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Robust model predictive control for train regulation in underground railway transportation

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***Abstract*—This paper investigates the robust model predictive control for train regulation in underground railway transportation. By considering the uncertain passenger arrival flow, a constrained state-space model for the train traffic of a metro loop line is developed. The goal of the paper is to design a state-feedback control law at each decision step to optimize a metro system cost function subject to safety constraints on the control input. Based on Lyapunov function theory, the problem of optimizing an upper bound on the system cost function subject to input constraints is reduced to a convex optimization problem involving linear matrix inequalities (LMIs). Moreover, for the inevitable disturbances leading to the delays, the robust model predictive control strategy of train regulation is designed for a metro loop line such that it ensures the minimization of an upper bound on metro system cost function, and meanwhile guarantees a disturbance attenuation level with respect to the disturbances. Numerical examples are given to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Train regulation, metro loop line, inevitable disturbance, robust model predictive control, linear matrix inequalities.**

I. Introduction

In modern large cities, urban underground railway transportation is an attractive mode of transport for relieving the traffic pressure in an eco-friendly and sustainable manner. On a high-frequency line, the train delays increase at each station with the accumulation of passengers. Additionally, the inevitable disturbances such as equipment failure and inadequate driver/passenger action will also lead to deviations from the nominal time schedule. Since any deviation with respect to the nominal time schedule of a given train will be amplified with time, a high-frequency metro line is naturally unstable [1], [2]. Therefore, train regulation by manipulating the running time and the staying time of each train is necessary to recover from delays and to prevent the instability of metro line operations [3], [4].

Usually, the buffer times or supplements in timetable are designed to absorb the train delays resulting from disturbances [5], [6]. However, the buffer times allocation is static and cannot be used dynamically and flexibly from a system-wide point of view, which may reduce system capacity utilization. In particular, automatic train regulation (ATR) can be applied to recover the schedule/headway deviations resulting from disturbances by dynamically adjusting the running time and the staying time of each train. In [2], a state feedback control algorithm based on linear quadratic regulator approach was adopted to ensure the system stability and the minimization of a given performance index. From the passenger perspective, a constrained nonlinear programming problem for the metro traffic regulation was considered in [7]. In [8], a genetic algorithm was applied to solve the optimal train regulation problem efficiently. By using dual heuristic dynamic programming, the authors in [9] proposed an automatic train regulation approach, in which a near-optimal regulation can be obtained more rapidly and accurately. Besides, in [10], an adaptive-optimal-control algorithm was devised to optimize the train regulator through reinforcement learning. However, with the increase of the variables and constraints, the computation burden of the existing linear or nonlinear programming methods will increase, and as a result they are not suited for the computation of schedules for a whole day of operations.

As one of the most powerful directions of the modern control, model predictive control (MPC) is able to efficiently handle large scale optimization problems with hard physical constraints [11]–[15]. This feature makes MPC an ideal candidate for real-time metro traffic regulation too. In [16], a linear-programming-based MPC methodology for the computation of optimal train schedules in metro lines was proposed, which can effectively generate a metro line train schedule for a whole day. Besides, a predictive traffic regulation model for metro loop lines to optimize a cost function along a time horizon was addressed in [17]. Moreover, it is important to consider the uncertain passenger flow fluctuation and the inevitable disturbances leading to the delays of trains. MPC that takes into account such uncertainties/disturbances within its formulation, is called robust model predictive control scheme [18], [19]. Therefore, it is necessary to study the train regulation problem in underground railway transportation within the framework of robust MPC to deal with the uncertain passenger flow fluctuations and the inevitable disturbances.

Motivated by the above discussions, in this paper, we will design a new methodology for train regulation in underground railway transportation based on the robust MPC scheme. By considering the safety constraints, we develop a constrained state-space model for the train traffic of a metro operation. Based on Lyapunov function theory, the problem of minimizing an upper bound on the system cost function subject to input constraints is reduced to a convex optimization problem involving linear matrix inequalities (LMIs). Moreover, a robust

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MPC train regulation strategy is designed that ensures the minimization of an upper bound on the metro system cost function, and meanwhile guarantees a disturbance attenuation level with respect to the inevitable disturbance. The proposed robust MPC law is computed through a sequence of low-dimensional LMIs optimizations which help to reduce the online computation burden. Numerical examples are given to illustrate the effectiveness of the proposed methods.

The rest of this paper is organized as follows. In Section 2, a constrained train traffic model of a metro loop line with uncertain passenger arrival flow and disturbances is presented. In Section 3, the robust MPC law for train regulation of a metro loop line is designed. In Section 4, numerical examples are provided to demonstrate the effectiveness of the proposed methods. We conclude this paper in Section 5.

II. Problem formulation

Consider a set of M trains that are operating simultaneously on a metro loop line with N stations ($M < N$). The considered structure of the metro loop line is shown in Figure 1.

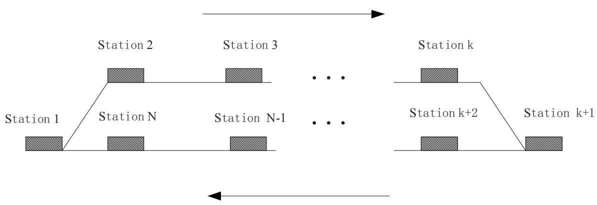

Figure 1. The structure of the metro line.

To describe the dynamic evolution of the traffic behavior, we use a two-index notation to identify the variables relative to a given train at a given station, in which the superscript i denotes the train number and the subscript k denotes the station number. For the metro loop line with M trains operating on N stations, the trains are running periodically, where train M is followed by train 1. To describe the train regulation problem within the framework of a state-space model, we adopt the number $i-1$ to denote the train ahead of train i. According to the above definition, the train ahead of train 1 is train 0, the ahead train of train 0 is train -1, and so on. Thus to facilitate the study, we adopt the virtual train number $\{2-M, 3-M, \ldots, -1, 0, 1, 2, \ldots, M\}$ to denote the physical train index, which is explained in the definition of the state vector $\bar{t}_j^i$ in the sequel. Additionally, to describe the state-transfer from one station to the next station, we use the virtual station number $\{1, 2, \ldots, N, N+1, \ldots\}$ to denote the physical station index.

A. The train traffic model of metro line

According to the practical operation of the metro line, the departure time of train i from station $k+1$ is given as [2]

$$t_{k+1}^i = t_k^i + r_k^i + s_{k+1}^i, \tag{1}$$

where t_k^i is the departure time of train i from station k, r_k^i is running time of train i from station k to $k+1$, and s_k^i is the staying time of train i at station k.

