*
$$\mathbb{D} = \Omega_1 \cup \cdots \cup \Omega_{\hat{N}}.$$
We call $\Omega_i, i = 1, \ldots, \hat{N}$ regions.
- *In each region Ω_i, the function f equals a local affine function ℓ_j with $j \in \{1, \ldots, M\}$. The affine functions ℓ_j and ℓ_k are different if $j \neq k$; hence, M is the number of distinct affine functions.*

According to [21], the region Ω_i can be further divided into so called nonempty, nonoverlapping base regions, in the interior of which no other affine functions intersect with the local affine function. Therefore, we then have

$$\mathbb{D} = \cup_{i=1}^{N} \mathbb{D}_i$$

with $\text{int}(\mathbb{D}_i) \neq \emptyset$ and $\text{int}(\mathbb{D}_i) \cap \text{int}(\mathbb{D}_j) = \emptyset, i, j \in \{1, \ldots, N\}, i \neq j$.

Besides, in each base region $\mathbb{D}_i, i = 1, \ldots, N$, we have

$$\{x | \ell_j(x) = \ell_{\text{act}(i)}(x), j \neq \text{act}(i)\} \cap \text{int}(\mathbb{D}_i) = \emptyset, \quad (1)$$

where $\ell_{\text{act}(i)}$ is called the active affine function in $\mathbb{D}_i$, i.e.,

$$f(x) \equiv \ell_{\text{act}(i)}(x), \forall x \in \mathbb{D}_i.$$

From Definition 1, it is clear that $\text{act}(i) \in \{1, \ldots, M\}$.

Analogous to the conjunctive normal form of Boolean functions, considering two binary operations "min" and "max", we define the conjunctive normal expression (CNE) for continuous PWA functions as follows:

Definition 2. *The CNE of a continuous PWA function f as defined in Definition 1 is defined as*

$$\min_{i \in \mathcal{I}} \{\max_{j \in I_i} \{\ell_j\}\} \quad (2)$$

where $\mathcal{I}$ and I_i are index sets. We call $\max_{j \in I_i}\{\ell_j\}$ with $i \in \mathcal{I}$ a term, and the affine functions $\ell_j, j \in I_i$ are called literals.

In fact, the CNE appears in [14], [15], [22], [23], in which different names are given for (2). It has also been proved in these papers that any continuous PWA function can be expressed as a CNE (2).

For a continuous PWA function, there may exist many CNEs. Then following gives a simple example to illustrate this.

Example 1. *Consider the following 1-dimensional continuous PWA function defined on $[0, 4]$:*

$$f(x) = \begin{cases} \ell_1(x) = -0.5x + 2.5, \ x \in [0, \ 1], \\ \ell_2(x) = -2x + 4, \ x \in [1, \ 2], \\ \ell_3(x) = 2x - 4, \ x \in [2, \ 2.5], \\ \ell_4(x) = 1, \ x \in [2.5, \ 4]. \end{cases} \quad (3)$$

Fig. 1 shows the plot of this function. We can see that there are 4 regions and 6 base regions with $\Omega_1 = \mathbb{D}_1$, $\Omega_2 = \mathbb{D}_2 \cup \mathbb{D}_3$, $\Omega_3 = \mathbb{D}_4$ and $\Omega_4 = \mathbb{D}_5 \cup \mathbb{D}_6$.

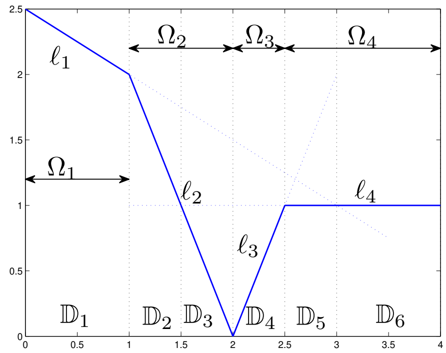

Fig. 1. A 1-dimensional example of a continuous PWA function.