The running time of train i from station k to station $k+1$ is

$$r_k^i = R_k + {u_1}_k^i + {w_1}_k^i, \tag{2}$$

where R_k is the nominal running time from station k to $k+1$, ${u_1}_k^i$ represents the control strategy to adjust the running time of train i between station k and $k+1$, and ${w_1}_k^i$ is the uncertain disturbance term to the running time.

Suppose that the staying time of the trains at the station increases proportionally to the number of passengers getting on the train [2], [9]. According to this, the staying time s_k^i is modeled as

$$s_k^i = a\lambda_k(t_k^i - t_k^{i-1}) + S_k + {u_2}_k^i + {w_2}_k^i, \tag{3}$$

where a is the average boarding time per passenger, λ_k is the average passenger arrival rate at station k, S_k is the minimal staying time at station k when no passenger gets on the train, ${u_2}_k^i$ is the staying time adjustment on train i at station k, and ${w_2}_k^i$ is the uncertain disturbance term to the staying time. In practice, the average passenger arrival rate λ_k will change with the time. We assume the realized values of λ_k vary within a range of values that are symmetric around the known nominal value λ with half-length d. Then the realized values of the average passenger arrival rate $\tilde{\lambda}_k$ are given by

$$\tilde{\lambda}_k = \lambda + \alpha_k d, \ -1 \le \alpha_k \le 1. \tag{4}$$

where α_k will be changing at different stations. For the sake of simplicity, the known nominal value λ and the half-length d are assumed to be the same for each station. Since α_k is changing with the different stations k, the considered average passenger arrival rates $\tilde{\lambda}_k$ are different for different stations, which are more general and practical than the case in [2] that assumed that the average passenger arrival rates are identical for all stations. In addition, by adjusting the nominal value λ and the half-length value d, we can ensure a maximum allowance passengers arrival rate, so as to satisfy the limited capacity of the train for carrying passengers.

Combining (1)–(4), the train traffic model for the operation of the metro loop line is described by

$$\begin{aligned}&(1 - a\tilde{\lambda}_{k+1})t_{k+1}^i \\ &= t_k^i - a\tilde{\lambda}_{k+1}t_{k+1}^{i-1} + S_{k+1} + R_k + u_k^i + w_k^i,\end{aligned} \tag{5}$$

where $u_k^i = {u_1}_k^i + {u_2}_{k+1}^i$, $w_k^i = {w_1}_k^i + {w_2}_{k+1}^i$. For the security requirement to prevent collisions between trains, the control input u_k^i satisfies the bounded constraint $-\check{u}_k^i \le u_k^i \le \hat{u}_k^i$.

In order to study the train regulation problem in the metro loop line from a system-theoretic standpoint, we will define the matrix form of the train traffic model in the metro loop line.

B. The matrix form of train traffic model

To study the train regulation problems by using the control theory conveniently, we develop a state-space formulation for the train traffic of metro loop line. Firstly, the state vector and the control vector for each train i are defined according to [2] as follows:

$$\bar{t}_j^i = [t_{j-N}^i, t_{j-N+1}^i, \ldots, t_{j-M-1}^i, t_{j-M}^i, t_{j-M+1}^{i-1}, \ldots, t_{j-1}^{i-M+1}]^T, \ i \in \{1, 2, \ldots, M\}, \tag{6}$$

$$\bar{u}_j^i = [u_{j-M}^i, u_{j-M+1}^{i-1}, \ldots, u_{j-1}^{i-M+1}]^T, \tag{7}$$

where the subscripts $j-N, j-N+1, \ldots, j-1$ represent the station numbers. The state vector $\bar{t}_j^i$ is composed of N components, which includes all the departure times at the N stations (not only referred to the $(N-M)$ successive stations), in which the first $(N-M)$ components are the departure time of train i at $(N-M)$ successive stations (index $j-N$ to $j-M-1$), and the last M components are the departure time relative to the M trains at other M successive stations. It should be pointed out that the considered state vector $\bar{t}_j^i$ has an interesting property that the components of the state vector are known nearly simultaneously. This property makes possible real time practical implementation of a state feedback control policy, which has been also adopted in the existing literature [2]. According to (6), the superscript of the components of $\bar{t}_j^i$ is changing from $2-M$ to M. Thus, we adopt the virtual train number $\{2-M, 3-M, \ldots, -1, 0, 1, 2, \ldots, M\}$ to denote the physical train index,

Let $\tilde{c}_{1,k} = \frac{1}{1-a\tilde{\lambda}_k}$ and $\tilde{c}_{2,k} = \frac{-a\tilde{\lambda}_k}{1-a\tilde{\lambda}_k}$. Note that by the practical data in reality [2], [9], it always holds that $0 < a\tilde{\lambda}_k < 1$. According to $\tilde{\lambda}_k = \lambda + \alpha_k d, \ -1 \leq \alpha_k \leq 1$, $\tilde{c}_{1,k}$ can be rewritten as $\tilde{c}_{1,k} = c_1 + \beta_{1,k} d_1, \ -1 \leq \beta_{1,k} \leq 1$, where $c_1 = \frac{1}{2}(\frac{1}{1-a(\lambda-d)} + \frac{1}{1-a(\lambda+d)})$ and $d_1 = \frac{1}{2}(\frac{1}{1-a(\lambda+d)} - \frac{1}{1-a(\lambda-d)})$. Similarly, $\tilde{c}_{2,k} = c_2 + \beta_{2,k} d_2, \ -1 \leq \beta_{2,k} \leq 1$, where $c_2 = \frac{1}{2}(\frac{-a(\lambda+d)}{1-a(\lambda+d)} + \frac{-a(\lambda-d)}{1-a(\lambda-d)})$ and $d_2 = \frac{1}{2}(\frac{-a(\lambda-d)}{1-a(\lambda-d)} - \frac{-a(\lambda+d)}{1-a(\lambda+d)})$. To formulate a convenient alternative model, we introduce the following matrices: $A_{11} = [a_{ij}]_{(N-M)\times(N-M)}, a_{ij} = \begin{cases} 1, & i = j-1, \\ 0, & \text{otherwise} \end{cases}$, $A_{12} = [b_{ij}]_{(N-M)\times M}, b_{ij} = \begin{cases} 1, & i = N-M, \ j = M, \\ 0, & \text{otherwise} \end{cases}$, $A_{21} = [c_{ij}]_{M\times(N-M)}, c_{ij} = \begin{cases} c_2, & i = M, \ j = N-M, \\ 0, & \text{otherwise} \end{cases}$, and $A_{22} = [d_{ij}]_{M\times M}, d_{ij} = \begin{cases} c_1, & i = j, \\ c_2, & i = j-1 \\ 0, & \text{otherwise} \end{cases}$.