For the two CNEs

$$f_1(x) = \min\{\max\{-0.5x + 2.5, 1\}, \max\{-2x + 4, 2x - 4\}, \max\{-2x + 4, 1\}\}, \quad (4)$$

$$f_2(x) = \min\{\max\{-0.5x + 2.5, 2x - 4, 1\}, \max\{-2x + 4, 2x - 4\}, \max\{-2x + 4, 1\}\}, \quad (5)$$

we can verify that both of them equal f in $[0, 4]$. However, there is one more literal in (5).

In the following, we derive minimal CNEs for continuous PWA functions.

III. Implicants and prime implicants

The notions "implicant" and "prime implicant" play an important role in the derivation of the conjunctive normal form for Boolean functions. Hence here similarly, we define these two in the context of continuous PWA functions.

Definition 3. *For a continuous PWA function f as defined in Definition 1, we say $T_i = \max_{j \in I_i}\{\ell_j\}$ is an implicant of f, if*

$$T_i(x) \geq f(x), \forall x \in \mathbb{D},$$

and there is some base region $\mathbb{D}_k$ such that $T_i \equiv f$ in $\mathbb{D}_k$. The implicant $T_i = \max_{j \in I_i}\{\ell_j\}$ is a prime implicant if there exists no other implicant $T_r = \max_{j \in I_r}\{\ell_j\}$ of f such that $I_r \subsetneq I_i$.

Next we explain how to identify an implicant.

Lemma 1. *Assume $T_{i_k} = \max_{j \in I_{i_k}}\{\ell_j\}$ is an implicant of f. Then there must exist some index $i \in \{1, \ldots, N\}$ such that*

$$I_{i_k} \subseteq I_{\leq, i}, \quad (6)$$

in which $I_{\leq, i} = \{v \in \{1, \ldots, M\} | \ell_v(x) \leq \ell_{\text{act}(i)}(x), \forall x \in \mathbb{D}_i\}$. Besides, $T_{i_k} \equiv f$ in $\mathbb{D}_i$.

Proof. As a dual result of Lemma 2 of [21], we have

$$\max_{j \in I_{\leq, i}} \{\ell_j(x)\} \geq f(x), \forall x \in \mathbb{D}$$

and

$$\max_{j\in I_{\le,i}}\{\ell_j(x)\} = f(x), \forall x \in \mathbb{D}_i.$$

Assume (6) does not hold, i.e., for all $t = 1,\dots,N$, we have

$$I_{i_k} \nsubseteq I_{\le,t}.$$

Then for each $t = 1,\dots,N$, there exists some $t_k \in I_{i_k}$ such that $t_k \notin I_{\le,t}$. According to (1) we have

$$T_{i_k}(x) \ge \ell_{t_k}(x) > \ell_{\text{act}(t)}(x) = f(x), \forall x \in \text{int}(\mathbb{D}_t).$$

Hence in all base regions $\mathbb{D}_1,\dots,\mathbb{D}_N$, we have $T_{i_k} \not\equiv f$. This contradicts that T_{i_k} is an implicant of f. Therefore, there exists some index $i \in \{1,\dots,N\}$ such that $I_{i_k} \subseteq I_{\le,i}$.

In the base region $\mathbb{D}_i$ with $I_{i_k} \subseteq I_{\le,i}$, we now have

$$f = \max_{j\in I_{\le,i}}\{\ell_j\} \ge T_{i_k} \ge f.$$

Thus $T_{i_k} \equiv f$ in the base region $\mathbb{D}_i$. □

According to Lemma 1, now we can calculate $I_{\le,i}$ for each base region $\mathbb{D}_i, i = 1,\dots,N$, and then obtain all prime implicants by searching for all the index sets $I_{i_k} \subseteq I_{\le,i}$ such that $T_{i_k} = \max_{j\in I_{i_k}}\{\ell_j\}$ is an implicant of f and any $\max_{j\in I_{i_r}}\{\ell_j\}$ with $I_{i_r} \subsetneq I_{i_k}$ is not an implicant of f. We denote all prime implicants of f defined in Definition 1 as $T_i = \max_{j\in I_i}\{\ell_j\}, i \in \mathcal{S}$.

IV. Deriving minimal CNEs of continuous PWA functions

A. Minimal CNE

We define the minimal CNE with the least number of parameters, i.e., the number of integers in the index sets $I_i, i \in \mathcal{I}$ plus the number of real parameters in the affine functions.