Then let $\bar{w}_j^i = [\frac{\tilde{c}_{1,j-M}}{c_1} w_{j-M}^i, \frac{\tilde{c}_{1,j-M+1}}{c_1} w_{j-M+1}^{i-1}, \ldots, \frac{\tilde{c}_{1,j-1}}{c_1} w_{j-1}^{i-M+1}]^T$, $\bar{R} = [R_{j-M}, R_{j-M+1}, \ldots, R_{j-1}]^T$, and $\bar{S} = [S_{j-M}, S_{j-M+1}, \ldots, S_{j-1}]^T$. The matrix form of the train traffic model from the basic model (5) can be expressed as

$$\begin{aligned}
\bar{t}_{j+1}^i = & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bar{t}_j^i + I_1 E_{1,j} \begin{bmatrix} 0 & 0 \\ 0 & D_{11} \end{bmatrix} \bar{t}_j^i \\
& + I_1 E_{2,j} \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \end{bmatrix} \bar{t}_j^i + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \bar{u}_j^i \\
& + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} E_{3,j} D_{11} \bar{u}_j^i + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \bar{w}_j^i \\
& + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \bar{R} + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \bar{S},
\end{aligned} \tag{8}$$

where $E_{1,j} = \text{diag}\{\beta_{1,j-N}, \beta_{1,j-N+1}, \ldots, \beta_{1,j-1}\}_{N\times N}$, $E_{2,j} = \text{diag}\{\beta_{2,j-N}, \beta_{2,j-N+1}, \ldots, \beta_{2,j-1}\}_{N\times N}$, $E_{3,j} = \text{diag}\{\beta_{1,j-M}, \beta_{1,j-M+1}, \ldots, \beta_{1,j-1})\}_{M\times M}$, $I_1 = I_{N\times N}$, $I_2 = I_{M\times M}$, $B_1 = \text{diag}\{c_1, c_1, \ldots, c_1\}_{M\times M}$, $D_{11} = \text{diag}\{d_1, d_1, \ldots, d_1\}_{M\times M}$, $D_{21} = [f_{ij}]_{M\times(N-M)}, f_{ij} = \begin{cases} d_2, & i = M, \ j = N-M, \\ 0, & \text{otherwise} \end{cases}$, $D_{22} = [g_{ij}]_{M\times(N-M)}, g_{ij} = \begin{cases} d_2, & i = j-1, \\ 0, & \text{otherwise} \end{cases}$.

In addition, we define the nominal time schedule representing the operation of the system with $u_k^i = 0$ and $w_k^i = 0$, which is presented as

$$T_{k+1}^i = T_k^i + R_k + a\tilde{\lambda}_{k+1}(T_{k+1}^i - T_{k+1}^{i-1}) + S_{k+1}, \tag{9}$$

where the nominal time schedule is characterized by a constant time interval H between two successive trains, i.e.,

$$H = T_k^{i+1} - T_k^i. \tag{10}$$

Due to the periodic circulation of the trains on the metro loop line, the following equation should be satisfied

$$MH = \sum_{k=1}^{N} (R_k + a\tilde{\lambda}_k H + S_k). \tag{11}$$

To guarantee that the trains on the metro loop line are operating according to the nominal time schedule, let e_k^i be the deviation of the actual departure time t_k^i from the nominal value T_k^i, i.e., $e_k^i = t_k^i - T_k^i$. According to (8) and (9), the error state-space model is obtained as

$$\begin{aligned}
e_{j+1}^i = & A e_j^i + I_1 E_{1,j} D_1 e_j^i + I_1 E_{2,j} D_2 e_j^i + B \bar{u}_j^i \\
& + L E_{3,j} D_{11} \bar{u}_j^i + B \bar{w}_j^i, \ i \in \{1, 2, \ldots, M\},
\end{aligned} \tag{12}$$

where $e_j^i = [e_{j-N}^i, e_{j-N+1}^i, \ldots, e_{j-M-1}^i, e_{j-M}^i, e_{j-M+1}^{i-1}, \ldots, e_{j-1}^{i-M+1}]^T$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $D_1 = \begin{bmatrix} 0 & 0 \\ 0 & D_{11} \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}$, $L = \begin{bmatrix} 0 \\ I_1 \end{bmatrix}$.

According to the definition of the state vector (6), it is obvious that the form of the error state-space model (12) for any train $i, i \in \{1, 2, \ldots, M\}$ is equivalent. Therefore, we only need to study one of the error state-space model (12) for any train i. Without loss of generality, we choose the error state-space model (12) of the M-th train, i.e., $i = M$, and design the robust train regulation. For simplicity, we rewrite the error state-space model (12) of the M-th train as the following matrix form

$$\begin{aligned}
e(j+1) = & A e(j) + I_1 E_{1,j} D_1 e(j) + I_1 E_{2,j} D_2 e(j) + B \bar{u}(j) \\
& + L E_{3,j} D_{11} \bar{u}(j) + B \bar{w}(j),
\end{aligned} \tag{13}$$

where $e(j) = [e_{j-N}^M, e_{j-N+1}^M, \ldots, e_{j-M-1}^M, e_{j-M}^M, e_{j-M+1}^{M-1}, \ldots, e_{j-1}^1]^T$, $\bar{u}(j) = [u_{j-M}^M, u_{j-M+1}^{M-1}, \ldots, u_{j-1}^1]^T$, $\bar{w}(j) = [\frac{\tilde{c}_{1,j-M}}{c_1} w_{j-M}^M, \frac{\tilde{c}_{1,j-M+1}}{c_1} w_{j-M+1}^{M-1}, \ldots, \frac{\tilde{c}_{1,j-1}}{c_1} w_{j-1}^1]^T$, the index j represents each decision step j, and the system parameters are the same to that in (12).

We choose the state feedback control for the train regulation as

$$\bar{u}(j) = K(j) e(j), \tag{14}$$

where $K(j)_{M\times N}$ is the control parameter to be determined.