Definition 4. *The CNE*

$$g^*(x) = \min_{i\in\mathcal{I}^*}\{T_i(x)\} = \min_{i\in\mathcal{I}^*}\{\max_{j\in I_i}\{\ell_j\}\}, \forall x \in \mathbb{D} \tag{7}$$

is called a minimal CNE of f in $\mathbb{D}$ if $g^* \equiv f$ in $\mathbb{D}$ and $\sum_{i\in\mathcal{I}^*}|I_i| \le \sum_{i\in\mathcal{I}}|I_i|$ for all $g_i = \min_{i\in\mathcal{I}}\{\max_{j\in I_i}\{\ell_j\}\} = f$. Here $|I_i|$ is the cardinality of I_i and $|I_i| \le M$.

It is obvious that all the distinct affine functions appear in the expression of g^* and g. In fact, the number of parameters is $\sum_{i\in\mathcal{I}^*}|I_i| + M\cdot(n+1)$, where $M\cdot(n+1)$ is the number of parameters in the M affine functions. As $M\cdot(n+1)$ is constant, we only consider $\sum_{i\in\mathcal{I}^*}|I_i|$.

The following theorem shows that the terms in g^* are prime implicants.

Theorem 1. *For a minimal CNE $g^* = \min_{i\in\mathcal{I}^*}\{T_i\}$ of the continuous PWA function f as defined in Definition 1, the terms $T_i, i \in \mathcal{I}^*$ must be prime implicants of f, i.e., $\mathcal{I}^* \subseteq \mathcal{S}$.*

Proof. First we show that T_i with $i \in \mathcal{I}^*$ is an implicant. If this is not true, then there would exist some term $T_\alpha = \max_{j\in I_\alpha}\{\ell_j\}, \alpha \in \mathcal{I}^*$ such that T_α is not an implicant of f.

According to Definition 3, there could be two cases. The first one is that there exists some $\hat{x} \in \mathbb{D}$ such that $T_\alpha(\hat{x}) < f(\hat{x})$. The second is that $T_\alpha(x) \ge f(x), \forall x \in \mathbb{D}$ but $T_\alpha \not\equiv f$ in each $\mathbb{D}_k, k = 1,\dots,N$.

Case 1: There exists an $\hat{x} \in \mathbb{D}$ such that $T_\alpha(\hat{x}) < f(\hat{x})$. In this case, we have

$$g^*(\hat{x}) \le T_\alpha(\hat{x}) < f(\hat{x}),$$

contradicting that $g^* \equiv f$ in $\mathbb{D}$.

Case 2: $T_\alpha(x) \ge f(x), \forall x \in \mathbb{D}$ but $T_\alpha \not\equiv f$ in each $\mathbb{D}_k, k = 1,\dots,N$. In this case, as (1) holds, we have

$$T_\alpha(x) > \ell_{\text{act}(k)}(x), \forall x \in \text{int}(\mathbb{D}_k), \forall k = 1,\dots,N. \tag{8}$$

As $g^* = \min_{i\in\mathcal{I}^*}\{T_i\}$ satisfies $g^* \equiv f$ in each $\mathbb{D}_k, k = 1,\dots,N$, according to (8), we have

$$\min_{i\in\mathcal{I}^*\backslash\{\alpha\}}\{T_i(x)\} \equiv f(x), \forall x \in \text{int}(\mathbb{D}_k), \forall k = 1,\dots,N.$$

Then it follows from continuity that

$$\min_{i\in\mathcal{I}^*\backslash\{\alpha\}}\{T_i(x)\} \equiv f(x), \forall x \in \mathbb{D}_k, \forall k = 1,\dots,N. \tag{9}$$

As (9) holds and $\sum_{i\in\mathcal{I}^*}|I_i| > \sum_{i\in\mathcal{I}^*\backslash\{\alpha\}}|I_i|$, the expression g^* is not a minimum CNE, yielding contradiction.

Now we prove that every term should also be a prime implicant. Assume $T_\beta = \max_{j\in I_\beta}\{\ell_j\}$ is not a prime implicant, then we can obtain a prime implicant $\bar{T}_\beta = \max_{j\in\bar{I}_\beta}\{\ell_j\}$ with $\bar{I}_\beta \subsetneq I_\beta$ such that

$$T_\beta(x) \ge \bar{T}_\beta(x) \ge f(x), \forall x \in \mathbb{D}.$$

Hence, we have

$$\begin{aligned} f(x) = g^*(x) &= \min\left(\min_{i\in\mathcal{I}^*\backslash\{\beta\}}\{T_i(x)\}, T_\beta(x)\right) \\ &\ge \min\left(\min_{i\in\mathcal{I}^*\backslash\{\beta\}}\{T_i(x)\}, \bar{T}_\beta(x)\right) \\ &\ge \min\left(\min_{i\in\mathcal{I}^*\backslash\{\beta\}}\{T_i(x)\}, f(x)\right) \ge f(x), \forall x \in \mathbb{D}. \end{aligned}$$

So $\min\left(\min_{i\in\mathcal{I}^*\backslash\{\beta\}}\{T_i\}, \bar{T}_\beta\right) \equiv f$ in $\mathbb{D}$. As $\sum_{i\in\mathcal{I}^*}|I_i| > \sum_{i\in\mathcal{I}^*\backslash\{\beta\}}|I_i| + |\bar{I}_\beta|$, the expression g^* is not a minimum CNE, yielding contradiction. Therefore, every term in g^* is a prime implicant, i.e., $\mathcal{I}^* \subseteq \mathcal{S}$. □

According to Theorem 1, now the problem of finding a minimal CNE consists of choosing $\mathcal{I}^* \subseteq \mathcal{S}$ such that $g^* = \min_{i\in\mathcal{I}^*}\{T_i\} = f$ and g^* has the smallest number of parameters.

Suppose $\mathcal{I} \subseteq \mathcal{S}$, then we first investigate the conditions under which $\min_{i\in\mathcal{I}}\{T_i\}$ is equivalent to f. To facilitate the derivation, we define the coverage vector of an implicant T_i as

$$\text{cov}(T_i) = [\text{cov}(T_i)_1,\dots,\text{cov}(T_i)_N]^T,$$

where the elements $\text{cov}(T_i)_\upsilon, \upsilon \in \{1,\dots,N\}$ are given by

$$\text{cov}(T_i)_\upsilon = \begin{cases} 1 & \text{if } T_i(x) = \ell_{\text{act}(\upsilon)}(x) = f(x), \forall x \in \mathbb{D}_\upsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Lemma 2. *The CNE $g = \min_{i\in\mathcal{I}}\{T_i\}$ with $\mathcal{I} \subseteq \mathcal{S}$ satisfies*

$$g(x) = f(x),\ \forall x \in \mathbb{D} \tag{11}$$

if and only if

$$\sum_{i\in\mathcal{I}} \mathrm{cov}(T_i) \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{12}$$

Proof. The condition (12) means that $\sum_{i\in\mathcal{I}} \mathrm{cov}(T_i)_k \geq 1$ for all $k \in \{1, \ldots, N\}$. According to the definition of $\mathrm{cov}(T_i)$, this means that for each $k \in \{1, \ldots, N\}$, there exists at least one $i_k \in \mathcal{I}$ such that $\mathrm{cov}(T_{i_k})_k = 1$, i.e.,

$$T_{i_k}(x) = \ell_{\mathrm{act}(k)}(x) = f(x), \forall x \in \mathbb{D}_k. \tag{13}$$

Necessity. Assume (11) is valid, i.e., $g(x) = f(x)$ for all $x \in \mathbb{D}_k, k = 1, \ldots, N$. Since each $T_i, i \in \mathcal{I}$ is a prime implicant of f, due to (1), for all $k = 1, \ldots, N$, we have either

$$T_i(x) > f(x), \forall x \in \mathrm{int}(\mathbb{D}_k)$$

or

$$T_i(x) = f(x), \forall x \in \mathrm{int}(\mathbb{D}_k).$$

Then for all $k = 1, \ldots, N$, as $g = f$ in $\mathbb{D}_k$, there exists at least one $i_k \in \mathcal{I}$ such that (13) holds, i.e., (12) is valid.