C. The robust model predictive control problem

For the train regulation, one should guarantee an adequate level of train operating performance. Then associated with the train regulation objective is the following cost function

$$G = \sum_{j=j_0}^{j_f} \left\{ e^T(j)Qe(j) + \bar{u}^T(j)R\bar{u}(j) \right\}, \tag{15}$$

where Q and R are given positive definite weighted matrices, j_0 and j_f are the initial state number and the terminal state number, respectively. The first part of (15) represents the deviations from the nominal time schedule and the second part is the amplitude of the control force.

For the train operation problem, the robust MPC approach uses the current train dynamic state, train model, and operational limits to calculate future changes in the control variable so that the train operating performance (15) is optimized for all admissible uncertainties. Assume that exact measurement of the state of the system $e(j)$ is available at each sampling decision step j. Then associated with (15) is the following optimization problem to compute the control input at each prediction step j.

$$\begin{aligned} &\min_{\bar{u}(j+i|j)=K(j)e(j+i|j), i=0,1,2,\ldots,j_f-j} G(j) \\ &= \sum_{i=0}^{j_f-j} \left\{ e(j+i|j)^T Q e(j+i|j) \right. \\ &\quad \left. + \bar{u}^T(j+i|j)R\bar{u}(j+i|j) \right\}, \end{aligned} \tag{16}$$

where $e(j+i|j)$ is the error state at step $j+i$ predicted based on the measurements at step j, $\bar{u}(j+i|j)$ is the control move at step $j+i$ computed by the optimization problem (16) at step j.

Associated with the optimization problem (16) is the following constraints for the control input $\bar{u}(j)$.

$$-\check{u} \leq \bar{u}(j+i|j) \leq \hat{u}, \tag{17}$$

where $\check{u}$ and $\hat{u}$ are two known positive vectors.

In all practical applications, some disturbances invariably enter the system and hence it is meaningful to study their effect on the closed-loop response. The H_∞ control method can be used to study the disturbance rejection of the system to the uncertain disturbance [20], [21]. By combining the H_∞ control method, the robust model predictive control for train regulation problem in metro line with disturbances considered in this paper can be formulated as follows: for the error state-space model (13), given a prescribed H_∞ disturbance attenuation level $\gamma > 0$, obtain the control gains $K(j)$ such that the following conditions hold.

1) In the case when $\bar{w}_j = 0$, the cost function (16) with the constraints (17) is minimized for all admissible uncertainties.

2) under the zero initial condition $e(j) = 0$, the error states satisfy

$$\begin{aligned} &\sum_{i=0}^{j_f-j} (e(j+i|j)^T e(j+i|j))^{1/2} \\ &\leq \gamma \sum_{i=0}^{j_f-j} (\bar{w}(j+i|j)^T \bar{w}(j+i|j))^{1/2}. \end{aligned} \tag{18}$$

The goal of robust model predictive control approach formulated in this paper is to develop the state feedback control law that satisfies all the constraints to track the nominal schedule with a minimized cost function by rejecting the effect of the uncertainties and disturbances.

III. Robust Model Predictive Control for Train Regulation

In this section, we will study the robust model predictive control for train regulation in underground railway transportation with uncertain passenger arrival rates. At first, to reformulate the optimization problem (16) into an efficiently solvable form, we will present the optimized condition for the optimization problem (16) when the uncertain disturbance $\bar{w}_k = 0$ as the following theorem.

Theorem 3.1: Consider the error state-space model (13) for the metro loop line with N stations and $w_k = 0$. Let the passengers average arrival rate in the stations of metro line satisfy the condition (4). At each sampling step j, the state feedback matrix $K(j)$ in the control law $\bar{u}(j+i|j) = K(j)e(j+i|j),\ i \geq 0$ which minimizes an upper bound $\alpha(j)$ on the worst case of the objective function $G(j)$, is given by $K(j) = Y(j)X^{-1}(j)$, where $Y(j)$ and $X(j)$ are obtained from the solution of the following LMIs:

$$\min_{(\tilde{\varepsilon}_1(j)>0,\ \tilde{\varepsilon}_2(j)>0,\ \tilde{\varepsilon}_3(j)>0,\ \alpha(j)>0,\ X(j)>0,\ Y(j),\ Z(j)>0)} \alpha(j) \tag{19}$$

subject to

$$\begin{bmatrix} -X(j) & \Phi_1(j) & X(j)D_1^T & X(j)D_2^T & Y^T(j)D_{11}^T & X(j) & Y^T(j) \\ \Phi_1^T(j) & \Phi_2(j) & 0 & 0 & 0 & 0 & 0 \\ D_1X(j) & 0 & -\tilde{\varepsilon}_1(j)I & 0 & 0 & 0 & 0 \\ D_2X(j) & 0 & 0 & -\tilde{\varepsilon}_2(j)I & 0 & 0 & 0 \\ D_{11}Y(j) & 0 & 0 & 0 & -\tilde{\varepsilon}_3(j)I & 0 & 0 \\ X(j) & 0 & 0 & 0 & 0 & -\alpha(j)Q^{-1} & 0 \\ Y(j) & 0 & 0 & 0 & 0 & 0 & -\alpha(j)R^{-1} \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} 1 & e^T(j|j) \\ e(j|j) & X(j) \end{bmatrix} \geq 0, \tag{21}$$

$$\begin{bmatrix} Z(j) & Y(j) \\ Y^T(j) & X(j) \end{bmatrix} \geq 0, \tag{22}$$

$$Z_{ll}(j) \leq \bar{u}_l^2, \tag{23}$$

where $\Phi_1(j) = X(j)A^T + Y^T(j)B^T$, $\Phi_2(j) = -X(j) + \tilde{\varepsilon}_1(j)I_1I_1^T + \tilde{\varepsilon}_2(j)I_1I_1^T + \tilde{\varepsilon}_3(j)LL^T$, $\bar{u}_l = \frac{\hat{u}_l+\check{u}_l}{2} - |\frac{\hat{u}_l-\check{u}_l}{2}|$, $Z_{ll}(j)$ denotes the l-th diagonal element of the matrix $Z(j)$.

Proof: For the error state-space model (13) with $w_k = 0$, at the sampling step j, construct the following Lyapunov function candidate

$$V(e(j)) = e^T(j)P(j)e(j). \tag{24}$$

Suppose the following robust stability constraint is satisfied:

$$\begin{aligned} &V(e(j+i+1|j)) - V(e(j+i|j)) \\ &\leq -[e^T(j+i|j)Qe(j+i|j) + u^T(j+i|j)Ru(j+i|j)], \end{aligned} \tag{25}$$

where $i = 0, 1, \ldots, j_f - j$.