Sufficiency. If (12) is valid, for each $k \in \{1, \ldots, N\}$ we have at least one $i_k \in \mathcal{I}$ such that (13) holds, which directly yields the validity of (11). $\square$

Now we show how to derive a minimal CNE g^*.

Analogous to the minimization of Boolean functions [24], the derivation of minimal CNEs of a continuous PWA function f can then be cast as the following binary programming problem:

$$\begin{aligned} \min_{s_i, i\in\mathcal{S}} \quad & J(s_i) \\ s.t. \quad & \sum_{i\in\mathcal{S}} s_i \mathrm{cov}(T_i) \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \\ & s_i = 0 \text{ or } 1, \forall i \in \mathcal{S}, \end{aligned} \tag{14}$$

where the optimized variables are the binary variables $s_i, i \in \mathcal{S}$. Each s_i corresponds to a prime implicant T_i. The case $s_i = 1$ is equivalent to the inclusion of T_i in the CNE, while $s_i = 0$ corresponds to the exclusion of T_i in the CNE. The cost function $J(s_i)$ can be expressed as

$$J(s_i) = \sum_{i\in\mathcal{S}} s_i |I_i|.$$

The optimal $s_i^*, i \in \mathcal{S}$ follows from solving the optimization problem (14). Let $\mathcal{I}^* = \{k | s_k^* = 1\}$, then $g^* = \min_{i\in\mathcal{I}^*}\{T_i\}$ is a minimal CNE according to Definition 4.

It can be seen in (14) that the number of binary variables s_i, i.e., $|\mathcal{S}|$, determines the complexity of the optimization problem. So next we will discuss a bound on $|\mathcal{S}|$ with respect to the number of distinct affine functions M in the continuous PWA function f.

B. Bound on the number of prime implicants

Theorem 2. *Given a continuous PWA function f with M distinct affine functions, the number of prime implicants satisfies*

$$|\mathcal{S}| \leq \begin{pmatrix} M \\ \lfloor \frac{M}{2} \rfloor \end{pmatrix}, \tag{15}$$

in which $\lfloor \frac{M}{2} \rfloor$ is the largest integer no greater than $\frac{M}{2}$. The bound is sharp, i.e., for each M, we can construct a continuous PWA function f such that $|\mathcal{S}| = \begin{pmatrix} M \\ \lfloor \frac{M}{2} \rfloor \end{pmatrix}$.

Proof. From Lemma 1, for a prime implicant $T_i = \max_{j\in I_i}\{\ell_j\}$ with $I_i \subseteq \{1, \ldots, M\}$, there is no $I_r \subsetneq I_i$ such that $T_r = \max_{j\in I_r}\{\ell_j\}$ is also an implicant of f. Hence, for two prime implicants T_i and T_k, neither $I_i \subseteq I_k$ nor $I_k \subseteq I_i$. We say that I_i and I_k are incomparable. Therefore, all index sets of prime implicants of f are pairwise incomparable subsets of $\{1, \ldots, M\}$.

It is demonstrated by Sperner's theorem [25], [26] that the maximal number of pairwise incomparable subsets of $\{1, \ldots, M\}$ does not exceed $\begin{pmatrix} M \\ \lfloor \frac{M}{2} \rfloor \end{pmatrix}$. Besides, all subsets of $\{1, \ldots, M\}$ with $\lfloor \frac{M}{2} \rfloor$ elements constitute such pairwise incomparable subsets. Therefore, $|\mathcal{S}| \leq \begin{pmatrix} M \\ \lfloor \frac{M}{2} \rfloor \end{pmatrix}$.

Now we prove that there exists a continuous PWA function f with M distinct affine functions such that $|\mathcal{S}| = \begin{pmatrix} M \\ \lfloor \frac{M}{2} \rfloor \end{pmatrix}$.

It is shown in [14] that if $M \leq n + 1$, the intersections of M distinct affine functions can divide the domain $\mathbb{D}$ into at most $M!$ distinct regions $\mathbb{D}_1, \ldots, \mathbb{D}_{M!}$, in the interior of which no intersections of any two affine functions occur. Therefore, those regions satisfy (1), and hence can be regarded as base regions.