Summing (25) from $i=0$ to $i=j_f-j$, it follows that

$$\begin{aligned} G(j) =& \sum_{i=0}^{j_f-j}\left\{e^T(j+i|j)Qe(j+i|j)+u^T(j+i|j)Ru(j+i|j)\right\} \\ \leq & \; V(e(j|j))-V(e(j_f+1|j)) \\ \leq & \; V(e(j|j)). \end{aligned} \tag{26}$$

Let $V(e(j|j)) \leq \alpha(j)$, which gives an upper bound $\alpha(j)$ on $G(j)$. Then the optimization problem (16) is relaxed by a convex optimization problem to minimize $\alpha(j)$.

The condition $V(e(j|j)) \leq \alpha(j)$ can be expressed equivalently as the LMI (21), where $X(j)=\alpha(j)P^{-1}(j)$. Next, we will give the sufficient conditions for (25) and the control constraints (17).

At sampling step j, one can further obtain that

$$\begin{aligned} & V(e(j+i+1|j))-V(e(j+i|j)) \\ =& \; e^T(j+i|j)[(A+I_1E_{1,j+i}D_1+I_1E_{2,j+i}D_2+BK(j) \\ & +LE_{3,j+i}D_{11}K(j))^TP(j)(A+I_1E_{1,j+i}D_1 \\ & +I_1E_{2,j+i}D_2+BK(j)+LE_{3,j+i}D_{11}K(j))]e(j+i|j) \\ & -e^T(j+i|j)P(j)e(j+i|j) \end{aligned} \tag{27}$$

Then the robust stability constraint (25) holds if and only if

$$\begin{aligned} & (A+I_1E_{1,j+i}D_1+I_1E_{2,j+i}D_2+BK(j) \\ & +LE_{3,j+i}D_{11}K(j))^TP(j)(A+I_1E_{1,j+i}D_1 \\ & +I_1E_{2,j+i}D_2+BK(j)+LE_{3,j+i}D_{11}K(j)) \\ & -P(j)+Q+K^T(j)RK(j)<0. \end{aligned} \tag{28}$$

According to the Schur complement [22], condition (28) is equivalent to

$$\begin{aligned} & \begin{bmatrix} -P(j)+Q+K^T(j)RK(j) & (A+BK(j))^T \\ A+BK(j) & -P^{-1}(j) \end{bmatrix} \\ & +2\begin{bmatrix} D_1^T \\ 0 \end{bmatrix} E_{1,j+i}\begin{bmatrix} 0 & I_1^T \end{bmatrix} + 2\begin{bmatrix} D_2^T \\ 0 \end{bmatrix} E_{2,j+i}\begin{bmatrix} 0 & I_1^T \end{bmatrix} \\ & +2\begin{bmatrix} K^T(j)D_{11}^T \\ 0 \end{bmatrix} E_{3,j+i}\begin{bmatrix} 0 & L^T \end{bmatrix}<0. \end{aligned} \tag{29}$$

By the Cauchy inequality that $A^TB+B^TA \leq \varepsilon^{-1}A^TA+\varepsilon B^TB, \forall\ \varepsilon>0$, and matrices A and B with appropriate dimensions, together with the condition (4), it holds that

$$\begin{aligned} & 2\begin{bmatrix} D_1^T \\ 0 \end{bmatrix} E_{1,j+i}\begin{bmatrix} 0 & I_1^T \end{bmatrix} \\ & \leq \varepsilon_1\begin{bmatrix} 0 \\ I_1 \end{bmatrix}\begin{bmatrix} 0 & I_1^T \end{bmatrix}+\varepsilon_1^{-1}\begin{bmatrix} D_1^T \\ 0 \end{bmatrix}\begin{bmatrix} D_1 & 0 \end{bmatrix}, \end{aligned} \tag{30}$$

$$\begin{aligned} & 2\begin{bmatrix} D_2^T \\ 0 \end{bmatrix} E_{2,j+i}\begin{bmatrix} 0 & I_1^T \end{bmatrix} \\ & \leq \varepsilon_2\begin{bmatrix} 0 \\ I_1 \end{bmatrix}\begin{bmatrix} 0 & I_1^T \end{bmatrix}+\varepsilon_2^{-1}\begin{bmatrix} D_2^T \\ 0 \end{bmatrix}\begin{bmatrix} D_2 & 0 \end{bmatrix}, \end{aligned} \tag{31}$$

$$\begin{aligned} & 2\begin{bmatrix} K^T(j)D_{11}^T \\ 0 \end{bmatrix} E_{3,j+i}\begin{bmatrix} 0 & L^T \end{bmatrix} \leq \varepsilon_3\begin{bmatrix} 0 \\ L \end{bmatrix}\begin{bmatrix} 0 & L^T \end{bmatrix} \\ & +\varepsilon_3^{-1}\begin{bmatrix} K^T(j)D_{11}^T \\ 0 \end{bmatrix}\begin{bmatrix} D_{11}K(j) & 0 \end{bmatrix}. \end{aligned} \tag{32}$$

Thus, by combining (29)–(32), we can obtain that the following matrix inequality

$$\begin{bmatrix} -P(j)+Q+K^T(j)RK(j) & (A+BK(j))^T & D_1^T & D_2^T & K^T(j)D_{11}^T \\ A+BK(j) & \Omega(j) & 0 & 0 & 0 \\ D_1 & 0 & -\varepsilon_1 I & 0 & 0 \\ D_2 & 0 & 0 & -\varepsilon_2 I & 0 \\ D_{11}K(j) & 0 & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0 \tag{33}$$

implies that (28) holds, where $\Omega(j)=-P^{-1}(j)+\varepsilon_1I_1I_1^T+\varepsilon_2I_1I_1^T+\varepsilon_3LL^T$.

In addition, by variable substitution, let $X(j)=\alpha(j)P^{-1}(j)$, $Y(j)=K(j)X(j)$, $\tilde{\varepsilon}_1(j)=\alpha(j)\varepsilon_1$, $\tilde{\varepsilon}_2(j)=\alpha(j)\varepsilon_2$, $\tilde{\varepsilon}_3(j)=\alpha(j)\varepsilon_3$, where $\alpha(j)>0$. According to the Schur complement [22], pre and post-multiplying both sides of (20) by $\mathrm{diag}\{\alpha^{1/2}(j)X^{-1}(j), \alpha^{-1/2}(j)I, \alpha^{-1/2}(j)I, \ldots, \alpha^{-1/2}(j)I\}$, one can obtain that the inequality (20) is equivalent to the inequality (33). Thus the condition (20) ensures the robust stability constraint (25).