Besides, it is also shown in [14] that for each region $\mathbb{D}_i$, there is a tuple $(a_{i,1} \cdots a_{i,M})$ with $a_{i,1}, \ldots, a_{i,M} \in \{1, \ldots, M\}$ such that

$$\ell_{a_{i,1}}(x) < \cdots < \ell_{a_{i,M}}(x), \forall x \in \mathrm{int}(\mathbb{D}_i).$$

Moreover, for any $i \neq k$, the tuples $(a_{i,1} \cdots a_{i,M})$ and $(a_{k,1} \cdots a_{k,M})$ are different. As there are $M!$ regions, the tuples $(a_{1,1} \cdots a_{1,M}), \ldots, (a_{M!,1} \cdots a_{M!,M})$ correspond to all possible permutations of $\{1, \ldots, M\}$.

Consider the arrangement of $\ell_1, \ldots, \ell_M$ yielding $M!$ base regions as described above. Let f be the $\lfloor \frac{M}{2} \rfloor$-th minimum of $\ell_1, \ldots, \ell_M$, i.e.,

$$f(x) = \ell_{\tau(\lfloor \frac{M}{2} \rfloor)}(x) \tag{16}$$

with $\ell_{\tau(1)}(x) \leq \ell_{\tau(2)}(x) \leq \cdots \leq \ell_{\tau(M)}(x)$. It is shown in [27] that f is a continuous PWA function.

In each base region $\mathbb{D}_k, k \in \{1, \ldots, M!\}$, the index $\tau(\lfloor \frac{M}{2} \rfloor)$ may be different. In fact, for each $k \in \{1, \ldots, M!\}$, we have

$$f(x) = \ell_{\tau(\lfloor \frac{M}{2} \rfloor)}(x) = \ell_{a_{k,\lfloor \frac{M}{2} \rfloor}}(x), \forall x \in \mathbb{D}_k,$$

i.e.,

$$\tau\left(\lfloor \frac{M}{2} \rfloor\right) = a_{k,\lfloor \frac{M}{2} \rfloor} = \mathrm{act}(k), \forall x \in \mathbb{D}_k.$$

We now prove that the number of prime implicants of the continuous function described in (16) is $\left(\begin{array}{c} M \\ \lfloor \frac{M}{2} \rfloor \end{array}\right)$.

For each base region $\mathbb{D}_k, k = 1, \ldots, M!$, let

$$T_k = \max_{j \in I_{\leq,k}} \{\ell_j\}$$

then there are $\lfloor \frac{M}{2} \rfloor$ indices in $I_{\leq,k}$. According to Lemma 1, T_k is an implicant of f. Next we will show that T_k is also a prime implicant of f.

For each $t \in I_{\leq,k}$, we collect the base regions $\mathbb{D}_{t_k}$ with tuples $(a_{t_k,1} \cdots a_{t_k,M})$ such that

$$a_{t_k,\lfloor \frac{M}{2} \rfloor} = t \tag{17}$$

and

$$\{a_{t_k,1}, \ldots, a_{t_k,\lfloor \frac{M}{2} \rfloor}\} = \{a_{k,1}, \ldots, a_{k,\lfloor \frac{M}{2} \rfloor}\}. \tag{18}$$

As there are $(\lfloor \frac{M}{2} \rfloor - 1)!$ possible permutations of $\{a_{t_k,1}, \ldots, a_{t_k,\lfloor \frac{M}{2} \rfloor - 1}\}$ and $(M - \lfloor \frac{M}{2} \rfloor)!$ permutations of $\{a_{t_k,\lfloor \frac{M}{2} \rfloor + 1}, \ldots, a_{t_k,M}\}$, the number of base regions satisfying (17) and (18) is $(\lfloor \frac{M}{2} \rfloor - 1)!(M - \lfloor \frac{M}{2} \rfloor)!$.