Moreover, the control constraints (17) can be expressed as $-\frac{\hat{u}_l+\check{u}_l}{2} \leq \bar{u}(j+i|j)_l-\frac{\hat{u}_l-\check{u}_l}{2} \leq \frac{\hat{u}_l+\check{u}_l}{2}$, i.e., $|\bar{u}(j+i|j)_l-\frac{\hat{u}_l-\check{u}_l}{2}| \leq \frac{\hat{u}_l+\check{u}_l}{2}$, where $\bar{u}(j+i|j)_l$ is the l-th element in the control input $\bar{u}(j+i|j)$, and $\hat{u}_l$ and $\check{u}_l$ are the l-th element of $\hat{u}$ and $\check{u}$ respectively. According to $|\bar{u}(j+i|j)_l-\frac{\hat{u}_l-\check{u}_l}{2}| \leq |\bar{u}(j+i|j)_l|+|\frac{\hat{u}_l-\check{u}_l}{2}|$, thus the control constraints (17) can be guaranteed by the inequality $|\bar{u}(j+i|j)_l| \leq \frac{\hat{u}_l+\check{u}_l}{2}-|\frac{\hat{u}_l-\check{u}_l}{2}|$.

Then according to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\bar{u}(j+i|j)_l|^2 =& \; |K_l(j)P^{-\frac{1}{2}}(j)P^{\frac{1}{2}}(j)e(j+i|j)|^2 \\ \leq & \; K_l(j)P^{-1}(j)K_l^T(j)e^T(j+i|j)P(j)e(j+i|j) \\ \leq & \; K_l(j)P^{-1}(j)K_l^T(j)\alpha(j), \end{aligned} \tag{34}$$

where $K_l(j)$ represents the $l-$th row of the matrix $K(j)$.

Pre and post-multiplying both sides of (22) by $\mathrm{diag}\{I, X^{-1}(j)\}$, one can get that (22) is equivalent to

$$\begin{bmatrix} Z(j) & K(j) \\ K^T(j) & \alpha^{-1}(j)P(j) \end{bmatrix} \geq 0. \tag{35}$$

It is shown from (23) and (35) that

$$K_l(j)P^{-1}(j)K_l^T(j)\alpha(j) \leq Z(j), Z_{ll}(j) \leq \bar{u}_l^2, \tag{36}$$

where $\bar{u}_l=\frac{\hat{u}_l+\check{u}_l}{2}-|\frac{\hat{u}_l-\check{u}_l}{2}|$, which implies from (34) that $|\bar{u}(j+i|j)_l| \leq \frac{\hat{u}_l+\check{u}_l}{2}-|\frac{\hat{u}_l-\check{u}_l}{2}|$, i.e., the control constraints (17) are satisfied.

Therefore, the minimized condition for the optimization problem (16) when the uncertain disturbance $w_k=0$ is relaxed to the convex optimization problem (19) with the constraints (20)–(23). ∎

Remark 3.1: It should be pointed out that the convex optimization problem (19) in Theorem 3.1 takes the form of linear matrix inequalities, which can be easily solved by using the Matlab LMI toolbox. Consequently, the proposed robust model predictive control scheme has a low computational complexity since the LMI-based optimization problems can be solved in polynomial time [18].

Next, based on Theorem 3.1, we will design the robust train regulation law for a metro line with uncertain passenger arrival rates and uncertain disturbances by using robust MPC method.

The following theorem will give a sufficient condition for the existence of a robust MPC law for a metro loop line with uncertain disturbances.

Theorem 3.2: For a given constant $\gamma > 0$ at each sampling step j, if the following LMI-based minimization problem:

$$\min_{(\tilde{\varepsilon}_1(j)>0,\ \tilde{\varepsilon}_2(j)>0,\ \tilde{\varepsilon}_3(j)>0,\ \alpha(j)>0,\ X(j)>0,\ Y(j),\ Z(j)>0)} \alpha(j) \quad (37)$$

subject to $(21)-(23)$,

$$\begin{bmatrix} \Xi_1(j) & \Xi_2^T(j) \\ \Xi_2(j) & \Xi_3(j) \end{bmatrix} < 0, \quad (38)$$

where

$$\Xi_1(j) = \begin{bmatrix} -X(j) & 0 & \Phi_1(j) & X(j)D_1^T & X(j)D_2^T & Y^T(j)D_{11}^T \\ 0 & -\alpha(j)\gamma^2 I & \alpha(j)B^T & 0 & 0 & 0 \\ \Phi_1^T(j) & \alpha(j)B & \Phi_2(j) & 0 & 0 & 0 \\ D_1X(j) & 0 & 0 & -\tilde{\varepsilon}_1 I & 0 & 0 \\ D_2X(j) & 0 & 0 & 0 & -\tilde{\varepsilon}_2 I & 0 \\ D_{11}Y(j) & 0 & 0 & 0 & 0 & -\tilde{\varepsilon}_3 I \end{bmatrix},$$

$$\Xi_2(j) = \begin{bmatrix} X(j)\ 0\ 0\ 0\ 0\ 0 \\ X(j)\ 0\ 0\ 0\ 0\ 0 \\ Y(j)\ 0\ 0\ 0\ 0\ 0 \end{bmatrix},$$

$$\Xi_3(j) = \begin{bmatrix} -\alpha(j)I & 0 & 0 \\ 0 & -\alpha(j)Q^{-1} & 0 \\ 0 & 0 & -\alpha(j)R^{-1} \end{bmatrix},$$

admits a set of solutions $(\tilde{\varepsilon}_1(j),\ \tilde{\varepsilon}_2(j),\ \tilde{\varepsilon}_3(j),\ \alpha(j),\ X(j),\ Y(j),\ Z(j))$, then the robust model predictive control law $\bar{u}(j+i|j) = Y(j)X^{-1}(j)e(j+i|j)$ can be obtained to ensure the minimization of the upper bound on the train operating performance $G(j)$, and meanwhile guarantee a disturbance attenuation level γ with respect to the uncertain disturbances.