From (17) and (18), we have

$$T_k(x) = \max_{j \in I_{\leq,k}} \{\ell_j(x)\} = \ell_t(x) = f(x), \forall x \in \mathbb{D}_{t_k}.$$

Thus for all $x \in \mathrm{int}(\mathbb{D}_{t_k})$, we have $\max\limits_{j \in I_{\leq,k} \setminus \{t\}} \{\ell_j(x)\} < f(x)$, meaning that $\max\limits_{j \in I_{\leq,k} \setminus \{t\}} \{\ell_j\}$ is not an implicant of f. Therefore, the term $T_k = \max\limits_{I_{\leq,k}} \{\ell_j\}$ is a prime implicant.

As the number of distinct $I_{\leq,k}$ is $\left(\begin{array}{c} M \\ \lfloor \frac{M}{2} \rfloor \end{array}\right)$, the number of prime implicants of (16) is

$$|\mathcal{S}| = \left(\begin{array}{c} M \\ \lfloor \frac{M}{2} \rfloor \end{array}\right).$$

□

C. Complexity analysis

In this paper, complexity refers to storage requirement, offline processing and online evaluation complexity. Similar to the results in [21] and [16], the minimal CNE has to store $(n+1) \cdot M$ real numbers and $\sum\limits_{i=1}^{\tilde{N}} |I_i|$ integer numbers, in which $\tilde{N}$ denotes the number of terms. Besides, the online evaluation complexity is $O(\tilde{N}^2)$.

Assuming that the continuous PWA function is given in the form of polyhedra and affine functions defined on them, the offline processing step includes calculating the base regions and index sets $I_{\leq,i}, i = 1, \ldots, N$; getting all prime implicants; and solving the binary programming problem (14). It is indicated in Lemma 6 of [21] that the worst-case offline complexity for the first step can be approximated by $O(\sum\limits_{i=1}^{\hat{N}} 2^{M-1}(r_i + M - 1)^3 L_i)$, in which $\hat{N}$ is the number of polyhedra, r_i is the number of linear inequalities defining the i-th polyhedron and L_i denotes the bit length of the linear programming (A.11) in [21].

To get all prime implicants, we search for all the index sets I_{i_k} with $I_{i_k} \subsetneq I_{\leq,i}, i = 1, \ldots, N$ such that $\max\limits_{j \in I_{i_k}} \{\ell_j\}$ is an implicant of f, and any $\max\limits_{j \in I_{i_r}} \{\ell_j\}$ with $I_{i_r} \subsetneq I_{i_k}$ is not an implicant of f. As each subset of $I_{\leq,i}$ should contain at least one element, there are at most $\sum\limits_{i=1}^{N} 2^{|I_i|-1}$ subsets. Since the worst-case complexity for checking whether a term is an implicant is $O(M^2 N)$ according to [21], the worst-case time complexity for deriving all prime implicants is $O\left(2^{M-1} M^2 N^2\right)$.

According to Theorem 2, the number of binary variables in the binary programming problem (14) is no more than $\left(\begin{array}{c} M \\ \lfloor \frac{M}{2} \rfloor \end{array}\right)$. According to Stirling's formula [28], when M is large, we have $M! \approx \sqrt{2\pi M} \left(\frac{M}{e}\right)^M$, and then $\left(\begin{array}{c} M \\ \lfloor \frac{M}{2} \rfloor \end{array}\right) \approx c\frac{2^M}{\sqrt{M}}$, where c is a constant. We use $O(\mathrm{bp}(N, c\frac{2^M}{\sqrt{M}}))$ to denote the complexity of solving (14), which has N constraints and $c\frac{2^M}{\sqrt{M}}$ binary variables. According to [29], the problem (14) is strongly NP-complete and $O(\mathrm{bp}(N, c\frac{2^M}{\sqrt{M}})) \geq O((\frac{c \cdot 2^M}{\sqrt{M}})^{2N+2} \cdot N^{(N+1)(2N+1)})$.

Generally speaking, $O\left(2^{M-1} M^2 N^2\right), O(\sum\limits_{i=1}^{\hat{N}} 2^{M-1}(r_i + M - 1)^3 L_i) \ll O\left((\frac{c \cdot 2^M}{\sqrt{M}})^{2N+2} \cdot N^{(N+1)(2N+1)}\right)$, then the worst-case offline complexity for deriving the minimal CNE is $O(\mathrm{bp}(N, c\frac{2^M}{\sqrt{M}}))$.