Proof: At first, according to the Schur complement [22], it can be derived that the inequality (38) implies that condition (20) holds. So according to Theorem 3.1, under the conditions (21)-(23), the cost function (16) with the constraints (17) is minimized for all admissible uncertainties.

Next for any nonzero disturbance $\bar{w}(j)$ with finite energy, one can get

$$V(e(j+i+1|j)) - V(e(j+i|j)) \quad (39)$$
$$= \begin{bmatrix} e(j+i|j) \\ \bar{w}(j+i|j) \end{bmatrix}^T \begin{bmatrix} -P(j)+\Theta^T(j)P(j)\Theta(j) & \Theta^T(j)P(j)B \\ B^T P\Theta(j) & B^T P(j)B \end{bmatrix} \begin{bmatrix} e(j+i|j) \\ \bar{w}(j+i|j) \end{bmatrix},$$

where $\Theta(j) = A + I_1E_{1j}D_1 + I_1E_{2j}D_2 + BK(j) + LE_{3j}D_{11}K(j)$.

Additionally, let $X(j) = \alpha(j)P^{-1}(j)$, $Y(j) = K(j)X(j)$. Pre and post-multiplying both sides of (38) by $\mathrm{diag}\{\alpha^{1/2}(j)X^{-1}(j), \alpha^{-1/2}(j)I, \alpha^{-1/2}(j)I, \ldots, \alpha^{-1/2}(j)I\}$ and combining the results with conditions (30)–(32), we can get that the condition (38) implies that

$$\begin{bmatrix} -P(j)+\Theta^T(j)P(j)\Theta(j)+I+Q+K^T(j)RK(j) & \Theta^T(j)P(j)B \\ B^T P\Theta(j) & B^T P(j)B-\gamma^2 I \end{bmatrix}$$
$$< 0, \quad (40)$$

which shows that

$$\begin{bmatrix} -P(j)+\Theta^T(j)P(j)\Theta(j)+I & \Theta^T(j)P(j)B \\ B^T P\Theta(j) & B^T P(j)B-\gamma^2 I \end{bmatrix}$$
$$< \begin{bmatrix} -Q-K^T(j)RK(j) & 0 \\ 0 & 0 \end{bmatrix} \leq 0. \quad (41)$$

It is obviously follows from (39)–(41) that

$$V(e(j+i+1|j)) - V(e(j+i|j)) + [e(j+i|j)^T e(j+i|j)]$$
$$-\gamma^2 \bar{w}(j+i|j)^T \bar{w}(j+i|j) < 0. \quad (42)$$

Thus, under the zero initial condition $e(j) = 0$, summing (42) from $i = 1$ to $i = j_f - j$ gives that

$$\sum_{i=1}^{j_f-j} (e(j+i|j)^T e(j+i|j) - \gamma^2 \bar{w}(j+i|j)^T \bar{w}(j+i|j))$$
$$< 0, \quad (43)$$

which shows that

$$\sum_{i=1}^{j_f-j} (e^T(j+i|j)e(j+i|j))^{1/2}$$
$$\leq \gamma \sum_{i=1}^{j_f-j} (w^T(j+i|j)w(j+i|j))^{1/2} \quad (44)$$

Therefore, according to Definition 2.1, the robust MPC law $\bar{u}(j+i|j) = Y(j)X^{-1}(j)e(j+i|j)$ is obtained, which ensures the minimization of the upper bound on the train operating performance $G(j)$, and meanwhile guarantees a disturbance attenuation level γ with respect to the uncertain disturbances. ■

According to Theorem 3.2, the main algorithm of the robust model predictive control for the train regulation in metro loop line is summarized as follows.

Algorithm 3.1:

- Step 1. At sampling step j, obtain the measure state $e(j) = e(j|j)$ for the error state-space model (13) of the train regulation in the metro loop line with uncertain parameters and disturbances.
- Step 2. For a given disturbance attenuation level $\gamma > 0$, according to Theorem 3.2, by solving the optimization problem (37), get the robust MPC gain $K(j) = Y(j)X^{-1}(j)$ and apply it to the train traffic model to obtain the next value $e(j+1)$.
- Step 3. Based on the measured value $e(j+1)$, repeat Step 1 and 2 until the step horizon j_f.

IV. Numerical Examples

In this section, we will give an example to illustrate the effectiveness of the proposed methods for the train regulation of underground railway transportation. Consider a metro loop line with 18 stations (i.e., $N = 18$) where a set of 14 trains (i.e., $M = 14$) are operating simultaneously on the loop line. For the cost function (15), the weight matrices are chosen as $Q = 0.1I_{18}$ and $R = 0.1I_{14}$. We choose the operating condition of the metro system during the peak hours from $7:00$am to $10:00$am.

According to the passenger flow of Yizhuang Line in Beijing metro system on one day in Figure 2 [23], we choose that the passenger arrival rates vary within a range of values that are symmetric around the known nominal value $\lambda = 0.35$ with a half-length $d = 0.15$ at each station during the peak hours from $7:00$am to $10:00$am. The average boarding time per passenger during the peak hours is chosen as $a = 0.2s$.

Additionally, the uncertain disturbances to the running time and staying time of trains are assumed to be varying in the interval $[2\mathrm{s}, 7\mathrm{s}]$.

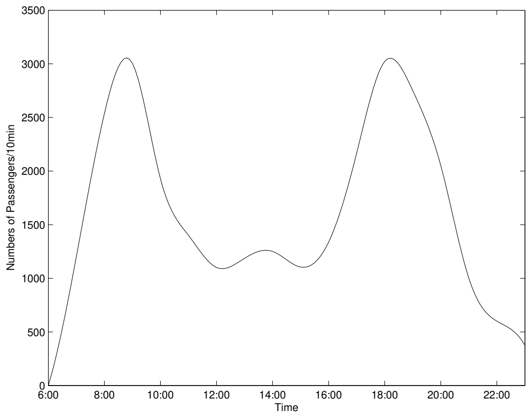

Figure 2. The passenger flow of Yizhuang Line in Beijing metro system on one day.

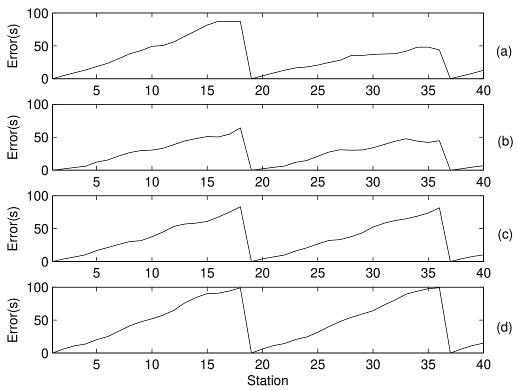

Figure 3. The error state for the metro system with $u_k^i = 0$ for train 1 (a), train 3 (b), train 10 (c) and train 14 (d).

For the error state-apace model of train traffic with $u_k^i = 0$, we apply the traditional train regulation method by using of time margins (resetting the time deviations to zero at selected stations) [2]. For this method, one of these stations is the terminus where the train staying time is adapted in order to ensure the periodicity of the nominal time schedule, which however will require more trains in standby at the terminus. Consider that the total staying time at the terminus is $6\mathrm{min}$ constituted by the minimal time $3\mathrm{min}$ and the time margin $3\mathrm{min}$ that is used for resetting the time deviations to zero. Under this case, the error state evolutions for the time deviations of trains 1, 3, 10 and 14 with uncertain passenger arrival flow and uncertain disturbances are plotted in Figure 3. Figure 3 shows that due to the uncertain passenger arrival flow and disturbances, the departure times of all the trains are delayed and the delays of the trains are propagated from one station to the next one, i.e., the deviation with respect to the nominal time schedule is amplified with time. At the terminus, the time margin is allowed to recover the nominal time schedule for all the trains. However, at each loop for each train, the maximum delays is exceed $50\mathrm{s}$. This unstable behavior is quite uncomfortable for passengers.

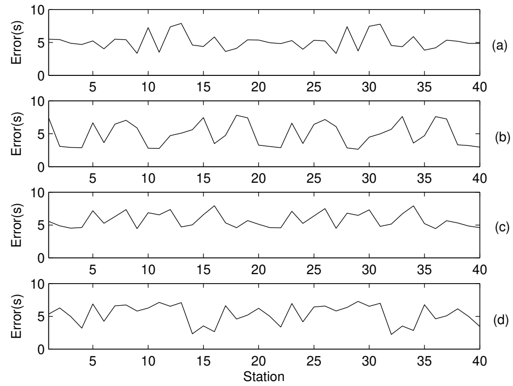

Figure 4. The error state for the metro system with robust MPC for train 1 (a), train 3 (b), train 10 (c) and train 14 (d).

Next, according to the robust MPC approach for train regulation in the metro loop line proposed in this paper, we will design the robust MPC law to suppress the effect of the uncertain passenger arrival flows and the uncertain disturbances to the nominal schedule of the metro system. Suppose that the state feedback control u_k^i is subject to the constraint $-20\mathrm{s} \leq u_k^i \leq 25\mathrm{s}$, i.e., the increase of the adjusting running time and staying time for each train from one station to the next station is not allowed to exceed 25s and the decrease is not exceed 20s. Choose a small H_∞ disturbance attenuation as $\gamma = 2.9$, then based on the Theorem 3.2 and performing Algorithm 3.1 with Matlab LMIs Toolbox, the corresponding state feedback control gains for each decision step can be obtained. To keep the paper concise, the controller gains $K \in R^{M\times N}$ at each decision step are not presented here for the dimensions are very large.

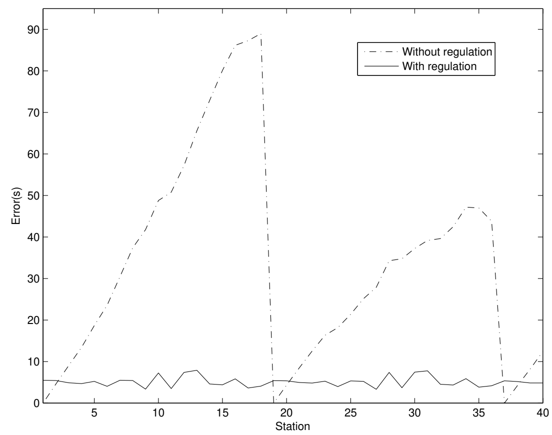

Figure 5. The error state for the operation of train 1.

TABLE I
THE COMPUTATIONAL TIMES FOR THE DIFFERENT NUMBERS OF STATIONS AND TRAINS

	Case 1	Case 2	Case 3	Case 4	Case 5
(N, M)	(18,14)	(22,16)	(26,18)	(30,20)	(34,22)
Computational time	6.1s	14.2s	24.1s	42.6s	74.2s

Under the robust regulation, the simulation results of the time deviations of trains 1, 3, 10, and 14 are given in Figure 4. From Figure 4, we find that the maximum delays of each train are effectively controlled in $8\mathrm{s}$. By comparing Figure

3 and Figure 4, it is obvious that the proposed robust train regulation significantly reduces the delays of all the trains, and improves the operation efficiency of the metro loop line system. In particular, for the operation of train 1, and the corresponding error state evolution for the metro system is shown in Figure 5, in which the dashed line represents the result of the metro system without train regulation and the solid line is the case of the metro system with robust train regulation. When the train operation is without train regulation, the delay of the trains is increasing from 0s to 90s at the first loop and from 0s to 50s at the second loop. By comparison, the solid line shows that under the robust train regulation, the delays of the trains are controlled in a reasonable range of 8s. By calculation, the total delay of train 1 is reduced 83% compared by the case without train regulation. Clearly the robust train regulation strategy significantly improves the operation efficiency of metro loop line system. Additionally, it should be pointed out that the computational time for each decision step is calculated as 6.1s, which shows that the computational time is short enough for the real-time control of underground railway transportation. Since the matrix variable $K \in R^{M\times N}$, the number of the stations and trains will affect the computational times. By increasing the numbers of stations and trains, a set of computational time are calculated in table 1, which shows that the computational time increases with the number of the stations and trains in a polynomial time. Therefore the proposed train regulation method can be implemented on-line for the usual metro lines with dozens of stations and trains.

V. Conclusion

In this paper, the robust MPC for train regulation in underground railway transportation was investigated. By considering the uncertain disturbances to the train operation, the robust model predictive control law for train regulation was designed that ensures the minimization of the upper bound on metro system cost function, and meanwhile guarantees a given disturbance attenuation level with respect to the uncertain disturbances. In addition, it is easy to extend the proposed robust MPC method to deal with the metro line with merging and branching. If the number of the stations and trains for this case is too large, one may resort to the distributed robust MPC design method, which needs to be investigated in the future.

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