V. Examples

Example 1 (Continued): Reconsidering Example 1, first we calculate the index sets $I_{\leq,k}$ for each base region $\mathbb{D}_k, k = 1, \ldots, 6$:

$$I_{\leq,1} = \{1,3,4\},\ I_{\leq,2} = \{2,3,4\},\ I_{\leq,3} = \{2,3\},$$
$$I_{\leq,4} = \{2,3\},\ I_{\leq,5} = \{2,4\},\ I_{\leq,6} = \{1,2,4\}.$$

We obtain all 4 prime implicants:

$$T_1 = \max\{\ell_1, \ell_4\}, T_2 = \max\{\ell_1, \ell_3\},$$
$$T_3 = \max\{\ell_2, \ell_3\}, T_4 = \max\{\ell_2, \ell_4\}.$$

The corresponding coverage vectors are

$$\mathrm{cov}(T_1) = [1\ 0\ 0\ 0\ 0\ 1]^T, \mathrm{cov}(T_2) = [1\ 0\ 0\ 0\ 0\ 0]^T,$$
$$\mathrm{cov}(T_3) = [0\ 1\ 1\ 1\ 0\ 0]^T, \mathrm{cov}(T_4) = [0\ 1\ 0\ 0\ 1\ 1]^T.$$

Solving the binary programming (14) yields $s_1^* = 1, s_2^* = 0, s_3^* = 1, s_4^* = 1$. This corresponds to the CNE

$$\min\{\max\{\ell_1, \ell_4\}, \max\{\ell_2, \ell_3\}, \max\{\ell_2, \ell_4\}\},$$

which coincides with (4).

Moreover, $s_1^* = 0, s_2^* = 1, s_3^* = 1, s_4^* = 1$ is also an optimal solution and this corresponds to the CNE

$$\min\{\max\{\ell_1, \ell_3\}, \max\{\ell_2, \ell_3\}, \max\{\ell_2, \ell_4\}\}.$$

Hence for this problem, we have two minimal CNEs. In fact, these two are the only minimal CNEs for the continuous PWA function (3).

Example 2. *In this example we consider a continuous PWA state-feedback control law. In [30], the authors have formulated and solved a constrained finite time optimal control problem. The resulting explicit control laws* $u = [u_1, u_2]^T$ *as computed through the Multi-Parametric Toolbox 3 [31] consists of 674 polyhedra.*

For u_1*, there are 9 different control laws, and the 674 polyhedra are divided into 712 base regions, i.e.,* $M = 9, N = 712$*. It takes 7.9198s for the division and calculation of* $I_{\leq,i}, i = 1, \ldots, 712$*, then 0.2297s to obtain all the 10 prime implicants, and 0.5669s to solve the binary integer programming through the "intlinprog" function in MATLAB R2014b on a 2.4 GHz Intel Core i5 computer. A minimal CNE is obtained containing only 6 prime implicants.*

For u_2*, there are 5 different control laws, and the 674 polyhedra are divided into 846 base regions, i.e.,* $M = 5, N = 846$*. It takes 11.1268s to divide the polyhedral and calculate* $I_{\leq,i}, i = 1, \ldots, 846$*, then 0.0460s to obtain all the 3 prime implicants, and 0.0325s to solve the binary integer programming on the same computer. All of the 3 prime implicants are included in the resulting minimal CNE.*

In this example, as the number of distinct affine functions is small, the complexity of deriving a minimal CNE is small.

VI. Conclusions and Future work

In this paper, we have considered deriving a minimal conjunctive normal expression (CNE), or in other words, a minimal min-max expression, for general continuous piecewise affine (PWA) functions. Analogous to Boolean algebra, we have defined implicants and prime implicants for the continuous PWA function. The minimal CNE is proved to contain only prime implicants. A binary programming problem has been formulated to obtain a minimal CNE. Besides, a sharp bound on the number of binary variables is given. The complexity has also been analyzed. We have presented two examples, in which the minimal CNEs are derived for continuous PWA functions.

For a continuous PWA function with large n, M and N, the time complexity for deriving minimal CNEs is large. In the future we will consider some suboptimal strategy like [32] to trade-off between time complexity and optimality.

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